Integration and Differentiation Limit Interchange Theorems

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Outline

Theorem

Let (x_n) be a sequence of Riemann Integrable functions on $[a, b]$. Let $x_n \xrightarrow{unif} x$ on $[a, b]$. Then x is Riemann integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b x_n(t)dt = \int_a^b (\lim_{n\to\infty} x_n(t))dt = \int_a^b x(t)dt.$$

Proof

If x is Riemann Integrable on $[a, b]$, the argument we presented in the proof of first integral interchange theorem works nicely. So to prove this theorem it is enough to show x is Riemman Integrable on $[a, b]$.

Proof

Let $\epsilon > 0$ be given. since the convergence is uniform, there is N so that

$$|x_n(t) - x(t)| \quad < \quad \frac{\epsilon}{5(b-a)}, \quad \forall\, n > N, \quad t \in [a,b] \qquad (\alpha)$$

Pick any $\hat{n} > N$. Since $x_{\hat{n}}$ is integrable, there is a partition $\boldsymbol{\pi_0}$ so that

$$U(x_{\hat{n}}, \boldsymbol{\pi}) - L(x_{\hat{n}}, \boldsymbol{\pi}) < \frac{\epsilon}{5} \qquad (\beta)$$

when $\boldsymbol{\pi}$ is a refinement of $\boldsymbol{\pi_0}$.

Since $x_n \xrightarrow{unif} x$, we can easily show x is bounded. So we can define

$$M_j \quad = \quad \sup_{t_{j-1} \leq t \leq t_j} x(t), \quad m_j = \inf_{t_{j-1} \leq t \leq t_j} x(t)$$

$$\hat{M}_j \quad = \quad \sup_{t_{j-1} \leq t \leq t_j} x_{\hat{n}}(t), \quad \hat{m}_j = \inf_{t_{j-1} \leq t \leq t_j} x_{\hat{n}}(t)$$

where $\{t_0, t_1, \ldots, t_p\}$ are the points in the partition $\boldsymbol{\pi}$.

Proof

Using the infimum and supremum tolerance lemma, we can find points $s_j \in [t_{j-1}, t_j]$ *and* $u_j \in [t_{j-1}, t_j]$ *so that*

$$M_j - \frac{\epsilon}{5(b-a)} < x(s_j) \leq M_j \qquad (\gamma)$$

$$m_j \leq x(u_j) < m_j + \frac{\epsilon}{5(b-a)} \qquad (\delta)$$

Thus

$$\begin{aligned} U(x,\pi) - L(x,\pi) &= \sum_{\pi} (M_j - m_j)\Delta t_j \\ &= \sum_{\pi} \{M_j - x(s_j) + x(s_j) - x_{\hat{n}}(s_j) + x_{\hat{n}}(s_j) \\ &\quad - x_{\hat{n}}(u_j) + x_{\hat{n}}(u_j) - x(u_j) + x(u_j) - m_j\}\Delta t_j \end{aligned}$$

Proof

Now break this into individual pieces:

$$\begin{aligned} U(x,\pi) - L(x,\pi) &= \sum_{\pi} (M_j - x(s_j))\Delta t_j + \sum_{\pi} (x(s_j) - x_{\hat{n}}(s_j))\Delta t_j \\ &+ \sum_{\pi} (x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j))\Delta t_j + \sum_{\pi} (x_{\hat{n}}(u_j) - x(u_j))\Delta t_j \\ &+ \sum_{\pi} (x(u_j) - m_j)\Delta t_j \end{aligned}$$

Using Equation γ and Equation δ we can overestimate the first and fifth term to get

$$\begin{aligned} U(x,\pi) - L(x,\pi) &< \sum_{\pi} \frac{\epsilon}{5(b-a)}\Delta t_j + \sum_{\pi} (x(s_j) - x_{\hat{n}}(s_j))\Delta t_j \\ &+ \sum_{\pi} (x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j))\Delta t_j + \sum_{\pi} (x_{\hat{n}}(u_j) - x(u_j))\Delta t_j \\ &+ \sum_{\pi} \frac{\epsilon}{5(b-a)}\Delta t_j \end{aligned}$$

