Properties of the Riemann Integral

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Outline

Theorem

If f is a bounded function on the finite interval [a, b], then

1. $\sup_{a \leq x \leq b} f(x) = -\inf_{a \leq x \leq b}(-f(x))$ *and* $-\sup_{a \leq x \leq b}(-f(x)) = \inf_{a \leq x \leq b}(f(x))$
2. $$\begin{aligned} \sup_{x,y \in [a,b]} (f(x) - f(y)) &= \sup_{y \in [a,b]} \sup_{x \in [a,b]} (f(x) - f(y)) \\ &= \sup_{x \in [a,b]} \sup_{y \in [a,b]} (f(x) - f(y)) = M - m \end{aligned}$$
   where $M = \sup_{a \leq x \leq b} f(x)$ *and* $m = \inf_{a \leq x \leq b} f(x)$.
3. $\sup_{x,y \in [a,b]} \mid f(x) - f(y) \mid = M - m$

Proof

First let $Q = \sup_{a \leq x \leq b}(-f)$ *and* $q = \inf_{a \leq x \leq b}(-f)$.

Proof

(1):
Let $(f(x_n)$ be a sequence which converges to M. Then since $-f(x_n) \geq q$ for all n, letting $n \to \infty$, we find $-M \geq q$.

Now let $(-f(z_n))$ be a sequence which converges to q. Then, we have $f(z_n) \leq M$ for all n and letting $n \to]infty$, we have $-q \leq M$ or $q \geq -M$.

Combining, we see $-q = M$ which is the first part of the statement; i.e. $\sup_{a \leq x \leq b} f(x) = -\inf_{a \leq x \leq b}(-f(x))$. Now just replace all the f's by $-f$'s in this to get $\sup_{a \leq x \leq b}(-f(x)) = -\inf_{a \leq x \leq b}(- - f(x))$ or $-\sup_{a \leq x \leq b}(-f(x)) = \inf_{a \leq x \leq b}(f(x))$ which is the other identity.

(2):
We know

$$f(x) - f(y) \quad \leq \quad \left(\sup_{a \leq x \leq b} f(x) \right) - f(y) = M - f(y)$$

But $f(y) \geq \inf_{y \in [a,b]} f(y) = m$, so

$$f(x) - f(y) \quad \leq \quad M - f(y) \leq M - m$$

Proof

Thus,

$$\sup_{x,y \in [a,b]} (f(x) - f(y)) \leq M - m$$

So one side of the inequality is clear. Now let $f(x_n)$ be a sequence converging to M and $f(y_n)$ be a sequence converging to m. Then, we have

$$f(x_n) - f(y_n) \leq \sup_{x,y \in [a,b]} (f(x) - f(y))$$

Letting $n \to \infty$, we see

$$M - m \leq \sup_{x,y \in [a,b]} (f(x) - f(y))$$

This is the other side of the inequality. We have thus shown that the equality is valid.

Proof

(3):
Note

$$| f(x) - f(y) | \quad = \quad \begin{cases} f(x) - f(y), & f(x) \geq f(y) \\ f(y) - f(x), & f(x) < f(y) \end{cases}$$

In either case, we have $| f(x) - f(y) | \leq M - m$ *for all* x, y *using Part (2) implying that* $\sup_{x,y} | f(x) - f(y) | \leq M - m$.

To see the reverse inequality holds, we first note that if $M = m$*, we see the reverse inequality holds trivially as*
$\sup_{x,y} | f(x) - f(y) | \geq 0 = M - m$.

Hence, we may assume without loss of generality that the gap $M - m$ *is positive.*

Then, using the STL and ITL, given $0 < 1/j < 1/2(M - m)$*, there exist,* $s_j, t_j \in [a, b]$ *such that* $M - 1/(2j) < f(s_j)$ *and* $m + 1/(2j) > f(t_j)$*, so that* $f(s_j) - f(t_j) > M - m - 1/j$*. By our choice of* j*, these terms are positive and so we also have* $| f(s_j) - f(t_j) | > M - m - 1/j$.

