One facility minimax location with Euclidean distance: 1 / *P* / - / l_2 / max

Given n distinct points $P_i = (a_i, b_i)$ in the plane, the problem is to find a point $X = (x, y)$ that minimizes the maximum Euclidean distance from X to the given points.

Let $f(X) = \max_{1 \le i \le n} l_2(X,P_i)$. The problem is to minimize $f(X)$, i.e.,

$$\min \max_{1 \le i \le n} l_2(X,P_i).$$

A standard transformation is to write the problem as follows:

$$\begin{aligned} &\min \; z \\ &\text{s.t.} \quad l_2(X,P_i) \le z \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

This version of the problem has the geometric intrepretation of finding a circle with center X and minimum radius z so that all the given points P_i are in the circle, called the minimum covering circle problem. See Figure 1.

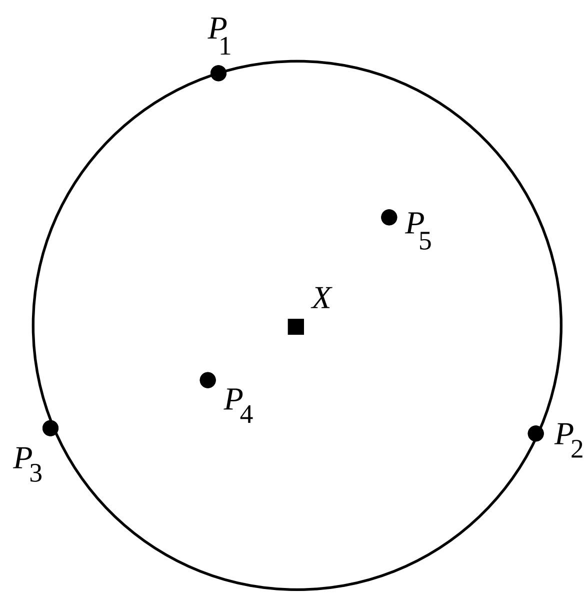

Figure 1

Notice that in the example of Figure 1, the minimum covering circle is determined by three points, P_1, P_2, and P_3. Alternatively, the minimum covering circle may be determined by two points, as in Figure 2.

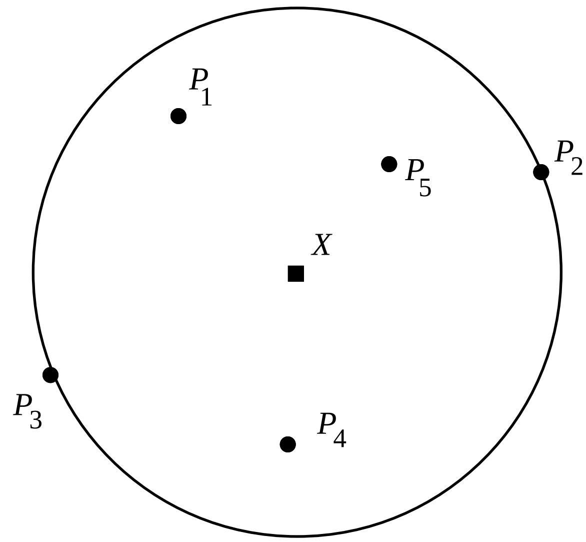

Figure 2

An alternate geometric interpretation is to find the minimum radius z so that the circles centered at P_i with radius z have nonempty intersection X. See Figure 3.

