More Properties of the Riemann Integral

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February 15, 2018

Outline

We first want to establish the familiar summation property of the Riemann integral over an interval $[a, b] = [a, c] \cup [c, b]$. Most of the technical work for this result is done in the following Lemma.

Lemma

Let $f \in B[a, b]$ and let $c \in (a, b)$. Let

$$\underline{\int_a^b} f(x)\,dx = L(f) \text{ and } \overline{\int_a^b} f(x)\,dx = U(f)$$

denote the lower and upper Darboux integrals of f on $[a, b]$, respectively. Then we have

$$\overline{\int_a^b} f(x)dx = \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx$$

$$\underline{\int_a^b} f(x)dx = \underline{\int_a^c} f(x)dx + \underline{\int_c^b} f(x)dx.$$

Proof

We prove the result for the upper integrals as the lower integral case is similar. Let π *be given by* $\pi = \{x_0 = a, x_1, \ldots, x_p = b\}$.

(Case (1): c is a partition point):
Thus, there is some index $1 \leq k_0 \leq p-1$ *such that* $x_{k_0} = c$. *For any interval* $[\alpha, \beta]$, *let* $U_\alpha^\beta(f, \pi)$ *denote the upper sum of f for the partition* π *over* $[\alpha, \beta]$.

Now, we can rewrite π *as* $\pi = \{x_0, x_1, \ldots, x_{k_0}\} \cup \{x_{k_0}, x_{k_0+1}, \ldots, x_p\}$. *Let* $\pi_1 = \{x_0, \ldots, x_{k_0}\}$ *and* $\pi_2 = \{x_{k_0}, \ldots, x_p\}$. *Then* $\pi_1 \in \boldsymbol{\Pi}[a, c]$, $\pi_2 \in \boldsymbol{\Pi}[c, b]$, *and*

$$U_a^b(f, \pi) \quad = \quad U_a^c(f, \pi_1) + U_c^b(f, \pi_2) \geq \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx,$$

by the definition of the upper sum.

Proof

(Case c is not in the partition):
Now, if c is not in π, we can refine π by adding c, obtaining the partition $\pi' = \{x_0, x_1, \ldots, x_{k_0}, c, x_{k_0+1}, \ldots, x_p\}$.
Splitting up π' at c as we did before into π_1 and π_2, we see that $\pi' = \pi_1 \vee \pi_2$ where $\pi_1 = \{x_0, \ldots, x_{k_0}, c\}$ and $\pi_2 = \{c, x_{k_0+1}, \ldots, x_p\}$.
Thus, by our properties of upper sums, we see that

$$U_a^b(f,\pi) \geq U_a^b(f,\pi') = U_a^c(f,\pi_1) + U_c^b(f,\pi_2) \geq \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx.$$

Now combine the cases to find for any partition π, we have

$$U_a^b(f,\pi) \geq \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx,$$

Proof

which implies that

$$\overline{\int_a^b} f(x)dx \geq \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx.$$

Now we want to show the reverse inequality. Let $\epsilon > 0$ *be given. By the definition of the upper integral, there exists* $\boldsymbol{\pi}_1 \in \mathbf{\Pi}[a, c]$ *and* $\boldsymbol{\pi}_2 \in \mathbf{\Pi}[c, b]$ *such that*

$$U_a^c(f, \boldsymbol{\pi}_1) \quad < \quad \overline{\int_a^c} f(x)dx + \frac{\epsilon}{2}, \quad U_c^b(f, \boldsymbol{\pi}_2) < \overline{\int_c^b} f(x)dx + \frac{\epsilon}{2}.$$

Let $\boldsymbol{\pi} = \boldsymbol{\pi}_1 \cup \boldsymbol{\pi}_2 \in \mathbf{\Pi}[a, b]$. *It follows that*

$$U_a^b(f, \boldsymbol{\pi}) = U_a^c(f, \boldsymbol{\pi}_1) + U_c^b(f, \boldsymbol{\pi}_2) < \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx + \epsilon.$$

Proof

But, by definition, we have

$$\overline{\int_a^b} f(x)dx \leq U_a^b(f, \boldsymbol{\pi})$$

for all $\boldsymbol{\pi}$. *Hence, we see that*

$$\overline{\int_a^b} f(x)dx < \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx + \epsilon.$$

Since ϵ *was arbitrary, this proves the reverse inequality we wanted. We can conclude, then, that*

$$\overline{\int_a^b} f(x)dx = \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx.$$

□

Theorem

If $f \in RI[a,b]$ and $c \in (a,b)$, then $f \in RI[a,c]$ and $f \in RI[c,b]$.

Proof

Let $\epsilon > 0$ be given. Then there is a partition $\pi_0 \in \Pi[a,b]$ such that $U_a^b(f,\pi) - L_a^b(f,\pi) < \epsilon$ for any refinement, π, of π_0.

Let π_0 be given by $\pi_0 = \{x_0 = a, x_1, \ldots, x_p = b\}$. Define $\pi_0' = \pi_0 \cup \{c\}$, so there is some index k_0 such that $x_{k_0} \leq c \leq x_{k_0+1}$.

