Completeness and Uniform Continuity

James K. Peterson

Department of Biological Sciences and Department of Mathematical Sciences
Clemson University

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Outline

1. The Completeness of $\Re$
2. Uniform Continuity and Compact Domains
3. Homework

Let's look more carefully at what is called the completeness of $\Re$. We prove this carefully by showing we can extend the field $\mathbb{Q}$ to another field $\widetilde{\mathbb{Q}}$ which is totally ordered, satisfires the Completeness axiom (i..e the least upper bound and greatest lower bound property) and in which Cauchy Sequences of objects converge to an object in $\widetilde{\mathbb{Q}}$. This new field is then identified with $\Re$.

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- There is a general process by which a metric space can be completed which we can illustrate by using the rational numbers as a guide. We actually don't use this process to construct the real numbers, but it will show you the steps we typically take.
- Let $(\mathbb{Q}, |\cdot|) = X$. This is a nice **metric space** where $|x - y|$ measures the **distance** between the two rational numbers x and y. We already know Cauchy Sequences of rational numbers need not converge to a rational number. A nice example is the sequence $x_n = (1 + 1/n)^n$ which we know converges to a number we call e.

- Let's define a new metric space which we will call Y. Y is the set of all Cauchy Sequences of rational numbers; i.e. the *objects* in our space are Cauchy Sequences! Note each rational number p/q forms a nice constant sequence $x_1 = p/q, x_2 = p/q, \ldots, x_n = p/q, \ldots$. We can denote this constant sequence by (p/q). So for example $(2/3)$ is the constant Cauchy Sequence whose entries are all $2/3$.

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- We need a metric for Y. Define the distance between two Cauchy Sequences in Y like this: $D(\,(x_n), (y_n)\,) = \lim_{n \to \infty} |x_n - y_n|$.

  The objects in Y divide naturally into **classes** called *equivalence classes*. Given any object from Y, (x_n), we let $[(x_n)]$ denote the collection of all other objects from Y, i.e. other Cauchy Sequences of rational numbers, whose distance to (x_n) is zero.

- We call this set of equivalence classes $\tilde{Y}$ and we define the distance, $\tilde{D}$, between two equivalence classes as follows: $\tilde{D}(\ [(x_n)], [(y_n)]\ ) = \lim_{n \rightarrow \infty} |x_n - y_n|$. We can show this limit exists when we construct the field $\widetilde{\mathbb{Q}}$. Of course, we would have to show the value of $\tilde{D}$ does not depend on the choice of representatives from the equivalence classes!

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- For example, the constant sequence $(3/5)$ is in Y and there are an infinite number of other sequences (a_n) so that $D((3/5), (a_n)) = 0$. Just let (b_n) be any sequence of rational numbers that converges to 0. Then $D((3/5), (3/5) + (b_n)) = 0$ and so $(3/5) + (b/n)$ is a member of $[(3/5)]$. This is the big point now! The sequence $((1 + 1/n)^n)$ does not converge to a rational number and so it can not be in the equivalence class associated to any rational number $[p/q]$. Another way of saying this is that $D((1 + 1/n)^n), (p/q)) \neq 0$ for all $p/q \in \mathbb{Q}$.

The equivalence class $[((1+1/n)^n)]$ is thus **different** from the equivalence classes formed from constant sequences of rationals. The collection of all equivalence classes of objects from Y can thus be identified in a natural way with the numbers we see in $\Re$.

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- Each constant rational sequence (p/q) is chosen as the representative of $[(p/q)]$.
- Each equivalence class that is different from the equivalence classes formed by constant rational sequences is identified with some representative from it. We call that α. Note it can not come from a constant rational sequence so it can not be a rational number. We generally call this an **irrational number**. But remember, in this context, it is really a Cauchy Sequence of rationals!

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- It is hard, but in a more advanced class, we can show Cauchy Sequences in $(\tilde{Y}, \tilde{D})$ converge to an object in $(\tilde{Y}, \tilde{D})$. So we can prove $(\tilde{Y}, \tilde{D})$ is a complete metric space.

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- It is hard, but in a more advanced class, we can show Cauchy Sequences in $(\tilde{Y}, \tilde{D})$ converge to an object in $(\tilde{Y}, \tilde{D})$. So we can prove $(\tilde{Y}, \tilde{D})$ is a complete metric space.
- We can do this construction process for any metric space (X, d) and build a new complete metric space $(\tilde{Y}, \tilde{D})$. We do this in the first course on linear analysis that follows this course.

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- We said $(C[0,1], ||\cdot||_2)$ is not complete. We also find the completion of $(C[0,1], ||\cdot||_2)$ can be done following this construction process and generates the space $(L_2, ||\cdot||_2)$ which is a space of equivalence classes of functions using Lebesgue Integration. This is real special and it turns out to be an inner product space which is complete. This is called a **Hilbert Space**.

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- The space $(C[0,1], ||\cdot||_\infty)$ is complete as we will show in a bit and so if we do the construction process as outlined earlier, we just get back the same space: (X, d) and (Y, D) will be the same here.

Let's look more carefully at continuous functions on compact domains. We can prove a nice theorem:

Theorem

If f is continuous on the compact interval $I = [a, b]$, then f is uniformly continuous on I.

