The Diabetes Detection Model

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November 23, 2014
Outline

1. The Diabetes Detection Model
This Lecture discusses the diabetes detection model.
Our model approximation is

\[ g'(t) = \frac{\partial F_1}{\partial g}(G_0, H_0) \cdot g + \frac{\partial F_1}{\partial h}(G_0, H_0) \cdot h \]

\[ h'(t) = \frac{\partial F_2}{\partial g}(G_0, H_0) \cdot g + \frac{\partial F_2}{\partial h}(G_0, H_0) \cdot h \]
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Next, we can reason out the algebraic signs are

\[ \frac{\partial F_1}{\partial g}(G_0, H_0) = - \]
\[ \frac{\partial F_1}{\partial h}(G_0, H_0) = - \]
\[ \frac{\partial F_2}{\partial g}(G_0, H_0) = + \]
\[ \frac{\partial F_2}{\partial h}(G_0, H_0) = - \]

The arguments for these algebraic signs come from our understanding of the physiological processes that are going on here.
Let’s look at a small positive deviation $g$ from the optimal value $G_0$ while letting the net hormone concentration be fixed at $H_0$. Then our model approximation is

$$g'(t) = \frac{\partial F_1}{\partial g}(G_0, H_0) g$$

since we do not have an external input here. At a state where we have an increase in blood sugar levels over optimal, i.e. $g > 0$, the other hormones such as cortisol and glucagon will try to regulate the blood sugar level down by increasing their concentrations and for example storing more sugar into glycogen.
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Hence, the term $\frac{\partial F_1}{\partial g}(G_0, H_0)$ should be negative as here $g'$ is negative as $g$ should be decreasing.
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So we model this as $\frac{\partial F_1}{\partial g}(G_0, H_0) = -m_1$ for some positive number $m_1$. 
Now consider a positive change in $h$ from the optimal level while keeping at the optimal level $G_0$. Then the model is

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and since $h > 0$, this means the net hormone concentration is up which we interpret as insulin above normal. This means blood sugar levels go down which implies $g'$ is negative again.
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Thus, $\frac{\partial F_1}{\partial h}(G_0, H_0)$ must be negative which means we model it as $\frac{\partial F_1}{\partial h}(G_0, H_0) = -m_2$ for some positive $m_2$. 
If we have a small positive deviation \( g \) from the optimal value \( G_0 \) while letting the net hormone concentration be fixed at \( H_0 \), we have

\[
h'(t) = \frac{\partial F_2}{\partial g}(G_0, H_0) g.
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$$h'(t) = \frac{\partial F_2}{\partial g}(G_0, H_0) g.$$ 

Again, since $g$ is positive, this means we are above normal blood sugar levels which implies mechanisms are activated to bring the level down. Hence $h' > 0$ as we have increasing net hormone levels.
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Thus, we must have $\frac{\partial F_2}{\partial g}(G_0, H_0) = m_3$ for some positive $m_3$. 

Finally, if we have a positive deviation $h$ from optimal while blood sugar levels are optimal, the model is

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Since $h$ is positive, we have the concentrations of the hormones that pull glucose out of the blood stream are above optimal. This means that too much sugar is being removed as so the regulatory mechanisms will act to stop this action implying $h' < 0$. 
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This tells us $\frac{\partial F_2}{\partial g}(G_0, H_0) = -m_4$ for some positive constant $m_4$. 
Hence, the four partial derivatives at the optimal points can be defined by four positive numbers $m_1$, $m_2$, $m_3$ and $m_4$ as follows:

\[
\begin{align*}
\frac{\partial F_1}{\partial g}(G_0, H_0) &= -m_1 \\
\frac{\partial F_1}{\partial h}(G_0, H_0) &= -m_2 \\
\frac{\partial F_2}{\partial g}(G_0, H_0) &= +m_3 \\
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\end{align*}
\]

Our model dynamics are thus approximated by

\[
\begin{align*}
g'(t) &= -m_1 g - m_2 h + J(t) \\
h'(t) &= m_3 g - m_4 h
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This implies

\[
g''(t) = -m_1 g' - m_2 h' + J'(t)
\]
Now plug in the formula for $h'$ to get

$$g''(t) = -m_1 g' - m_2 (m_3 g - m_4 h) + J'(t)$$

$$= -m_1 g' - m_2 m_3 g + m_2 m_4 h + J'(t).$$
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But we can use the $g'$ equation to solve for $h$. This gives

$$m_2 h = -g'(t) - m_1 g + J(t)$$
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$$g''(t) = -m_1 g' - m_2 m_3 g + m_4 (-g'(t) - m_1 g + J(t)) + J'(t)$$

$$= -(m_1 + m_4) g' - (m_1 m_4 + m_2 m_3) g + m_4 J(t) + J'(t).$$
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$$= -(m_1 + m_4) g' - (m_1 m_4 + m_2 m_3) g + m_4 J(t) + J'(t).$$

So our final model is

$$g''(t) + (m_1 + m_4) g' + (m_1 m_4 + m_2 m_3) g = m_4 J(t) + J'(t).$$
Let $\alpha = (m_1 + m_4)/2$ and $\omega^2 = m_1 m_4 + m_2 m_3$ and we can rewrite as

$$g''(t) + 2\alpha g' + \omega^2 g = S(t).$$

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Now the right hand side here is zero except for the very short time interval when the glucose load is being ingested. Hence, we can simply search for the solution to the homogeneous model

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The roots of the characteristic equation here are

$$r = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}.$$
The most interesting case is if we have complex roots. In that case, \( \alpha^2 - \omega^2 < 0 \). Let \( \Omega^2 = |\alpha^2 - \omega^2| \). Then, the general phase shifted solution has the form \( g = Re^{-\alpha t} \cos(\Omega t - \delta) \) which implies

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G = G_0 + Re^{-\alpha t} \cos(\Omega t - \delta).
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Hence, our model has five unknowns to find: \( G_0, R, \alpha, \Omega \) and \( \delta \).

The easiest way to do this is to measure \( G_0 \), the patient’s initial blood glucose concentration, when the patient arrives. Then measure the blood glucose concentration \( N \) more times giving the data pairs \((t_1, G_1), (t_2, G_2)\) and so on out to \((t_N, G_N)\).
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Then form the least squares error function

\[
E = \sum_{i=1}^{N} \left( G_i - G_0 - Re^{-\alpha t_i \cos(\Omega t_i - \delta)} \right)^2
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Numerous experiments have been done with this model and if we let $T_0 = 2\pi/\Omega$, it has been found that if $T_0 < 4$ hours, the patient is normal and if $T_0$ is much larger than that, the patient has mild diabetes.
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And that wraps up this course. I hope you take more courses involving mathematics, computation and science!!!