Hölder's and Minkowski's Inequality

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Outline

We say the positive numbers p and q are **conjugate exponents** if $p > 1$ and $1/p + 1/q = 1$. If $p = 1$, we define its conjugate exponent to be $q = \infty$. Conjugate exponents satisfy some fundamental identities. Clearly, if $p > 1$,

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Longrightarrow \quad 1 = \frac{p+q}{p\,q}$$

From that it follows

$$pq \quad = \quad p + q$$

and from that using factoring

$$(p-1)\,(q-1) \quad = \quad 1$$

We will use these identies quite a bit.

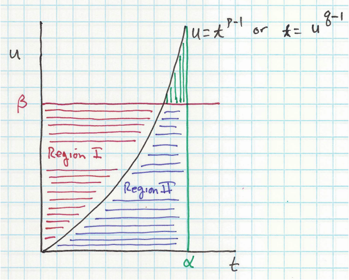
Lemma

The $\alpha - \beta$ Lemma:
Let α and β be positive real numbers and p and q be conjugate exponents. Then $\alpha\,\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$.

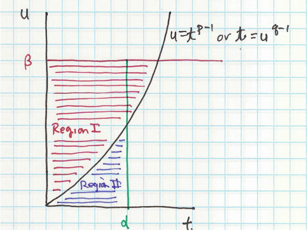
Proof

To see this is a straightforward integration. We haven't discussed Riemann integration in depth yet, but all of you know how to integrate from your earlier courses. Let $u = t^{p-1}$. Then, $t = u^{1/(p-1)}$ and using the identity $(p-1)\,(q-1) = 1$, we have $t = u^{q-1}$. Now we are going to draw the curve $u = t^{p-1}$ or $t = u^{q-1}$ in the first quadrant and show you how this inequality makes sense. We will draw the curve $u = t^{p-1}$ as if it was concave up (i.e. like when $p = 3$) even though it could be concave down (i.e. like when $p = 3/2$). Whether the curve is concave up or down does not change how the argument goes. So make sure you can see that. A lot of times our pictures are just aids to helping us think through an argument. A placeholder, so to speak!

Here the area of the rectangle $\alpha\,\beta \leq$ the area of Region I + the area of Region II + the area marked in green.



Here the area of the rectangle $\alpha\,\beta \leq$ the area of Region I + the area of Region II.



Proof

In the first picture,
(1): the area of Region I is the area under the curve $t = u^{q-1}$ *from* $u = 0$ *to* $u = \beta$*. This is* $\int_0^\beta u^{q-1}\,du$*.*
(2): the area of Region II + the area marked in green is the area under the curve $u = t^{p-1}$ *from* $t = 0$ *to* $t = \alpha$*. This is* $\int_0^\alpha t^{p-1}\,dt$*.*

In the second picture,
(1): the area of Region I is still the area under the curve $t = u^{q-1}$ *from* $u = 0$ *to* $u = \beta$*. This is* $\int_0^\beta u^{q-1}\,du$*.*
(2): the area of Region II is the area under the curve $u = t^{p-1}$ *from* $t = 0$ *to* $t = \alpha$*. This is* $\int_0^\alpha t^{p-1}\,dt$*.*

So in both cases

$$\alpha\,\beta \quad \leq \quad \int_0^\beta u^{q-1}\,du + \int_0^\alpha t^{p-1}\,dt = \frac{\beta^q}{q} + \frac{\alpha^p}{p}.$$

□

Definition

Let $p \geq 1$. The collection of all sequence, $(a_n)_{n=1}^\infty$ for which $\sum_{n=1}^\infty |a_n|^p$ converges is denoted by the symbol ℓ^p.

(1) $\ell^1 = \{(a_n)_{n=1}^\infty : \sum_{n=1}^\infty |a_n| \ \text{converges.}\ \}$

(2) $\ell^2 = \{(a_n)_{n=1}^\infty : \sum_{n=1}^\infty |a_n|^2 \ \text{converges.}\ \}$

We also define $\ell^\infty = \{(a_n)_{n=1}^\infty : \sup_{n\geq 1} |a_n| < \infty\}$.

Theorem

Hölder's Inequality:
Let $p > 1$ *and* p *and* q *be conjugate exponents. If* $x \in \ell^p$ *and* $y \in \ell^q$*, then*

$$\sum_{n=1}^\infty |x_n\,y_n| \quad \leq \quad \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \left(\sum_{n=1}^\infty |y_n|^q\right)^{1/q}$$

where $x = (x_n)$ *and* $y = (y_n)$*.*

Proof

This inequality is clearly true if either of the two sequences x and y are the zero sequence. So we can assume both x and y have some nonzero terms in them. Then $x \in \ell^p$, we know

$$0 < u = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty, \quad 0 < v = \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q} < \infty$$