Proof

We know $\sum_\pi \Delta t_j = b - a$ *and so simplifying a bit, we have*

$$\begin{aligned} U(x,\pi) - L(x,\pi) &< 2\frac{\epsilon}{5} + \sum_\pi (x(s_j) - x_{\hat{n}}(s_j))\Delta t_j \\ &+ \sum_\pi (x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j))\Delta t_j + \sum_\pi (x_{\hat{n}}(u_j) - x(u_j))\Delta t_j \end{aligned}$$

Now use Equation α *in piece two and four above to get*

$$\begin{aligned} U(x,\pi) - L(x,\pi) &< 2\frac{\epsilon}{5} + \sum_\pi \frac{\epsilon}{5(b-a)}\Delta t_j \\ &+ \sum_\pi (x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j))\Delta t_j + \sum_\pi \frac{\epsilon}{5(b-a)}\Delta t_j \\ &< 4\frac{\epsilon}{5} + \sum_\pi (x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j))\Delta t_j \end{aligned}$$

since like before $\sum_\pi \frac{\epsilon}{5(b-a)}\Delta t_j = \frac{\epsilon}{5}$.

Proof

Finally, $|x_{\hat{n}}(s_j) - x_{\hat{n}}(u_j)| \leq \hat{M}_j - \hat{m}_j$ *and so*

$$\begin{aligned} U(x,\pi) - L(x,\pi) &< 4\frac{\epsilon}{5} + \sum_{\pi} (\hat{M}_j - \hat{m}_j)\Delta t_j \\ &= 4\frac{\epsilon}{5} + U(x_{\hat{n}},\pi) - L(x_{\hat{n}},\pi) < \epsilon \end{aligned}$$

using Equation β. *We conclude* $U(x,\pi) - U(x,\pi) < \epsilon$ *for all refinements of* π_0 *which shows* x *is Riemann Integrable on* $[a,b]$.

It remains to show the limit interchange portion of the theorem. As mentioned at the start of this proof, this argumen is the same as the one given in the first integral interchange theorem and so it does not have to be repeated.

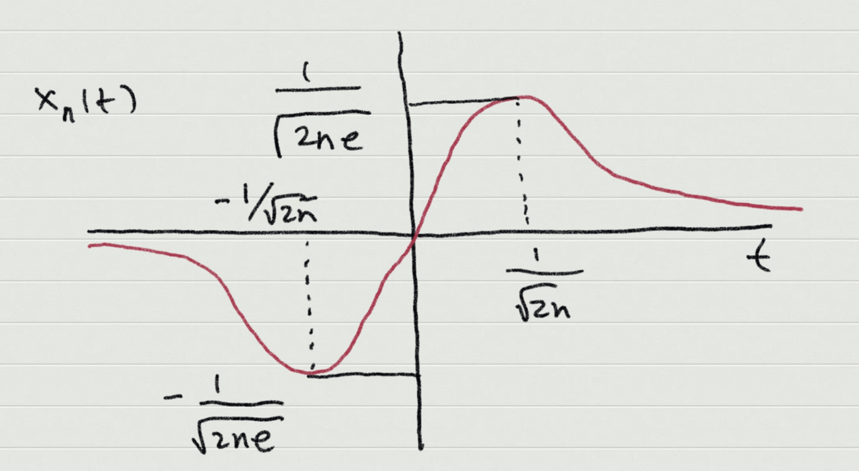
We now consider another important theorem about the interchange of integration and limits of functions. We start with an illuminating example.

Consider the sequence (x_n) on $[-1, 1]$ where $x_n(t) = t/e^{nt^2}$. It is easy to see $x_n \xrightarrow{\text{ptws}} x$ where $x(t) = 0$ on $[-1, 1]$. Taking the derivative, we see

$$x_n'(t) = \frac{1 - 2nt^2}{e^{nt^2}}$$

and the critical points of x_n are when $1 - 2nt^2 = 0$ or at $t = \pm 1/\sqrt{2n}$.