Proof

It follows that

$$\sup_{x,y\in[a,b]} \mid f(x)-f(y) \mid \;\; \geq \;\; \mid f(s_j)-f(t_j) \mid > M-m-1/j \mid .$$

Since we can make $1/j$ arbitrarily small, this implies that

$$\sup_{x,y\in[x_{j-1},x_j]} \mid f(x)-f(y) \mid \;\; \geq \;\; M-m.$$

This establishes the reverse inequality and proves the claim. □

We are now ready to look at some of the properties of the Riemann Integral.

Theorem

Let $f, g \in RI[a,b]$. *Then*

(1) $|\ f\ | \in RI[a,b]$*;*

(2) $|\int_a^b f(x)dx| \leq \int_a^b |\ f\ |\ dx$*;*

(3) $f^+ = \max\{f, 0\} \in RI[a,b]$*;*

(4) $f^- = \max\{-f, 0\} \in RI[a,b]$*;*

(5) $\int_a^b f(x)dx = \int_a^b [f^+(x) - f^-(x)]dx \ = \ \int_a^b f^+(x)dx - \int_a^b f^-(x)dx$

$\int_a^b |\ f(x)\ |\ dx = \int_a^b [f^+(x) + f^-(x)]dx \ = \ \int_a^b f^+(x)dx + \int_a^b f^-(x)dx$*;*

(6) $f^2 \in RI[a,b]$*;*

(7) $fg \in RI[a,b]$*;*

(8) If there exists m, M *such that* $0 < m \leq |\ f\ | \leq M$, *then* $1/f \in RI[a,b]$.

Proof

(1) $|\, f \,| \in RI[a,b]$:
Note given a partition $\boldsymbol{\pi} = \{x_0 = a, x_1, \ldots, x_p = b\}$, *for each* $j = 1, \ldots, p$, *using the result above, we know*

$$\sup_{x,y \in [x_{j-1}, x_j]} (f(x) - f(y)) \quad = \quad M_j - m_j$$

Now, let m'_j *and* M'_j *be defined by*

$$m'_j \quad = \quad \inf_{[x_{j-1}, x_j]} |\, f(x) \,|, \quad M'_j = \sup_{[x_{j-1}, x_j]} |\, f(x) \,| \,.$$

Then, applying the first Theorem to $|f|$, *we have*

$$M'_j - m'_j \quad = \quad \sup_{x,y \in [x_{j-1}, x_j]} \Big(|\, f(x) \,| - |\, f(y) \,| \Big).$$

Proof

For each $j = 1, \ldots, p$, we have

$$M_j - m_j = \sup_{x,y \in [x_{j-1}, x_j]} |\, f(x) - f(y) \,| \,.$$

So, since $|\, f(x) \,| - |\, f(y) \,| \leq |\, f(x) - f(y) \,|$ for all x, y, it follows that $M_j' - m_j' \leq M_j - m_j$.

This implies $\sum_\pi (M_j' - m_j') \Delta x_j \leq \sum_\pi (M_j - m_j) \Delta x_j$.

This means $U(|f|, \pi) - L(|f|, \pi) \leq U(f, \pi) - L(f, \pi)$ for the chosen π.

Since f is integrable by hypothesis, we know the Riemann criterion must also hold for f.

Thus, given $\epsilon > 0$, there is a partition π_0 so that $U(f, \pi) - L(f, \pi) < \epsilon$ for any refinement π of π_0. Therefore $|f|$ also satisfies the Riemann Criterion and so $|\, f \,|$ is Riemann integrable.