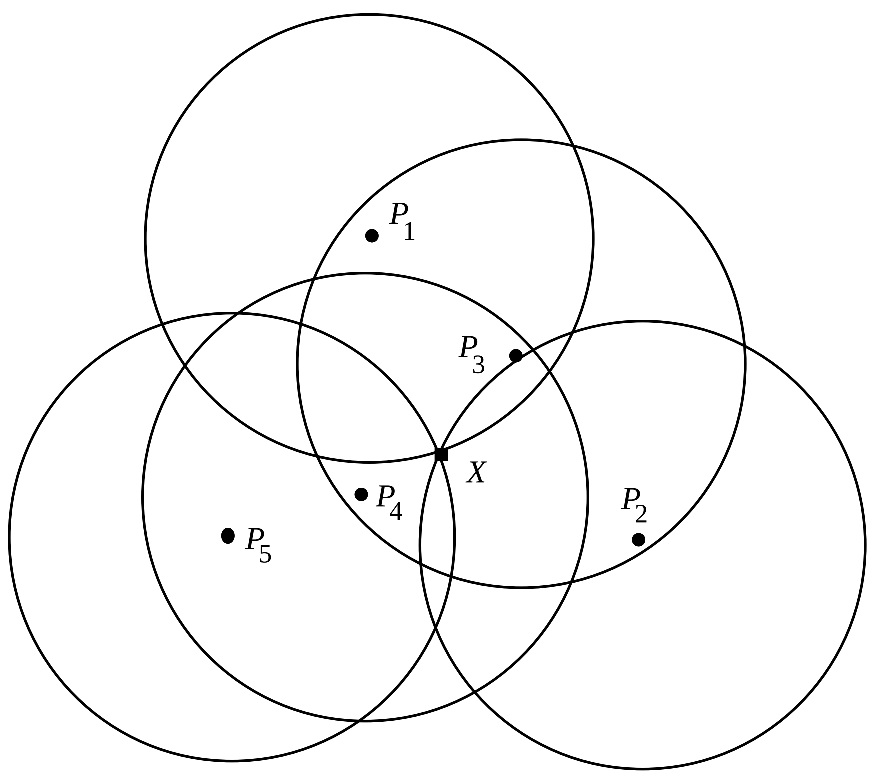

Figure 3

Elzinga and Hearn (1972) give a geometric algorithm for solving the one center problem with Euclidean distances and they prove the correctness of the algorithm.

1. Choose any two points, P_i and P_j
2. Construct the circle whose diameter is $l_2(P_i, P_j)$.
   If this circle contains all points, then the center of the circle is the optimal X.
   Else, choose a point P_k outside the circle.
3. If the triangle determined by P_i, P_j and P_k is a right triangle or an obtuse triangle, rename the two points oposite the right angle or the obstuse angle as P_i and P_j and go to step 2.
   Else, the three points determine an acute triangle. Construct the circle passing through the three points. (The center is the intersection of the perpendicular bisectors of two sides of the triangle.) If the circle contains all the points, stop, else, go to 4.
4. Choose some point P_l not in the circle, and let Q be the point among $\{P_i, P_j, P_k\}$ that is greatest distance from P_l. Extend the diameter through the point Q to a line that divides the plane into two half planes. Let the point R be the point among $\{P_i, P_j, P_k\}$that is in the half plane opposite P_l. With the points Q, R, and P_l, go to step 3.

Example: Consider the points $P_1, \ldots P_5$ as shown in Figure 4. Starting the algorithm with P_2 and P_4, Figure 4 shows the circle whose diameter is the line segment from P_2 to P_4.

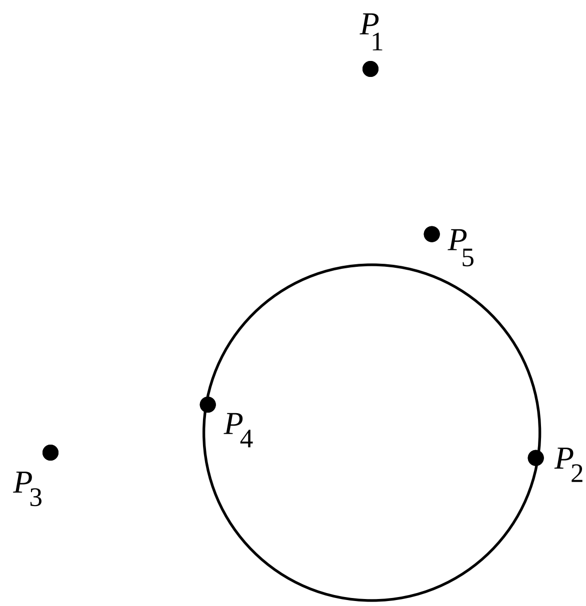

Figure 4.