We have studied this kind of sequence before but the other sequences had high peaks proportional to n which meant we could not get uniform convergence on any interval $[c, d]$ containing 0. In this sequence, the extreme values occur at $\pm 1/\sqrt{2ne}$. Hence, the *peaks* hear decrease. We could illustrate this behavior with some carefully chosen Octave plots, but this time we will simply sketch the graph labeling important points.


$x_n(t)$
$\frac{1}{\sqrt{2ne}}$
$-1/\sqrt{2n}$
$\frac{1}{\sqrt{2n}}$
t
$-\frac{1}{\sqrt{2ne}}$

Let

$$M_n = \sup_{-1\leq t\leq 1} |x_n(t) - x(t)| = \sup_{-1\leq t\leq 1} \left| \frac{t}{e^{nt^2}} \right| = \frac{1}{\sqrt{2ne}}$$

Since $M_n \to 0$, by the Weierstrass Theorem for Uniform Convergence, we have $x_n \xrightarrow{\text{unif}} x$.

At $t = 0$, $x_n'(0) = 1$ for all n and so $\lim_{n\to\infty} x_n'(t)\Big|_{t=0} = 1$. But $x'(0) = 0$ as $x(t) = 0$ on $[-1, 1]$.

Hence we conclude for this example that at $t = 0$ the differentiation interchange does not work:

$$\lim_{n\to\infty} x_n'(t) \neq \left(\lim_{n\to\infty} x_n(t)\right)',$$

that is the interchange of convegence and differentiation fails here **even though we have uniform convergence of the sequence**.

Finally, since $x_n'(t) = \frac{1-2nt^2}{e^{nt^2}}$, we see the pointwise limit of the derivative sequence is $y(0) = 1$ but $y(t) = 0$ on $[-1,0) \cup (0,1]$ which is not continuous. Since y is not continuous, $x_n' \overset{\text{unif}}{\not\longrightarrow} y$. What conditions are required to guarantee the derivative interchange result?

Definition

The sequence of functions (x_n) on the set S satisfies the **Uniform Cauchy Criterion** if

$$\forall \epsilon > 0, \; \exists \, N \; \ni \; |x_n(t) - x_m(t)| < \epsilon, \;\; \text{for} \;\; n > m > N \text{ and } \forall \, t \; \in \; S$$

This can also be stated like this:

$$\forall \epsilon > 0, \; \exists \, N \; \ni \; ||x_n - x_m||_\infty < \epsilon, \;\; \text{for} \;\; n > m > N.$$

We will abbreviate this with **UCC**.

Using the UCC, we can prove another test for uniform convergence which is often easier to use.

Theorem

Let (x_n) be a sequence of functions on the set S. Then

$$\left(\exists x : S \rightarrow \Re \ni x_n \stackrel{unif}{\rightarrow} x\right) \iff \left((x_n) \text{ satisfies the UCC}\right).$$

Proof

($\Rightarrow$:)

We assume there is an $x : S \rightarrow \Re$ so that $x_n \stackrel{unif}{\rightarrow} x$. Then, given $\epsilon > 0$, there is N so that $|x_n(t) - x(t)| < \epsilon/2$ for all $t \in S$.

Proof

Thus, if $n > m > N$, we have

$$\begin{aligned} |x_n(t) - x_m(t)| &= |x_n(t) - x(t) + x(t) - x_m(t)| \\ &\leq |x_n(t) - x(t)| + |x(t) - x_m(t)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Thus, (x_n) satisfies the UCC.
($\Leftarrow$:)
If (x_n) satisfies the UCC, then given $\epsilon > 0$, there is N so that $|x_n(t) - x_m(t)| < \epsilon$ for $n > m > N$.