Proof

(2) $|\int_a^b f(x)dx| \leq \int_a^b |f|dx$:
We have $f \leq |f|$ and $f \geq -|f|$, so that

$$\begin{aligned} \int_a^b f(x)dx &\leq \int_a^b |f(x)|\,dx \\ \int_a^b f(x)dx &\geq -\int_a^b |f(x)|\,dx, \end{aligned}$$

from which it follows that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|\,dx$$

and so

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Proof

(3) $f^+ = \max\{f, 0\} \in RI[a,b]$ *and (4)* $f^- = \max\{-f, 0\} \in RI[a,b]$*:*
This follows from the facts that $f^+ = \frac{1}{2}(\mid f \mid + f)$ *and* $f^- = \frac{1}{2}(\mid f \mid - f)$ *and the Riemann integral is a linear mapping.*

(5) $\int_a^b f(x)dx = \int_a^b [f^+(x) - f^-(x)]dx \; = \; \int_a^b f^+(x)dx - \int_a^b f^-(x)dx$
$\int_a^b \mid f(x) \mid dx = \int_a^b [f^+(x) + f^-(x)]dx \; = \; \int_a^b f^+(x)dx + \int_a^b f^-(x)dx$*:*
This follows from the facts that $f = f^+ - f^-$ *and* $\mid f \mid = f^+ + f^-$ *and the linearity of the integral.*

(6): $f^2 \in RI[a,b]$
Note that, since f *is bounded, there exists* $K > 0$ *such that* $\mid f(x) \mid \leq K$ *for all* $x \in [a,b]$*.*

Applying our infimum/ supremum properties theorem to f^2*, we have*

$$\sup_{x,y \in [x_{j-1}, x_j]} (f^2(x) - f^2(y)) \quad = \quad M_j(f^2) - m_j(f^2)$$

Proof

where $[x_{j-1}, x_j]$ *is a subinterval of a given partition* π *and* $M_j(f^2) = \sup_{x \in [x_{j-1}, x_j]} f^2(x)$ *and* $m_j(f^2) = \inf_{x \in [x_{j-1}, x_j]} f^2(x)$.

Thus, for this partition, we have

$$U(f^2, \pi) - L(f^2, \pi) = \sum_{\pi} \left(M_j(f^2) - m_j(f^2) \right) \Delta x_j$$

But we also know

$$\begin{aligned} \sup_{x,y \in [x_{j-1}, x_j]} (f^2(x) - f^2(y)) &= \sup_{x,y \in [x_{j-1}, x_j]} (f(x) + f(y))(f(x) - f(y)) \\ &\leq 2K \sup_{x,y \in [x_{j-1}, x_j]} ((f(x) - f(y)) \\ &= 2K\,(M_j - m_j). \end{aligned}$$

Proof

Thus,

$$\begin{aligned} U(f^2, \pi) - L(f^2, \pi) &= \sum_{\pi} \left(M_j(f^2) - m_j(f^2)\right) \Delta x_j \\ &\leq 2K \sum_{\pi} (M_j - m_j) \Delta x_j \\ &= 2K \left(U(f, \pi) - L(f \pi)\right). \end{aligned}$$

Now since f is Riemann Integrable, it satisfies the Riemann Criterion and so given $\epsilon > 0$, there is a partition π_0 so that $U(f, \pi) - L(f \pi) < \epsilon/(2K)$ for any refinement π of π_0. Thus, f^2 satisfies the Riemann Criterion too and so it is integrable.

Proof

(7): $fg \in RI[a, b]$
To prove that fg *is integrable when* f *and* g *are, simply note that*

$$fg = (1/2)\left((f+g)^2 - f^2 - g^2\right).$$

Property (6) *and the linearity of the integral then imply* fg *is integrable.*

(8) If there exists m, M *such that* $0 < m \leq \mid f \mid \leq M$*, then* $1/f \in RI[a, b]$*:*
Suppose $f \in RI[a, b]$ *and there exist* $M, m > 0$ *such that*
$m \leq \mid f(x) \mid \leq M$ *for all* $x \in [a, b]$*. Note that*

$$\frac{1}{f(x)} - \frac{1}{f(y)} = \frac{f(y) - f(x)}{f(x)f(y)}.$$

Proof

Let $\pi = \{x_0 = a, x_1, \ldots, x_p = b\}$ *be a partition of* $[a, b]$*, and define*

$$M'_j = \sup_{[x_{j-1}, x_j]} \frac{1}{f(x)}, \quad m'_j = \inf_{[x_{j-1}, x_j]} \frac{1}{f(x)}.$$