In Step 2, choose P_1 as the point outside the circle. The points $\{P_1, P_2, P_4\}$ determine an acute triangle, and Figure 5 shows the circle determined by the points $\{P_1, P_2, P_4\}$.

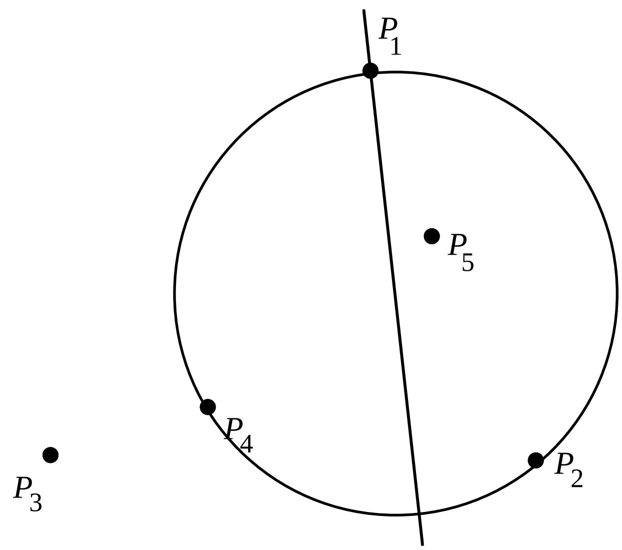

Figure 5

The point P_3 is not in the circle, and $Q = P_1$ is the point among $\{P_1, P_2, P_4\}$ that is greatest distance from P_3. Figure 5 shows the line extended from the diameter through $Q = P_1$ and that $R = P_2$ is the point among $\{P_1, P_2, P_4\}$ that is in the half plane opposite P_3.
With the points $\{Q, R, P_3\} = \{P_1, P_2, P_3\}$, go to step 3.
Figure 6 shows the circle determined by the points $\{P_1, P_2, P_3\}$, which includes all points.

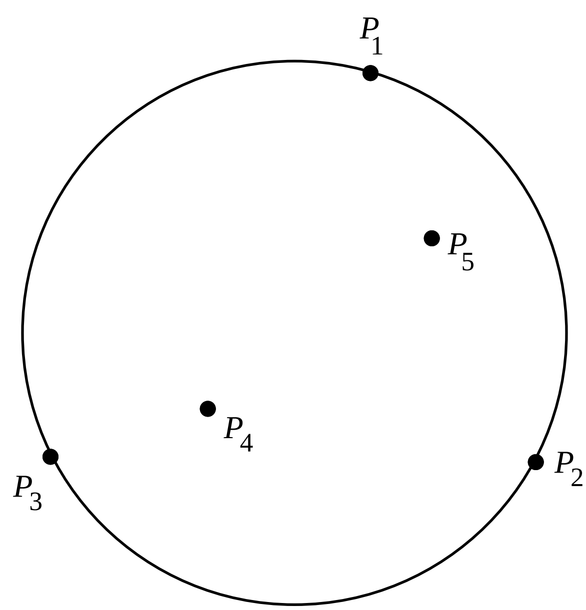

Figure 6

An alternative algorithm is given by the Chrystal-Peirce Algorithm found in Sylvester (1860), and Chrystal (1885):

0. Set $k = 1$. Construct a large circle which covers all the points P_i, and which passes through two points P_s and P_t. Define X_k as the center of the circle, and $S_k = \{P_s, P_t\}$.
1. Let $\angle P_s P_r P_t = \min\{ \angle P_s P_j P_t : P_j \notin S_k \}$. If $\angle P_s P_r P_t$ is obtuse, stop. The minimum circle has diameter $\frac{1}{2} l_2(P_s, P_t)$, and $X = \frac{1}{2}(P_s + P_t)$. Else, go to 2.
2. Compute the center of the circle, X_{k+1} passing through P_s, P_r, and P_t. If the triangle $\Delta P_s P_r P_t$ is not obtuse, stop; $X = X_{k+1}$. Else, drop the point among P_s, P_r, and P_t with the obtuse angle. Rename the remaining points P_s and P_t, set $S_{k+1} = \{P_s, P_t\}$, increment k and go to 1.

