The Existence of the Riemann Integral

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Outline

Although we have a definition for what it means for a bounded function to be Riemann integrable, we still do not actually know that $RI[a, b]$ is nonempty! We will now show how we prove that the set of Riemann integrable functions is quite rich and varied.

Definition

Darboux Upper and Lower Sums
Let $f \in B[a, b]$. Let $\boldsymbol{\pi} \in \boldsymbol{\Pi}[a, b]$ be given by $\boldsymbol{\pi} = \{x_0 = a, x_1, \ldots, x_p = b\}$. Define for $1 \leq j \leq p$

$$m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x) \quad 1 \leq j \leq p, \ M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) \quad .$$

We define the **Lower Darboux Sum** $L(f, \boldsymbol{\pi})$ and **Upper Darboux Sum** $U(f, \boldsymbol{\pi})$ by

$$L(f, \boldsymbol{\pi}) = \sum_{\boldsymbol{\pi}} m_j \Delta x_j, \ U(f, \boldsymbol{\pi}) = \sum_{\boldsymbol{\pi}} M_j \Delta x_j.$$

1. It is straightforward to see that

$$L(f,\pi) \;\leq\; S(f,\pi,\sigma) \leq U(f,\pi)$$

for all $\pi \in \mathbf{\Pi}[a,b]$.

2. We also have

$$U(f,\pi) - L(f,\pi) \;=\; \sum_{\pi}(M_j - m_j)\Delta x_j.$$

Theorem

If $\pi \preceq \pi'$, that is, if π' refines π, then $L(f,\pi) \leq L(f,\pi')$ and $U(f,\pi) \geq U(f,\pi')$.

Proof

The general result is established by induction on the number of points added. It is actually quite an involved induction.
Step 1:
We prove the proposition for inserting points $\{z_1, \ldots, z_q\}$ *into one subinterval of* π. *The argument consists of*

1. *The* Basis *Step where we prove the proposition for the insertion of a single point into one subinterval.*
2. *The* Induction *Step where we assume the proposition holds for the insertion of* q *points into one subinterval and then we show the proposition still holds if an additional point is inserted.*
3. *With the* Induction *Step verified, the Principle of Mathematical Induction then tells us that the proposition is true for any refinement of* π *which places points into one subinterval of* π.

Proof

Step 2:
Next, we allow the insertion of a finite number of points into a finite number of subintervals of π. The induction is now on the number of subintervals.

1. *The* Basis *Step where we prove the proposition for the insertion of points into one subinterval.*
2. *The* Induction *Step where we assume the proposition holds for the insertion of points into q subintervals and then we show the proposition still holds if an additional subinterval has points inserted.*
3. *With the* Induction *Step verified, the Principle of Mathematical Induction then tells us that the proposition is true for any refinement of π which places points into any number of subintervals of π.*

Proof

Now the arguments:
Step 1:
Basis:
Let $\pi \in \Pi[a, b]$ *be given by* $\{x_0 = a, x_1, \ldots, x_p = b\}$*. Suppose we form the refinement,* π'*, by adding a single point* x' *to* π*. into the interior of the subinterval* $[x_{k_0-1}, x_{k_0}]$*. Let*

$$m' = \inf_{[x_{k_0-1}, x']} f(x), \quad m'' = \inf_{[x', x_{k_0}]} f(x).$$

Note that $m_{k_0} = \min\{m', m''\}$ *and*

$$\begin{aligned} m_{k_0} \Delta x_{k_0} &= m_{k_0}(x_{k_0} - x_{k_0-1}) \\ &= m_{k_0}(x_{k_0} - x') + m_{k_0}(x' - x_{k_0-1}) \\ &\leq m''(x_{k_0} - x') + m'(x' - x_{k_0-1}) \\ &\leq m'' \Delta x'' + m' \Delta x', \end{aligned}$$

where $\Delta x'' = x_{k_0} - x'$ *and* $\Delta x' = x' - x_{k_0-1}$.

Proof

It follows that

$$\begin{aligned} L(f, \pi') &= \sum_{j \neq k_0} m_j \Delta x_j + m' \Delta x' + m'' \Delta x'' \\ &\geq \sum_{j \neq k_0} m_j \Delta x_j + m_{k_0} \Delta x_{k_0} \\ &\geq L(f, \pi). \end{aligned}$$

Induction:
We assume that q *points* $\{z_1, \ldots, z_q\}$ *have been inserted into the subinterval* $[x_{k_0 - 1}, x_{k_0}]$. *Let* π' *denote the resulting refinement of* π. *We assume that*

$$L(f, \pi) \leq L(f, \pi')$$

Let the additional point added to this subinterval be called x' *and call* π'' *the resulting refinement of* π'.