Now define new sequences, $\hat{x}$ and $\hat{y}$ by $\hat{x}_n = x_n/u$ and $\hat{y}_n = y_n/v$. Then, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |\hat{x}_n|^p &= \sum_{n=1}^{\infty} \frac{|x_n|^p}{u^p} = \frac{1}{u^p} \sum_{n=1}^{\infty} |x_n|^p = \frac{u^p}{u^p} = 1. \\ \sum_{n=1}^{\infty} |\hat{y}_n|^q &= \sum_{n=1}^{\infty} \frac{|y_n|^q}{v^q} = \frac{1}{v^q} \sum_{n=1}^{\infty} |y_n|^q = \frac{v^q}{v^q} = 1. \end{aligned}$$

Proof

Now apply the $\alpha - \beta$ Lemma to $\alpha = |\hat{x_n}|$ and $\beta = |\hat{y_n}|$ for any nonzero terms $\hat{x}_n$ and $\hat{y}_n$. Then $|\hat{x_n}\, \hat{y_n}| \leq |\hat{x_n}|^p/p + |\hat{y_n}|^q/q$.

This is also true, of course, if either $\hat{x}_n$ or $\hat{y}_n$ are zero although the $\alpha - \beta$ lemma does not apply!

Now sum over N terms to get

$$\sum_{n=1}^{N} |\hat{x_n}\, \hat{y_n}| \leq \frac{1}{p} \sum_{n=1}^{N} |\hat{x_n}|^p + \frac{1}{q} \sum_{n=1}^{N} |\hat{y_n}|^q$$

Since we know $x \in \ell^p$ and $y \in \ell^q$, we know

$$\begin{aligned} \sum_{n=1}^{N} |\hat{x_n}|^p \leq \sum_{n=1}^{\infty} |\hat{x_n}|^p &= 1 \\ \sum_{n=1}^{N} |\hat{y_n}|^q \leq \sum_{n=1}^{\infty} |\hat{y_n}|^q &= 1 \end{aligned}$$

Proof

So we have

$$\sum_{n=1}^{N} |\hat{x}_n \, \hat{y}_n| \quad \leq \quad \frac{1}{p} + \frac{1}{q} = 1$$

This is true for all N *the partial sums* $\sum_{n=1}^{N} |\hat{x}_n \, \hat{y}_n|$ *are bounded above. Hence, the partial sums converge to this supremum which is denoted by* $\sum_{n=1}^{\infty} |\hat{x}_n \, \hat{y}_n|$. *We conclude* $\sum_{n=1}^{\infty} |\hat{x}_n \, \hat{y}_n| \leq 1$. *But* $\hat{x}_n \hat{y}_n = 1/(u\, v)\, x_n y_n$ *and so we have* $\frac{1}{u\, v} \sum_{n=1}^{\infty} |x_n \, y_n| \leq 1$ *which implies the result as*

$$\sum_{n=1}^{\infty} |x_n \, y_n| \leq u\, v = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}$$

□

Theorem

Hölder's Theorem for $p = 1$ *and* $q = \infty$

if $x \in \ell^1$ *and* $y \in \ell^\infty$, *then* $\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n| \right) \sup_{n \geq 1} |y_n|$.

Proof

We know since $y \in \ell^\infty$, $|y_n| \leq \sup_{k \geq 1} |y_k|$. *Thus,*

$$\sum_{n=1}^{N} |x_n y_n| \quad \leq \quad \left(\sum_{n=1}^{N} |x_n| \right) \sup_{k \geq 1} |y_k|$$

Thus the sequence of partial sums $\sum_{n=1}^{N} |x_n y_n|$ *is bounded above by* $\left(\sum_{n=1}^{\infty} |x_n| \right) \sup_{k \geq 1} |y_k|$. *This gives us our result.* □

Theorem

Minkowski's Inequality
Let $p \geq 1$ and let x and y be in ℓ^p, Then,

$$\left(\sum_{n-1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n-1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n-1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

Proof

(1): $p = \infty$: We know $|x_n + y_n| \leq |x_n| + |y_n|$. So $|x_n + y_n| \leq \sup_{n \geq 1} |x_n| + \sup_{n \geq 1} |y_n|$. Thus, the right hand is an UB for the left. Hence $\sup_{n \geq 1} |x_n + y_n| \leq \sup_{n \geq 1} |x_n| + \sup_{n \geq 1} |y_n|$ which is the result for $p = \infty$.
(2): $p = 1$: Again, we know $|x_n + y_n| \leq |x_n| + |y_n|$. Then

$$\begin{aligned} \sum_{n=1}^{N} |x_n + y_n| &\leq \sum_{n=1}^{N} |x_n| + \sum_{n=1}^{N} |y_n| \\ &\leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \end{aligned}$$

Proof

The right hand side is an upper bound for the partial sums on the left. Hence, we have

$$\sum_{n=1}^{\infty} |x_n + y_n| \leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n|$$

(3) $1 < p < \infty$
We have

$$\begin{aligned} |x_n + y_n|^p &= |x_n + y_n| \, |x_n + y_n|^{p-1} \\ &\leq |x_n| \, |x_n + y_n|^{p-1} + |y_n| \, |x_n + y_n|^{p-1} \\ \sum_{n=1}^{N} |x_n + y_n|^p &\leq \sum_{n=1}^{N} |x_n| \, |x_n + y_n|^{p-1} + \sum_{n=1}^{N} |y_n| \, |x_n + y_n|^{p-1} \end{aligned}$$

Let $a_n = |x_n|$, $b_n = |x_n + y_n|^{p-1}$, $c_n = |y_n|$ and $d_n = |x_n + y_n|^{p-1}$. Hölder's Inequality applies just fine to finite sequences: i.e. sequences in $\Re^N$.