This is a primal algorithm in that the current circle always covers all the given points, and the radius decreases at each step.

The Kuhn-Tucker conditions for the minimax location problem:

For a general nonlinear programming problem with f and g_i convex, continuous and differentiable:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \geq 0 \quad i = 1, \ldots, n \end{aligned}$$

the Kuhn-Tucker optimality conditions state that x is an optimal solution if and only if, there exists λ_i such that:

$$\nabla f(x) = \sum_{i=1}^{n} \lambda_i \nabla g_i(x) \quad ,$$

$$g_i(x) \geq 0 \qquad i = 1, \ldots, n,$$

$$\lambda_i \, g_i(x) = 0 \qquad i = 1, \ldots, n,$$

$$\lambda_i \geq 0 \qquad i = 1, \ldots, n.$$

Observe that the minimax Euclidean distance problem is equivalent to the minimax squared Eulclidean distance problem:

$$\min \max_{1 \leq i \leq n} \; l_2(X, P_i)^2.$$

which is written in constrained form as:

$$\min \; z$$
$$\text{s.t.} \quad z \;\ge\; (x - a_i)^2 + (y - b_i)^2 \quad \text{for } i = 1, \ldots, n.$$

Then the Kuhn-Tucker state that (x, y) and z is an optimal solution if and only if there exists $\lambda_i \ge 0$ such that the following conditions hold:

$$1 = \sum_{i=1}^{n} \lambda_i \;, \tag{1}$$

$$0 = \sum_{i=1}^{n} \lambda_i (x - a_i) \quad \text{which gives} \quad x = \frac{\Sigma_i \lambda_i a_i}{\Sigma_i \lambda_i} = \Sigma_i \lambda_i a_i, \tag{2}$$

$$0 = \sum_{i=1}^{n} \lambda_i (y - b_i) \quad \text{which gives} \quad y = \frac{\Sigma_i \lambda_i b_i}{\Sigma_i \lambda_i} = \Sigma_i \lambda_i b_i, \tag{3}$$

$$z^* \;\ge\; (x - a_i)^2 + (y - b_i)^2 \quad \text{for } i = 1, \ldots, n, \text{ and} \tag{4}$$

$$\lambda_i \left(z^* - (x - a_i)^2 + (y - b_i)^2 \right) = 0 \quad \text{for } i = 1, \ldots, n. \tag{5}$$

These conditions are interpreted as follows: Conditions (1), (2), and (3) imply that the center of the circle, $X = (x, y)$, is a convex combination of the given points $P_i = (a_i, b_i)$, but conditions (5) state that the only λ_i allowed to be positive are associated with points P_i that are on the circle, i.e., where conditions (4) hold at equality. Thus, X is a convex combination of those points P_i that lie on the circle. Condition (4) requires all points P_i to lie inside or on the circle centered at X with radius z.

A theorem of Caratheodory states that to express a given point X in R^n as a convex combination of a given set of points, requires at most $n + 1$ of the given points. In the plane, with $n = 2$, this theorem implies that to express the center X of the minimum covering circle as a convex combination of the given points P_i requires at most 3 of the given points.

The minimax Euclidean distance problem requires either 2 or 3 points to specify the minimum covering circle.

A dual of the minimax Euclidean distance problem: $\min \max_{1 \le i \le n} l_2(X,P_i)$.

Consider the equivalent problem: $\min \max_{1 \le i \le n} l_2(X,P_i)^2$ in which the distance is squared.