Proof

We know that π' *has broken* $[x_{k_0-1}, x_{k_0}]$ *into* $q+1$ *pieces. For convenience of notation, let's label these* $q+1$ *subintervals as* $[y_{j-1}, y_j]$ *where* y_0 *is* x_{k_0-1} *and* y_{q+1} *is* x_{k_0} *and the* y_j *values in between are the original* z_i *points for appropriate indices. The new point* x' *is thus added to one of these* $q+1$ *pieces, call it* $[y_{j_0-1}, y_{j_0}]$ *for some index* j_0. *This interval plays the role of the original subinterval in the proof of the* Basis *Step.*

An argument similar to that in the proof of the Basis *Step then shows us that*

$$L(f, \pi') \quad \leq \quad L(f, \pi'')$$

Combining with the first inequality from the Induction *hypothesis, we establish the result. Thus, the* Induction *Step is proved.*

Proof

Step 2:
Basis:
Step 1 *above gives us the* Basis *Step for this proposition.*
Induction:
We assume the results holds for p subintervals and show it also holds when one more subinterval is added. Specifically, let π' *be the refinement that results from adding points to p subintervals of* π*. Then the* Induction *hypothesis tells us that*

$$L(f, \pi) \quad \leq \quad L(f, \pi')$$

Let π'' *denote the new refinement of* π *which results from adding more points into one more subinterval of* π*. Then* π'' *is also a refinement of* π' *where all the new points are added to one subinterval of* π'*. Thus, Step* 1 *holds for the pair* (π', π'')*. We see*

$$L(f, \pi') \quad \leq \quad L(f, \pi'')$$

and the desired result follows. A similar argument works for upper sums.

Theorem

Let π_1 and π_2 be any two partitions in $\Pi[a, b]$. Then $L(f, \pi_1) \leq U(f, \pi_2)$.

Proof

Let $\pi = \pi_1 \vee \pi_2$ be the common refinement of π_1 and π_2. Then, by the previous result, we have

$$L(f, \pi_1) \leq L(f, \pi) \leq U(f, \pi) \leq U(f, \pi_2).$$

These Theorems allows us to define a new type of integrability for the bounded function f. We begin by looking at the infimum of the upper sums and the supremum of the lower sums for a given bounded function f.

Theorem

Let $f \in B[a, b]$.
Let $\mathscr{L} = \{L(f, \pi) \mid \pi \in \mathbf{\Pi}[a, b]\}$
and
$\mathscr{U} = \{U(f, \pi) \mid \pi \in \mathbf{\Pi}[a, b]\}$.
Define $L(f) = \sup \mathscr{L}$, *and*
$U(f) = \inf \mathscr{U}$.
Then $L(f)$ *and* $U(f)$ *are both finite.*
Moreover, $L(f) \leq U(f)$.

Proof

We know the set $\mathscr{L}$ *is bounded above by any upper sum for* f*. Hence, it has a finite supremum and so* $\sup \mathscr{L}$ *is finite.*

Also, the set $\mathscr{U}$ *is bounded below by any lower sum for* f*. Hence,* $\inf \mathscr{U}$ *is finite.*

Finally, since $L(f) \leq U(f, \pi)$ *and* $U(f) \geq L(f, \pi)$ *for all* π*, by definition of the infimum and supremum of a set of numbers, we must have* $L(f) \leq U(f)$. □

Definition

Let f be in $B[a, b]$. The **Lower Darboux Integral** of f is defined to be the finite number $L(f) = \sup \mathscr{L}$, and the **Upper Darboux Integral** of f is the finite number $U(f) = \inf \mathscr{U}$.

We then define the Darboux integrability of a bounded function on $[a, b]$.

Definition

Let f be in $B[a, b]$. We say f is Darboux Integrable on $[a, b]$ if $L(f) = U(f)$. The common value is then called the Darboux Integral of f on $[a, b]$ and is denoted by the symbol $DI(f; a, b)$.

Comment

Not all bounded functions are Darboux Integrable. Consider

$$f(t) = \begin{cases} 1 & t \in [0,1] \text{ and is rational} \\ -1 & t \in [0,1] \text{ and is irrational} \end{cases}$$

For any partition of $[0, 1]$, the infimum of f on any subinterval is always -1 as any subinterval contains irrational numbers. Similarly, any subinterval contains rational numbers and so the supremum of f on a subinterval is 1. Thus $U(f, \pi) = 1$ and $L(f, \pi) = -1$ for any partition π of $[0, 1]$. It follows that $L(f) = -1$ and $U(f) = 1$. Thus, f is bounded but not Darboux Integrable.