Proof

So we have

$$\sum_{n=1}^{N} a_n b_n \quad \leq \quad \left(\sum_{n=1}^{N} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} b_n^q \right)^{\frac{1}{q}}$$

But $b_n^q = |x_n + y_n|^{q(p-1)} = |x_n + y_n|^p$ *using the conjugate exponents identities we established. So we have found*

$$\sum_{n=1}^{N} |x_n| \, |x_n + y_n|^{p-1} \quad \leq \quad \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{\frac{1}{q}}$$

We can apply the same reasoning to the terms c_n *and* d_n *to find*

$$\sum_{n=1}^{N} |y_n| \, |x_n + y_n|^{p-1} \quad \leq \quad \left(\sum_{n=1}^{N} |y_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{\frac{1}{q}}$$

Proof

We can use the inequalities we just figured out to get the next estimate

$$\begin{aligned} \sum_{n=1}^{N} |x_n + y_n|^p \quad \leq \quad & \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{\frac{1}{q}} \\ & + \left(\sum_{n=1}^{N} |y_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{\frac{1}{q}} \end{aligned}$$

Now factor out the common term to get

$$\sum_{n=1}^{N} |x_n + y_n|^p \leq \left(\left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p \right)^{\frac{1}{p}} \right) \left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{\frac{1}{q}}$$

Rewrite again as

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p \right)^{1-\frac{1}{q}} \leq \left(\sum_{n=1}^{N} |x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p \right)^{\frac{1}{p}}$$

Proof

But $1 - 1/q = 1/p$*, so we have*

$$\begin{aligned}\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}\end{aligned}$$

This says the right hand side is an upper bound for the partial sums on the left. Hence, we know

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

□

If $x \in \ell^p$, we can define a new function, called a **norm**, on ℓ^p like this:

$$\begin{aligned}||x||_p &= \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ ||x||_\infty &= \sup_{n \geq 1} |x_n|, \quad p = \infty\end{aligned}$$

The Minkowski Inequality can then be rephrased as

$$||x + y||_p \leq ||x||_p + ||y||_p$$

and so we can use $|| \cdot ||_p$ as a way to measure both size of x and the distance between x and y. Letting $d(x, y)$ denote the distance between x and y, we have

$$d(x, y) = ||x - y||_p = ||x + (-y)|_p \leq ||x||_p + || - y||_p = ||x||_p + ||y||_p$$

We note if $d(x, y) = 0$, then in the case $1 \leq p < \infty$, we have $\sum_{n=1} |x_n - y_n|^p = 0$. But here you are adding up non-negative terms, so this must imply $|x_n - y_n|^p = 0$ for all n. This tells us $x_n = y_n$ for all n; i..e $x = y$.

On the other hand, when $p = \infty$, we would have $\sup_{n \geq 1} |x_n - y_n| = 0$. But this says, $|x_n - y_n| \leq 0$ for all n. Hence, $x_n = y_n$ for all n telling us $x = y$ again.

Also, note the Minkowski Inequality tells us that ℓ^p is a vector space as if x and y are in ℓ^p, their sum $x + y$ defined by $x + y$ is the sequence $(x_n + y_n)$ has $\sum_{n=1}^{\infty} |x_n + y_n|^p < \infty$ because of Minkowski's Inequality.

A function like $|| \cdot ||_p$ on the set ℓ^p is called a **norm** and the set ℓ^p is called a **Normed Linear Space** or **Normed Vector Space**. A **vector** in ℓ^p is the sequence x and its magnitude or size in $||x||_p$.

This is not a finite dimensional vector space and is another example of such things to add to your collection along with $(C[0, 1], d_1)$ and so forth.

Homework 7

7.1 Prove a geometric series x with common ratio r in $(-1, 1)$ is in ℓ^1 and find $||x||_1$.

7.2 If $x \in \Re^2$, the usual Euclidean norm is $|| \cdot ||_2$. Note $||x||_3$ and so on is also a norm on $\Re^2$. Compute $||x||_1$, $||x||_2$, $||x||_3$, ..., $||x||_{10}$ for $x = [2, 5]'$. What do you think happens as $p \to \infty$?

7.3 The Hölder's inequality tells us that in $\Re^2$, $< x, y > /(||x||_p \, ||y||_q)$ is in $[-1, 1]$. So we can use this to define the angle between x and y. For $p = q = 2$ this is our usual angle, but for $p = 3$, $q = 3/2$ and so on the calculation changes. Calculate this angle for $p = 2$, $p = 3$, $p = 4$ and $p = 5$ where the q value is the exponent conjugate to p for the two vectors $x = [-1, 3]'$ and $y = [2, 4]'$.