

# Completeness and Uniform Continuity

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## Outline

The Completeness of  $\mathbb{R}$

Uniform Continuity and Compact Domains

Let's look more carefully at what is called the completeness of  $\mathfrak{R}$ . We prove this carefully by showing we can extend the field  $\mathbb{Q}$  to another field  $\tilde{\mathbb{Q}}$  which is totally ordered, satisfies the Completeness axiom (i.e. the least upper bound and greatest lower bound property) and in which Cauchy Sequences of objects converge to an object in  $\tilde{\mathbb{Q}}$ . This new field is then identified with  $\mathfrak{R}$ .

- ▶ There is a general process by which a metric space can be completed which we can illustrate by using the rational numbers as a guide. We actually don't use this process to construct the real numbers, but it will show you the steps we typically take.
- ▶ Let  $(\mathbb{Q}, |\cdot|) = X$ . This is a nice **metric space** where  $|x - y|$  measures the **distance** between the two rational numbers  $x$  and  $y$ . We already know Cauchy Sequences of rational numbers need not converge to a rational number. A nice example is the sequence  $x_n = (1 + 1/n)^n$  which we know converges to a number we call  $e$ .

- ▶ Let's define a new metric space which we will call  $Y$ .  $Y$  is the set of all Cauchy Sequences of rational numbers; i.e. the *objects* in our space are Cauchy Sequences! Note each rational number  $p/q$  forms a nice constant sequence  $x_1 = p/q, x_2 = p/q, \dots, x_n = p/q, \dots$ . We can denote this constant sequence by  $(p/q)$ . So for example  $(2/3)$  is the constant Cauchy Sequence whose entries are all  $2/3$ .

- ▶ We need a metric for  $Y$ . Define the distance between two Cauchy Sequences in  $Y$  like this:

$$D((x_n), (y_n)) = \lim_{n \rightarrow \infty} |x_n - y_n|.$$

The objects in  $Y$  divide naturally into **classes** called *equivalence classes*. Given any object from  $Y$ ,  $(x_n)$ , we let  $[(x_n)]$  denote the collection of all other objects from  $Y$ , i.e. other Cauchy Sequences of rational numbers, whose distance to  $(x_n)$  is zero.

- ▶ We call this set of equivalence classes  $\tilde{Y}$  and we define the distance,  $\tilde{D}$ , between two equivalence classes as follows:  
 $\tilde{D}([x_n], [y_n]) = \lim_{n \rightarrow \infty} |x_n - y_n|$ . We can show this limit exists when we construct the field  $\mathbb{Q}$ . Of course, we would have to show the value of  $\tilde{D}$  does not depend on the choice of representatives from the equivalence classes!
- ▶ For example, the constant sequence  $(3/5)$  is in  $Y$  and there are an infinite number of other sequences  $(a_n)$  so that  $D((3/5), (a_n)) = 0$ . Just let  $(b_n)$  be any sequence of rational numbers that converges to 0. Then  $D((3/5), (3/5) + (b_n)) = 0$  and so  $(3/5) + (b/n)$  is a member of  $[(3/5)]$ . This is the big point now! The sequence  $((1 + 1/n)^n)$  does not converge to a rational number and so it can not be in the equivalence class associated to any rational number  $[p/q]$ . Another way of saying this is that  $D((1 + 1/n)^n, (p/q)) \neq 0$  for all  $p/q \in \mathbb{Q}$ .

The equivalence class  $[(1 + 1/n)^n]$  is thus **different** from the equivalence classes formed from constant sequences of rationals. The collection of all equivalence classes of objects from  $Y$  can thus be identified in a natural way with the numbers we see in  $\mathfrak{R}$ .

- ▶ Each constant rational sequence  $(p/q)$  is chosen as the representative of  $[(p/q)]$ .
- ▶ Each equivalence class that is different from the equivalence classes formed by constant rational sequences is identified with some representative from it. We call that  $\alpha$ . Note it can not come from a constant rational sequence so it can not be a rational number. We generally call this an **irrational number**. But remember, in this context, it is really a Cauchy Sequence of rationals!
- ▶ It is hard, but in a more advanced class, we can show Cauchy Sequences in  $(\tilde{Y}, \tilde{D})$  converge to an object in  $(\tilde{Y}, \tilde{D})$ . So we can prove  $(\tilde{Y}, \tilde{D})$  is a complete metric space.
- ▶ We can do this construction process for any metric space  $(X, d)$  and build a new complete metric space  $(\tilde{Y}, \tilde{D})$ . We do this in the first course on linear analysis that follows this course.

- ▶ So  $\mathbb{R}$  is the completion of the metric space  $(\mathbb{Q}, |\cdot|)$  as outlined above.
- ▶ We said  $(C[0, 1], \|\cdot\|_1)$  is not complete. In the next graduate course on analysis, we find the completion of  $(C[0, 1], \|\cdot\|_1)$  can be done following this construction process and generates the space  $(L_1, \|\cdot\|_1)$  which is a space of equivalence classes of functions and to do this right we also have to extend our notion of Riemann Integration to something called Lebesgue Integration.
- ▶ We said  $(C[0, 1], \|\cdot\|_2)$  is not complete. We also find the completion of  $(C[0, 1], \|\cdot\|_2)$  can be done following this construction process and generates the space  $(L_2, \|\cdot\|_2)$  which is a space of equivalence classes of functions using Lebesgue Integration. This is real special and it turns out to be an inner product space which is complete. This is called a **Hilbert Space**.
- ▶ The space  $(C[0, 1], \|\cdot\|_\infty)$  is complete as we will show in a bit and so if we do the construction process as outlined earlier, we just get back the same space:  $(X, d)$  and  $(Y, D)$  will be the same here.