This says the seq $(x_n(\hat{t})$ is a Cauchy Sequence for each $\hat{t} \in S$. Since $\Re$ is complete, there is a number $a_{\hat{t}}$ so that $x_n(\hat{t}) \rightarrow a_{\hat{t}}$. The number $a_{\hat{t}}$ defines a function $x : S \rightarrow \Re$ by $x(t) = a_{\hat{t}}$. Clearly, $x_n \stackrel{ptws}{\rightarrow} x$ on S. From the UCC, we can therefore say for the given ϵ,

$$\lim_{n \rightarrow \infty} |x_n(t) - x_m(t)| \leq \epsilon/2, \text{ if } n > m > N, \ t \in S$$

Proof

But the absolute function is continuous and so

$$|\lim_{n\to\infty} x_n(t) - x_m(t)| \quad \leq \quad \epsilon/2, \quad \textit{if } \; n > m > N, \; t \in S$$

or $|x(t) - x_m(t)| < \epsilon$ *when* $m > N$. *This shows* $x_n \overset{unif}{\rightarrow} x$ *as the choice of index* m *is not important.*

Note this argument is essentially the same as the one we used in the proof that uniform convergence of continuous functions gives a continuous limit.

The difference here is that we do not know each x_n *is continuous. We are simply proving the existence of a limit function which is possible as* $\Re$ *is complete. But the use of the completeness of* $\Re$ *is common in both proofs.* □

We are ready to prove our next interchange theorem.

Theorem

Let (x_n) be a sequence of functions defined on the interval $[a, b]$. Assume

1. x_n *is differentiable on* $[a, b]$.
2. x_n' *is Riemann Integrable on* $[a, b]$.
3. *There is at least one point* $t_0 \in [a, b]$ *such that the sequence* $(x_n(t_0))$ *converges.*
4. $x_n' \xrightarrow{unif} y$ *on* $[a, b]$ *and the limit function* y *is continuous.*

Then there is $x : [a, b] \to \Re$ *which is differentiable on* $[a, b]$ *and* $x_n \xrightarrow{unif} x$ *on* $[a, b]$ *and* $x' = y$. *Another way of saying this is*

$$\left(\lim_{n \to \infty} x_n(t) \right)' = \lim_{n \to \infty} x_n'(t)$$

Proof

Since x_n' is integrable, by the recapture theorem,

$$x_n(t) = \int_{t_0}^{t} x_n'(s)\, ds + x_n(t_0).$$

Hence, for any n and m, we have

$$x_n(t) - x_m(t) = \int_{t_0}^{t} (x_n'(s) - x_m'(s))\, ds + x_n(t_0) - x_m(t_0).$$

If $t > t_0$, then

$$\begin{aligned} |x_n(t) - x_m(t)| &\leq \left| \int_{t_0}^{t} (x_n'(s) - x_m'(s))\, ds \right| + |x_n(t_0) - x_m(t_0)| \\ &\leq \int_{t_0}^{t} |x_n'(s) - x_m'(s)| + |x_n(t_0) - x_m(t_0)| \end{aligned}$$

Proof

Since $x_n' \xrightarrow{unif} y$ *on* $[a,b]$, (x_n') *satisfies the UCC.*

For a given $\epsilon > 0$, *we have there is* N_1 *so that*

$$|x_n'(s) - x_m'(s)| < \frac{\epsilon}{4(b-a)}, \quad \text{for} \quad n > m > N_1, \ s \in [a,b] \qquad (\alpha)$$

Thus, using Equation α, *for* $n > m > N_1, \ t > t_0 \in [a,b]$

$$\begin{aligned} |x_n(t) - x_m(t)| &< \frac{\epsilon}{4(b-a)} \int_{t_0}^{t} ds + |x_n(t_0) - x_m(t_0)| \\ &= \frac{\epsilon}{4} + |x_n(t_0) - x_m(t_0)| \end{aligned}$$

A similar argument for $t < t_0$ *gives the same result: for* $n > m > N_1, \ t < t_0 \in [a,b]$

$$|x_n(t) - x_m(t)| < \frac{\epsilon}{4} + |x_n(t_0) - x_m(t_0)|$$

Proof

Hence, since the statement if clearly true at $t = t_0$, we can say for $n > m > N_1,\ t > t_0 \in [a, b]$

$$|x_n(t) - x_m(t)| \quad < \quad \frac{\epsilon}{4} + |x_n(t_0) - x_m(t_0)|$$

We also know $(x_n(t_0))$ converges and hence is a Cauchy Sequence.