Let $\pi_1 = \{x_0, \ldots, x_{k_0}, c\}$ and $\pi_2 = \{c, x_{k_0+1}, \ldots, x_p\}$.

Proof

Then $\pi_1 \in \mathbf{\Pi}[a, c]$ *and* $\pi_2 \in \mathbf{\Pi}[c, b]$. *Let* π_1' *be a refinement of* π_1. *Then* $\pi_1' \cup \pi_2$ *is a refinement of* π_0, *and it follows that*

$$\begin{aligned} U_a^c(f, \pi_1') - L_a^c(f, \pi_1') &= \sum_{\pi_1'} (M_j - m_j)\Delta x_j \\ &\leq \sum_{\pi_1' \cup \pi_2} (M_j - m_j)\Delta x_j \\ &\leq U_a^b(f, \pi_1' \cup \pi_2) - L_a^b(f, \pi_1' \cup \pi_2). \end{aligned}$$

But, since $\pi_1' \cup \pi_2$ *refines* π_0, *we have*

$$U_a^b(f, \pi_1' \cup \pi_2) - L_a^b(f, \pi_1' \cup \pi_2) < \epsilon,$$

implying that

$$U_a^c(f, \pi_1') - L_a^c(f, \pi_1') < \epsilon$$

Proof

for all refinements, π_1'*, of* π_1*. Thus,* f *satisfies Riemann's criterion on* $[a, c]$*, and* $f \in RI[a, c]$*.*

The proof on $[c, b]$ *is done in exactly the same way.* □

Theorem

If $f \in RI[a, b]$ *and* $c \in (a, b)$*, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof

Since $f \in RI[a, b]$*, we know that* $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$*.*

Proof

Further, we also know that $f \in RI[a, c]$ *and* $f \in RI[c, b]$ *for any* $c \in (a, b)$. *Thus,*

$$\begin{aligned} \overline{\int_a^c} f(x)dx &= \underline{\int_a^c} f(x)dx \\ \overline{\int_c^b} f(x)dx &= \underline{\int_c^b} f(x)dx. \end{aligned}$$

So, applying the Lemma, we conclude that, for any $c \in (a, b)$,

$$\begin{aligned} \int_a^b f(x)dx &= \overline{\int_a^b} f(x)\, dx = \overline{\int_a^c} f(x)\, dx + \overline{\int_c^b} f(x)\, dx \\ &= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \end{aligned}$$

□

Theorem

Let $f \in RI[a,b]$. Define $F : [a,b] \to \Re$ by $F(x) = \int_a^x f(t)dt$. Then

(i) $F \in C[a,b]$;

(ii) *if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.*

Proof

First, note that $f \in RI[a,b] \Rightarrow f \in R[a,x]$ for all $x \in [a,b]$, by our previous results. Hence, F is well-defined. We will prove the results in order.

(i):

Now, let $x, y \in [a,b]$ be such that $x < y$. Then

$\inf_{[x,y]} f(t)\,(y-x) \leq \int_x^y f(t)dt \leq \sup_{[x,y]} f(t)\,(y-x)$

Proof

which implies that

$$| F(y) - F(x) | = \left| \int_x^y f(t)dt \right| \leq \| f \|_\infty \ (y - x).$$

A similar argument shows that if $y, x \in [a, b]$ *satisfy* $y < x$, *then*

$$| F(y) - F(x) | = \left| \int_x^y f(t)dt \right| \leq \| f \|_\infty \ (x - y).$$

Let $\epsilon > 0$ *be given. Then if*

$$| x - y | < \frac{\epsilon}{\| f \|_\infty + 1},$$

we have

$$| F(y) - F(x) | \leq \| f \|_\infty | y - x | < \frac{\| f \|_\infty}{\| f \|_\infty + 1} \epsilon < \epsilon.$$

Proof

Thus, F is continuous at x and, consequently, on $[a, b]$.
(ii):
Finally, assume f is continuous at $c \in [a, b]$, and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap [a, b]$ implies $\mid f(x) - f(c) \mid < \epsilon/2$.

Pick h such that $0 <\mid h \mid< \delta$ and $c + h \in [a, b]$. Let's assume, for concreteness, that $h > 0$.

Define

$$m = \inf_{[c,c+h]} f(t) \qquad \text{and} \qquad M = \sup_{[c,c+h]} f(t).$$

If $c < x < c + h$, then we have $x \in (c - \delta, c + \delta) \cap [a, b]$ and $-\epsilon/2 < f(x) - f(c) < \epsilon/2$. That is,

$$f(c) - \frac{\epsilon}{2} < f(x) < f(c) + \frac{\epsilon}{2} \qquad \forall x \in [c, c + h].$$

Proof

Hence, $m \geq f(c) - \epsilon/2$ *and* $M \leq f(c) + \epsilon/2$*. Now, we also know that*

$$mh \leq \int_c^{c+h} f(t)dt \leq Mh.$$

Thus, we have

$$\frac{F(c+h) - F(c)}{h} = \frac{\int_a^{c+h} f(t)dt - \int_a^c f(t)dt}{h} = \frac{\int_c^{c+h} f(t)dt}{h}.$$

Combining inequalities, we find

$$f(c) - \frac{\epsilon}{2} \leq m \leq \frac{F(c+h) - F(c)}{h} \leq M \leq f(c) + \frac{\epsilon}{2}$$

yielding

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \frac{\epsilon}{2} < \epsilon$$

if $x \in [c, c+h]$.