Definition

Let $f \in B[a, b]$. We say that **Riemann's Criteria** holds for f if for every positive ϵ there exists a $\pi_0 \in \mathbf{\Pi}[a, b]$ such that $U(f, \pi) - L(f, \pi) < \epsilon$ for any refinement, π, of π_0.

Theorem

The Equivalence Theorem:
Let $f \in B[a, b]$. Then the following are equivalent.

(i) $f \in RI[a, b]$.

(ii) *f satisfies Riemann's Criteria.*

(iii) *f is Darboux Integrable, i.e, $L(f) = U(f)$, and $RI(f; a, b) = DI(f; a, b)$.*

Proof

$(i) \Rightarrow (ii)$:

Assume $f \in RI[a,b]$, *and let* $\epsilon > 0$ *be given. Let* IR *be the Riemann integral of* f *over* $[a,b]$. *Choose* $\boldsymbol{\pi}_0 \in \boldsymbol{\Pi}[a,b]$ *such that* $| S(f, \boldsymbol{\pi}, \boldsymbol{\sigma}) - IR | < \epsilon/3$ *for any refinement,* $\boldsymbol{\pi}$, *of* $\boldsymbol{\pi}_0$ *and any* $\boldsymbol{\sigma} \subset \boldsymbol{\pi}$. *Let* $\boldsymbol{\pi}$ *be any such refinement, denoted by* $\boldsymbol{\pi} = \{x_0 = a, x_1, \ldots, x_p = b\}$, *and let* m_j, M_j *be defined as usual. Using the Infimum and Supremum Tolerance Lemmas, we can conclude that, for each* $j = 1, \ldots, p$, *there exist* $s_j, t_j \in [x_{j-1}, x_j]$ *such that*

$$M_j - \frac{\epsilon}{6(b-a)} < f(s_j) \leq M_j$$

$$m_j \leq f(t_j) < m_j + \frac{\epsilon}{6(b-a)}.$$

It follows that

$$f(s_j) - f(t_j) > M_j - \frac{\epsilon}{6(b-a)} - m_j - \frac{\epsilon}{6(b-a)}.$$

Proof

Thus, we have

$$M_j - m_j - \frac{\epsilon}{3(b-a)} < f(s_j) - f(t_j).$$

Multiply this inequality by Δx_j to obtain

$$(M_j - m_j)\Delta x_j - \frac{\epsilon}{3(b-a)}\Delta x_j < \big(f(s_j) - f(t_j)\big)\Delta x_j.$$

Now, sum over π to obtain

$$\begin{aligned} U(f,\pi) - L(f,\pi) &= \sum_{\pi}(M_j - m_j)\Delta x_j \\ &< \frac{\epsilon}{3(b-a)}\sum_{\pi}\Delta x_j + \sum_{\pi}\big(f(s_j) - f(t_j)\big)\Delta x_j. \end{aligned}$$

Proof

This simplifies to

$$\sum_{\pi}(M_j - m_j)\Delta x_j - \frac{\epsilon}{3} \quad < \quad \sum_{\pi}(f(s_j) - f(t_j))\Delta x_j. \qquad (*)$$

Now, we have

$$\begin{aligned}
&\mid \sum_{\pi}(f(s_j) - f(t_j))\Delta x_j \mid \\
=&\mid \sum_{\pi} f(s_j)\Delta x_j - \sum_{\pi} f(t_j)\Delta x_j \mid \\
=&\mid \sum_{\pi} f(s_j)\Delta x_j - IR + IR - \sum_{\pi} f(t_j)\Delta x_j \mid \\
\leq&\mid \sum_{\pi} f(s_j)\Delta x_j - IR \mid + \mid \sum_{\pi} f(t_j)\Delta x_j - IR \mid \\
=&\mid S(f, \pi, \sigma_s) - IR \mid + \mid S(f, \pi, \sigma_t) - IR \mid
\end{aligned}$$

Proof

where $\boldsymbol{\sigma}_s = \{s_1, \ldots, s_p\}$ *and* $\boldsymbol{\sigma}_t = \{t_1, \ldots, t_p\}$ *are evaluation sets of* $\boldsymbol{\pi}$*. Now, by our choice of partition* $\boldsymbol{\pi}$*, we know*

$$\begin{aligned} \mid S(f, \boldsymbol{\pi}, \boldsymbol{\sigma}_s) - IR \mid &< \frac{\epsilon}{3} \\ \mid S(f, \boldsymbol{\pi}, \boldsymbol{\sigma}_t) - IR \mid &< \frac{\epsilon}{3}. \end{aligned}$$

Thus, we can conclude that

$$\mid \sum_{\boldsymbol{\pi}} (f(s_j) - f(t_j)) \Delta x_j \mid < \frac{2\epsilon}{3}.$$

Applying this to the inequality in Equation $*$*, we obtain*

$$\sum_{\boldsymbol{\pi}} (M_j - m_j) \Delta x_j < \epsilon.$$

Proof

Now, π was an arbitrary refinement of π_0, and $\epsilon > 0$ was also arbitrary. So this shows that f satisfies Riemann's condition.