Thus, there is N_2 so that $|x_n(t_0) - x_m(t_0)| < \epsilon/4$ for $n > m > N_2$. We conclude if $N > \max(N_1, N_2)$, both conditions apply and we can say

$$|x_n(t) - x_m(t)| \quad < \quad \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \quad \textit{for} \quad n > m > N, \;\; t \in [a, b]$$

This shows (x_n) satisfies the UCC and there is $x : S \rightarrow \Re$ so that $x_n \xrightarrow{unif} x$. So far we know

$$x_n(t) \quad = \quad x_n(t_0) + \int_{t_0}^{t} x_n'(s)\, ds.$$

Proof

Since $x_n' \xrightarrow{unif} y$ *and each* x_n *is Riemann Integrable on* $[a, b]$, *by the integral interchange theorem* $\int_{t_0}^{t} x_n'(s)\, ds = \int_{t_0}^{t} y(s)\, ds$ *on* $[a, b]$.

Also, $x_n \xrightarrow{unif} x$ *implies* $\lim_{n\to\infty} x_n(t) = x(t)$.

We conclude

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} x_n(t_0) + \lim_{n\to\infty} \int_{t_0}^{t} x_n'(s)\, ds.$$

and so

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s)\, ds$$

Since we assume y *is continuous on* $[a, b]$, *by the FTOC,* $x'(t) = y(t)$ *on* $[a, b]$. □

Homework 23

The first three problems are based on this kind of thing: We know $\int_0^1 1/(1+x^2)dx = \arctan(1) = \pi/4$. We can write this an a uniform sum limit. Take uniformly spaced partitions of width $1/n$ of $[0,1]$. Then points in the partition have the form k/n and we take as evaluation points the RH endpoint. The Riemann sum is then

$$\begin{aligned} S(f(x) = 1/(1+x^2), P_n, S_n) &= \sum_{k=1}^{n} 1/(1+k^2/n^2)(1/n) \\ &= \sum_{k=1}^{n} 1/(1+k^2/n^2)(1/n) = \pi/4. \end{aligned}$$

So a cool question might be Prove $\lim_{n\to\infty} 4\sum_{k=1}^{n} n/(n^2+k^2) = \pi$. Note we know $f(x) = 1/(1+x^2)$ is RI on $[0,1]$ since it is continuous and our theorem says the limit of these Riemann sums converges to $\int_0^1 1/(1+x^2)dx$.

Homework 23

23.1 Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{k}/n^{3/2}$.

23.2 Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} 1/(n+k)$.

23.3 Prove $\lim_{n\to\infty} \sum_{k=1}^{n} 1/\sqrt{n^2 - k^2} = \pi/2$.

Hint

The function here is **not** *Riemann integrable on* $[0, 1]$ *but it is on* $[L, 1]$ *for any* $0 < L < 1$. *Start with uniform partitions of the interval* $[L, 1]$ *and go from there.*

23.4 Discuss the convergence of the sequence of functions (x_n)

$$x_n(t) = \begin{cases} 0, & 0 \leq t \leq 1/n \\ 1, & 1/n < t \leq 2/n \\ 0, & 2/n < t \leq 1 \end{cases}$$

23.5 Prove if g is a bounded function on $[a, b]$ and $x_n \xrightarrow{\text{unif}} x$ on $[a, b]$, then $g\, x_n \xrightarrow{\text{unif}} g\, x$ on $[a, b]$.