Let's look more carefully at continuous functions on compact domains. We can prove a nice theorem:

### Theorem

*If  $f$  is continuous on the compact interval  $I = [a, b]$ , then  $f$  is uniformly continuous on  $I$ .*

### Proof

We are going to prove this by **contradiction**. If  $f$  is not uc on  $I$ , there is an  $\epsilon_0$  so that

$$\forall \delta > 0, \exists x, y \in I \ni |x - y| < \delta \text{ and } |f(x) - f(y)| > \epsilon_0$$

## Proof

In particular for the choice  $\delta_n = 1/n$  for all  $n \geq 1$ , we have

$$\exists x_n, y_n \in I \ni |x_n - y_n| < 1/n \text{ and } |f(x_n) - f(y_n)| \geq \epsilon_0$$

Since  $(x_n)$  and  $(y_n)$  are contained in the compact set  $I$ , the Bolzano - Weierstrass Theorem tells us there are subsequences  $(x_n^1)$  and  $(y_n^1)$  and points  $x$  and  $y$  in  $I$  so that  $x_n^1 \rightarrow x$  and  $y_n^1 \rightarrow y$ .

*Claim 1:*  $x = y$

To see this, note for a tolerance  $\epsilon'$ , there are integers  $N_1$  and  $N_2$  so that

$$n > N_1 \implies |x_n^1 - x| < \epsilon'/6 \text{ when } n^1 > N_1$$

$$n > N_2 \implies |y_n^1 - y| < \epsilon'/6 \text{ when } n^1 > N_2$$

where  $n^1$  indicates the subsequence index.

Now pick any subsequence index greater than  $\max(N_1, N_2)$ . Call these subsequence elements  $x_{\hat{n}}^1$  and  $y_{\hat{n}}^1$ . Also choose the subsequence index so that  $1/\hat{n}^1 < \epsilon'/6$ . So both conditions hold for this choice.

## Proof

$$\begin{aligned} |x - y| &= |x - x_{\hat{n}}^1 + x_{\hat{n}}^1 - y_{\hat{n}}^1 + y_{\hat{n}}^1 - y| \\ &\leq |x - x_{\hat{n}}^1| + |x_{\hat{n}}^1 - y_{\hat{n}}^1| + |y_{\hat{n}}^1 - y| \end{aligned}$$

The first and last are less than  $\epsilon'/6$ , so we have

$$|x - y| \leq |x_{\hat{n}}^1 - y_{\hat{n}}^1| + \epsilon'/3$$

Now remember, we know  $|x_{\hat{n}}^1 - y_{\hat{n}}^1| < 1/\hat{n}^1$ . So we have

$$|x - y| \leq 1/\hat{n}^1 + \epsilon'/3 < \epsilon'/6 + \epsilon'/3 = 2\epsilon'/3 < \epsilon'$$

Since  $\epsilon'$  is arbitrary, we see  $x = y$ . Of course, this also means  $f(x) = f(y)$  which says  $|f(x) - f(y)| = 0$ .

*Claim 2:*  $|f(x) - f(y)| \geq 2\epsilon_0/3$ . Since  $x_n^1 \rightarrow x$  and  $y_n^1 \rightarrow y = x$  and  $f$  is continuous on  $I$ , we have

## Proof

$$\exists M_1 \ni |f(x_n^1) - f(x)| < \epsilon_0/6 \quad \forall n^1 > M_1$$

$$\exists M_2 \ni |f(y_n^1) - f(y)| < \epsilon_0/6 \quad \forall n^1 > M_2$$

where again the indices for these subsequences are denoted by  $n^1$ . Pick a fixed  $n^1 > \max(M_1, M_2)$  and then both conditions hold. We can say

$$\begin{aligned} \epsilon_0 &\leq |f(x_n^1) - f(y_n^1)| \\ &= |f(x_n^1) - f(x) + f(x) - f(y) + f(y) - f(y_n^1)| \\ &\leq |f(x_n^1) - f(x)| + |f(x) - f(y)| + |f(y) - f(y_n^1)| \\ &\leq |f(x) - f(y)| + \epsilon_0/3 \end{aligned}$$

This tells us  $|f(x) - f(y)| \geq 2\epsilon_0/3$ .

But we also know  $|f(x) - f(y)| = 0$ . This contradiction tells us our assumption that  $f$  is not uc on  $I$  is wrong. Thus  $f$  is uc on  $I$ .  $\square$

- ▶ This result is true for a continuous function on any compact set  $D$  of  $\mathbb{R}^n$  although we would have to use the Euclidean norm  $\|\cdot\|$  to do the proof.
- ▶ So continuity and compactness are linked again. Recall continuous functions on compact sets must have an absolute minimum and absolute maximum too.

## Homework 3

Provide a careful proof of this proposition.

- 3.1 Prove  $\sqrt{x}$  is not Lipschitz on  $[0, 1]$ .

Comment: the thing here is that you can't find an  $L > 0$  that will work. You know if it works you have  $|\sqrt{x} - \sqrt{y}| \leq L|x - y|$  holds for any  $x, y$  in  $[0, 1]$ . So let  $y = 0$  and see what is happening there. Note it is easy to see why it fails but your job is to write your argument mathematically clear.

- 3.2 Prove  $\sqrt{x}$  is continuous on  $[0, 1]$  using an  $\epsilon - \delta$  argument.

Comments: there are two cases here: the case  $p = 0$  and the others,  $p \in (0, 1]$ . for the first case, given  $\epsilon$ , just pick  $\delta = \epsilon^2$  (details left to you); for the other case, this is the Mean Value Theorem approach.

- 3.3 Prove  $\sqrt{x}$  is uniformly continuous on  $[0, 1]$  the easy way.