Then we have

$$\begin{aligned} M'_j - m'_j &= \sup_{x, y \in [x_{j-1}, x_j]} \frac{f(y) - f(x)}{f(x) f(y)} \\ &\leq \sup_{x, y \in [x_{j-1}, x_j]} \frac{\mid f(y) - f(x) \mid}{\mid f(x) \mid\mid f(y) \mid} \\ &\leq \frac{1}{m^2} \sup_{x, y \in [x_{j-1}, x_j]} \mid f(y) - f(x) \mid \\ &\leq \frac{M_j - m_j}{m^2}. \end{aligned}$$

Proof

Since $f \in RI[a,b]$, given $\epsilon > 0$ there is a partition π_0 such that $U(f,\pi) - L(f,\pi) < m^2\epsilon$ for any refinement, π, of π_0. Hence, the previous inequality implies that, for any such refinement, we have

$$\begin{aligned} U\left(\frac{1}{f},\pi\right) - L\left(\frac{1}{f},\pi\right) &= \sum_{\pi}(M'_j - m'_j)\Delta x_j \\ &\leq \frac{1}{m^2}\sum_{\pi}(M_j - m_j)\Delta x_j \\ &\leq \frac{1}{m^2}\Big(U(f,\pi) - L(f,\pi)\Big) \\ &< \frac{m^2\epsilon}{m^2} = \epsilon. \end{aligned}$$

Thus $1/f$ satisfies the Riemann Criterion and hence it is integrable. □

Now we need to show that the set $RI[a, b]$ is nonempty. We begin by showing that all continuous functions on $[a, b]$ will be Riemann Integrable.

Theorem

If $f \in C[a, b]$, then $f \in RI[a, b]$.

Proof

Since f is continuous on a compact set, it is uniformly continuous. Hence, given $\epsilon > 0$, there is a $\delta > 0$ such that $x, y \in [a, b]$, $| x - y | < \delta \Rightarrow | f(x) - f(y) | < \epsilon/(b - a)$. Let π_0 be a partition such that $|| \pi_0 || < \delta$, and let $\pi = \{x_0 = a, x_1, \ldots, x_p = b\}$ be any refinement of π_0. Then π also satisfies $|| \pi || < \delta$. Since f is continuous on each subinterval $[x_{j-1}, x_j]$, f attains its supremum, M_j, and infimum, m_j, at points s_j and t_j, respectively. That is, $f(s_j) = M_j$ and $f(t_j) = m_j$ for each $j = 1, \ldots, p$. □

Proof

Thus, the uniform continuity of f on each subinterval implies that, for each j,

$$M_j - m_j = | f(s_j) - f(t_j) | < \frac{\epsilon}{b-a}.$$

Thus, we have

$$U(f,\pi) - L(f,\pi) = \sum_{\pi} (M_j - m_j)\Delta x_j < \frac{\epsilon}{b-a} \sum_{\pi} \Delta x_j = \epsilon.$$

Since π was an arbitrary refinement of π_0, it follows that f satisfies Riemann's criterion. Hence, $f \in RI[a,b]$.

Theorem

If $f : [a,b] \to \Re$ is a constant function, $f(t) = c$ for all t in $[a,b]$, then f is Riemann Integrable on $[a,b]$ and $\int_a^b f(t)dt = c(b-a)$.