(ii) $\Rightarrow$ (iii):

Now, assume that f satisfies Riemann's criteria, and let $\epsilon > 0$ be given. Then there is a partition, $\pi_0 \in \Pi[a,b]$ such that $U(f,\pi) - L(f,\pi) < \epsilon$ for any refinement, π, of π_0. Thus, by the definition of the upper and lower Darboux integrals, we have

$$U(f) \leq U(f,\pi) < L(f,\pi) + \epsilon \leq L(f) + \epsilon.$$

Since ϵ is arbitrary, this shows that $U(f) \leq L(f)$. The reverse inequality has already been established. Thus, we see that $U(f) = L(f)$.

(iii) $\Rightarrow$ (i):

Finally, assume f is Darboux integral which means $L(f) = U(f)$. Let ID denote the value of the Darboux integral. We will show that f is also Riemann integrable according to the definition and that the value of the integral is ID.

Proof

Let $\epsilon > 0$ *be given. Now, recall that*

$$ID = L(f) = \sup_{\pi} L(f, \pi) = U(f) = \inf_{\pi} U(f, \pi)$$

Hence, by the Supremum Tolerance Lemma, there exists $\pi_1 \in \Pi[a, b]$ *such that*

$$ID - \epsilon = L(f) - \epsilon < L(f, \pi_1) \leq L(f) = ID$$

and by the Infimum Tolerance Lemma, there exists $\pi_2 \in \Pi[a, b]$ *such that*

$$ID = U(f) \leq U(f, \pi_2) < U(f) + \epsilon = ID + \epsilon.$$

Let $\pi_0 = \pi_1 \vee \pi_2$ *be the common refinement of* π_1 *and* π_2. *Now, let* π *be any refinement of* π_0, *and let* $\sigma \subset \pi$ *be any evaluation set.*

Proof

Then we have

$$\begin{aligned} ID - \epsilon &< L(f, \pi_1) \le L(f, \pi_0) \le L(f, \pi) \\ &\le S(f, \pi, \sigma) \le U(f, \pi) \le U(f, \pi_0) \\ &\le U(f, \pi_2) < ID + \epsilon. \end{aligned}$$

Thus, it follows that

$$ID - \epsilon < S(f, \pi, \sigma) < ID + \epsilon.$$

Since the refinement, π, of π_0 was arbitrary, as were the evaluation set, σ, and the tolerance ϵ, it follows that for any refinement, π, of π_0 and any $\epsilon > 0$, we have

$$| S(f, \pi, \sigma) - ID | < \epsilon.$$

This shows that f is Riemann Integrable and the value of the integral is ID. □

We now know that the Darboux and Riemann integral are equivalent. Hence, it is now longer necessary to use a different notation for these two different approaches to what we call integration. From now on, we will use this notation

$$RI(f; a, b) \equiv DI(f; a, b) \equiv \int_a^b f(t)\, dt$$

where the (t) in the new integration symbol refers to the name we wish to use for the independent variable and dt is a mnemonic to remind us that the $|| \boldsymbol{\pi} ||$ is approaching zero as we choose progressively finer partitions of $[a, b]$. This is, of course, not very rigorous notation. A better notation would be

$$RI(f; a, b) \equiv DI(f; a, b) \equiv I(f; a, b)$$

where the symbol I denotes that we are interested in computing the integral of f using the equivalent approach of Riemann or Darboux. Indeed, the notation $I(f; a, b)$ does not require the uncomfortable lack of rigor that the symbol dt implies. However, for historical reasons, the symbol $\int_a^b f(t)\, dt$ will be used.

Homework 11

11.1 If f is RI on $[1,3]$ and g is RI on $[1,3]$ prove $3f - 4g$ is RI on $[1,3]$.

11.2 If f is RI on $[-2,5]$ and g is RI on $[-2,5]$ prove $4f + 7g$ is RI on $[-2,5]$.

11.3 Prove the constant function $f(x) = 3$ is RI for any finite interval $[a, b]$ with value $3(b - a)$.