Proof

Thus, since ϵ *was arbitrary, this shows* $(F'^{+}(c) = f(c)$*. The case where* $h < 0$ *is handled in exactly the same way which tells us* $(F'^{-}(c) = f(c)$*. Combining, we have* $F'(c) = f(c)$*.*

Note that if $c = a$ *or* $c = b$*, we need only consider the definition of the derivative from one side.* □

Comment

We call $F(x)$ *the indefinite integral of* f*.* F *is always better behaved than* f*, since integration is a smoothing operation. We can see that* f *need not be continuous, but, as long as it is integrable,* F *is always continuous.*

The next result is one of the many mean value theorems in the theory of integration. It is a more general form of the standard mean value theorem given in beginning calculus classes.

Theorem

The Mean Value Theorem For Riemann Integrals:
Let $f \in C[a, b]$, and let $g \geq 0$ be integrable on $[a, b]$. Then there is a point, $c \in [a, b]$, such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Proof

Since f is continuous, it is also integrable. Hence, fg is integrable. Let m and M denote the lower and upper bounds of f on $[a, b]$, respectively. Then $mg(x) \leq f(x)g(x) \leq Mg(x)$ for all $x \in [a, b]$.

Since the integral preserves order, we have

Proof

$$m \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M \int_a^b g(x)dx.$$

If the integral of g *on* $[a, b]$ *is* 0*, then the integral of* fg *is* 0*. Hence, in this case, choose any* $c \in [a, b]$*. If the integral of* g *is not* 0*, then it must be positive, since* $g \ge 0$*. Hence, we have, in this case,*

$$m \quad \le \quad \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M.$$

Now, f *is continuous, so it attains* M *and* m *at some points. Hence, by the Intermediate Value Theorem there is a* $c \in [a, b]$ *with*

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}.$$

This implies the desired result. □

The next result is another standard mean value theorem from basic calculus. This result can be interpreted as stating that integration is an averaging process.

Theorem

Average Value For Riemann Integrals
If $f \in C[a, b]$, then there is a point $c \in [a, b]$ such that $\frac{1}{b-a} \int_a^b f(x)dx = f(c)$.

Proof

Apply $g(x) = 1$ in the previous theorem. □

We need better ways to calculate Riemann Integrals. A great way to do it starts with the idea of antiderivatives.

Definition

Let $f : [a,b] \to \Re$ be a bounded function. Let $G : [a,b] \to \Re$ be such that G' exists on $[a,b]$ and $G'(x) = f(x)$ for all $x \in [a,b]$. Such a function is called an **antiderivative** or a **primitive** of f.

The idea of an antiderivative is intellectually distinct from the Riemann integral of a bounded function f. Consider the following function f defined on $[-1,1]$.

$$f(x) = \begin{cases} x^2 \sin(1/x^2), & x \neq 0,\ x \in [-1,1] \\ 0, & x = 0 \end{cases}$$

It is easy to see that this function has a removable discontinuity at 0. Moreover, f is even differentiable on $[-1, 1]$ with derivative

$$f'(x) = \begin{cases} 2x\sin(1/x^2) - (2/x)\cos(1/x^2), & x \neq 0,\ x \in [-1,1] \\ 0, & x = 0 \end{cases}$$

Note f' is *not* bounded on $[-1, 1]$ and hence it can not be Riemann Integrable. Now to connect this to the idea of antiderivatives, just relabel the functions. Let g be defined by

$$g(x) = \begin{cases} 2x\sin(1/x^2) - (2/x)\cos(1/x^2), & x \neq 0,\ x \in [-1,1] \\ 0, & x = 0 \end{cases}$$

then define G by

$$G(x) = \begin{cases} x^2\sin(1/x^2), & x \neq 0,\ x \in [-1,1] \\ 0, & x = 0 \end{cases}$$

We see that G is the antiderivative of g even though g itself does not have a Riemann integral. Again, the point is that the idea of the antiderivative of a function is intellectually distinct from that of being Riemann integrable.

Homework 13

13.1 You can guess the antiderivative of x is $x^2/2 + C$. Use induction to proves the antiderivative of x^n for integers $n \geq 1$ is $x^{n+1}/(n+1) + C$.

13.2 You can guess the antiderivative of $1/x^2$ is $-1/x + C$. Use induction to proves the antiderivative of x^n for integers $n < -1$ is $x^{n+1}/(n+1) + C$.

13.3 What is the most general antiderivative of $1/x$?

13.4 Let f be defined by

$$f(x) = \begin{cases} x \sin(1/x), & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Prove $|\int_0^1 f(x)dx| \leq 1/2$.