Proof

For any partition π of $[a, b]$, since f is a constant, all the individual m_j's and M_j's associated with π take on the value c. Hence, $U(f, \pi) - L(f, \pi) = 0$ always. It follows immediately that f satisfies the Riemann Criterion and hence is Riemann Integrable. Finally, since f is integrable, by our fundamental integral estimates, we have

$$c(b - a) \leq RI(f; a, b) \leq c(b - a).$$

Thus, $\int_a^b f(t)dt = c(b - a)$. □

Theorem

If f is monotone on $[a, b]$, then $f \in RI[a, b]$.

Proof

As usual, for concreteness, we assume that f is monotone increasing. We also assume $f(b) > f(a)$, for if not, then f is constant and must be integrable by the previous theorem. Let $\epsilon > 0$ be given, and let π_0 be a partition of $[a, b]$ such that $\| \pi_0 \| < \epsilon/(f(b) - f(a))$. Let $\pi = \{x_0 = a, x_1, \ldots, x_p = b\}$ be any refinement of π_0. Then π also satisfies $\| \pi \| < \epsilon/(f(b) - f(a))$. Thus, for each $j = 1, \ldots, p$, we have

$$\Delta x_j < \frac{\epsilon}{f(b) - f(a)}.$$

Since f is increasing, we also know that $M_j = f(x_j)$ and $m_j = f(x_{j-1})$ for each j. Hence,

$$\begin{aligned} U(f, \pi) - L(f, \pi) &= \sum_{\pi} (M_j - m_j)\Delta x_j = \sum_{\pi} [f(x_j) - f(x_{j-1})]\Delta x_j \\ &< \frac{\epsilon}{f(b) - f(a)} \sum_{\pi} [f(x_j) - f(x_{j-1})]. \end{aligned}$$

Proof

But this last sum is telescoping and sums to $f(b) - f(a)$. So, we have

$$U(f, \pi) - L(f, \pi) < \frac{\epsilon}{f(b) - f(a)}(f(b) - f(a)) = \epsilon.$$

Thus, f satisfies Riemann's criterion.

Let f_n, for $n \geq 2$ be defined by

$$f_n(x) = \begin{cases} 1, & x = 1 \\ 1/(k+1), & 1/(k+1) \leq x < 1/k,\ 1 \leq k < n \\ 0, & 0 \leq x < 1/n \end{cases}$$

We know f_n is $RI[0,1]$ because it is monotonic although we do not know what the value of the integral is.

Define f by $f(x) = \lim_{n\to\infty} f_n(x)$. Then given x, we can find an integer N so that $1/(N+1) \leq x < 1/N$ telling us $f(x) = 1/(N+1)$.

Moreover $f(x) = f_{N+1}(x)$. So if $x < y$, y is either in the interval $[1/(N+1), 1/N)$ or $y \in [1/N, 1]$ implying $f(x) \leq f(y)$. Hence f is monotonic.

At each $1/N$, the right and left hand limits do not match and so f is not continuous at a countable number of points yet it is still Riemann Integrable.

12.1 If you didn't know $f(x) = x$ was continuous, why would you know f is $RI[a, b]$ for any $[a, b]$?

12.2 Use induction to prove $f(x) = x^n$ is $RI[a, b]$ for any $[a, b]$ without assuming continuity.

12.3 Use induction to prove $f(x) = 1/x^n$ is $RI[a, b]$ on any $[a, b]$ that does not contain 0 without assuming continuity.

12.4 For $f(x) = \sin(2x)$ on $[-2\pi, 2\pi]$, draw f^+ and f^-.

12.5 Prove f is $RI[0, 1]$ where f is defined by

$$f(x) = \begin{cases} x \sin(1/x), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

12.6 Let

$$f_n(x) = \begin{cases} 1, & x = 1 \\ 1/(k+1), & 1/(k+1) \leq x < 1/k,\ 1 \leq k < n \\ 0, & 0 \leq x < 1/n \end{cases}$$

Graph f_5 and f_8 and determine the cluster points $S(p)$.