Proof

We are going to prove this by **contradiction**. *If f is not uc on I, there is an ϵ_0 so that*

$$\forall \delta > 0,\ \exists x, y \in I \ni |x - y| < \delta \text{ and } |f(x) - f(y)| > \epsilon_0$$

Proof

In particular for the choice $\delta_n = 1/n$ *for all* $n \geq 1$, *we have*

$$\exists\, x_n, y_n \in I \ni |x_n - y_n| < 1/n \text{ and } |f(x_n) - f(y_n)| \geq \epsilon_0$$

Since (x_n) *and* (y_n) *are contained in the compact set* I, *the Bolzano - Weierstrass Theorem tells us there are subsequences* (x_n^1) *and* (y_n^1) *and points* x *and* y *in* I *so that* $x_n^1 \rightarrow x$ *and* $y_n^1 \rightarrow y$.

Claim 1: $x = y$
To see this, note for a tolerance ϵ', *there are integers* N_1 *and* N_2 *so that*

$$n > N_1 \Longrightarrow |x_n^1 - x| < \epsilon'/6 \text{ when } n^1 > N_1$$
$$n > N_2 \Longrightarrow |y_n^1 - y| < \epsilon'/6 \text{ when } n^1 > N_2$$

where n^1 *indicates the subsequence index.*

Now pick any subsequence index greater than $\max(N_1, N_2)$. *Call these subsequence elements* $x_{\hat{n}}^1$ *and* $y_{\hat{n}}^1$. *Also choose the subsequence index so that* $1/\hat{n}^1 < \epsilon'/6$. *So both conditions hold for this choice.*

Proof

$$\begin{aligned} |x - y| &= |x - x_{\hat{n}}^1 + x_{\hat{n}}^1 - y_{\hat{n}}^1 + y_{\hat{n}}^1 - y| \\ &\leq |x - x_{\hat{n}}^1| + |x_{\hat{n}}^1 - y_{\hat{n}}^1| + |y_{\hat{n}}^1 - y| \end{aligned}$$

The first and last are less than $\epsilon'/6$, so we have

$$|x - y| \leq |x_{\hat{n}}^1 - y_{\hat{n}}^1| + \epsilon'/3$$

Now remember, we know $|x_{\hat{n}}^1 - y_{\hat{n}}^1| < 1/\hat{n}^1$. So we have

$$|x - y| \leq 1/\hat{n}^1 + \epsilon'/3 < \epsilon'/6 + \epsilon'/3 = 2\epsilon'/3 < \epsilon'$$

Since ϵ' is arbitrary, we see $x = y$. Of course, this also means $f(x) = f(y)$ which says $|f(x) - f(y)| = 0$.
Claim 2: $|f(x) - f(y)| \geq 2\epsilon_0/3$. Since $x_n^1 \to x$ and $y_n^1 \to y = x$ and f is continuous on I, we have

Proof

$$\exists M_1 \ni |f(x_n^1) - f(x)| < \epsilon_0/6 \quad \forall n^1 > M_1$$
$$\exists M_2 \ni |f(y_n^1) - f(y)| < \epsilon_0/6 \quad \forall n^1 > M_2$$

where again the indices for these subsequences are denoted by n^1. *Pick a fixed* $n^1 > \max(M_1, M_2)$ *and then both conditions hold. We can say*

$$\begin{aligned} \epsilon_0 &\leq |f(x_n^1) - f(y_n^1)| \\ &= |f(x_n^1) - f(x) + f(x) - f(y) + f(y) - f(y_n^1)| \\ &\leq |f(x_n^1) - f(x)| + |f(x) - f(y)| + |f(y) - f(y_n^1)| \\ &\leq |f(x) - f(y)| + \epsilon_0/3 \end{aligned}$$

This tells us $|f(x) - f(y)| \geq 2\epsilon_0/3$.

But we also know $|f(x) - f(y)| = 0$. *This contradiction tells us our assumption that* f *is not uc on* I *is wrong. Thus* f *is uc on* I. $\square$

- This result is true for a continuous function on any compact set D of $\Re^n$ although we would have to use the Euclidean norm $||\cdot||$ to do the proof.

- This result is true for a continuous function on any compact set D of $\Re^n$ although we would have to use the Euclidean norm $|| \cdot ||$ to do the proof.
- So continuity and compactness are linked again. Recall continuous functions on compact sets must have an absolute minimum and absolute maximum too.

Homework 3

Provide a careful proof of this proposition.

3.1 Prove $\sqrt{x}$ is not Lipschitz on $[0, 1]$.
Comment: the thing here is that you can't find an $L > 0$ that will work. You know if it works you have $|\sqrt{x} - \sqrt{y}| \leq L|x - y|$ holds for any x, y in $[0, 1]$. So let $y = 0$ and see what is happening there. Note it is easy to see why it fails but your job is to write your argument mathematically clear.

3.2 Prove $\sqrt{x}$ is continuous on $[0, 1]$ using an $\epsilon - \delta$ argument.
Comments: there are two cases here: the case $p = 0$ and the others, $p \in (0, 1]$. for the first case, given ϵ, just pick $\delta = \epsilon^2$ (details left to you); for the other case, this is the Mean Value Theorem approach.

3.3 Prove $\sqrt{x}$ is uniformly continuous on $[0, 1]$ the easy way.