This problem is equivalent to

$$\begin{aligned} &\min\ z \\ &\text{s.t.}\quad z \ \ge\ l_2(X,P_i)^2 \quad i = 1, \ldots, n. \end{aligned}$$

The Lagrangian for this problem may be written as

$$\begin{aligned} L(z, x, \lambda) &= z - \sum_{i=1}^{n} \lambda_i \left(z - l_2(X,P_i)^2\right) \\ &= z\left(1 - \sum_{i=1}^{n} \lambda_i\right) + \sum_{i=1}^{n} \lambda_i \ l_2(X,P_i)^2. \end{aligned}$$

The Lagrangian dual is:

$$\max_{\lambda \ge 0} \ \min_{z\ X} \quad z\left(1 - \sum_{i=1}^{n} \lambda_i\right) + \sum_{i=1}^{n} \lambda_i \ l_2(X,P_i)^2$$

Observe that if $1 - \sum_{i=1}^{n} \lambda_i < 0$, then as $z \to +\infty$, $L(z, x, \lambda) \to -\infty$, and if $1 - \sum_{i=1}^{n} \lambda_i > 0$, then as $z \to -\infty$, $L(z, x, \lambda) \to -\infty$, so that in either case, the dual has no maximum. Thus the Lagrangian dual may be written as:

$$\begin{aligned} &\max \ \min_{X} \ \sum_{i=1}^{n} \lambda_i \ l_2(X,P_i)^2 \\ &\text{s.t.} \ \sum_{i=1}^{n} \lambda_i = 1, \quad \text{and} \quad \lambda \ \ge\ 0. \end{aligned}$$

Since $\sum_{i=1}^{n} \lambda_i \; l_2(X,P_i)^2$ is strictly convex, the minimum occurs if and only if the necessary conditions are met, i.e., $\sum_{i=1}^{n} \lambda_i \; (X - P_i) = 0$, or $X = \sum_{i=1}^{n} \lambda_i P_i$. Thus the minimum can be replaced with the constraint $X = \sum_{i=1}^{n} \lambda_i P_i$.

Therefore the dual is:

$$\max \quad \sum_{i=1}^{n} \lambda_i \; l_2(X,P_i)^2$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \lambda_i = 1,$$

$$X = \sum_{i=1}^{n} \lambda_i P_i ,$$

$$\lambda \geq 0.$$

This dual has the following interpretation. Assume the given points P_i are rigidly interconnected in a weightless lamina. Consider the dual variable λ_i as a weight to be assigned to the points P_i. The center of gravity of the points P_i with weights λ_i is $X = \sum_{i=1}^{n} \lambda_i P_i$. The objective function gives the moment of inertia of this system of points and weights about the center of gravity. The constraint $\sum_{i=1}^{n} \lambda_i = 1$ normalizes the assigned weights to 1. Thus the dual problem is to assign nonnegative weights λ_i to the points P_i in order to maximize the moment of inertia of this system about its center of gravity X.
The Kuhn-Tucker conditions state that the moment of inertia is maximized by assigning positive weight only to the points P_i that are at the maximum distance from the center X.

Another dual of the minimax Euclidean distance problem has a quadratic objective function and linear constraints. Standard quadratic programming approaches may be applied to its solution.

One facility minimax location with weighted Euclidean distance: $1 \,/\, P \,/\, w_i \,/\, l_2 \,/$ max

Given distinct points $P_i = (a_i, b_i)$ in the plane, and positive weights w_i for $i = 1, \ldots n$, the problem is to find a point $X = (x, y)$ that minimizes the maximum weighted Euclidean distance from X to the given points. Let $f(X) = \max_{1 \le i \le n} w_i\, l_2(X,P_i)$. The problem is to minimize $f(X)$, i.e.,

$$\min \max_{1 \le i \le n} w_i\, l_2(X,P_i).$$

For two points P_s and P_t, let $L(P_s,P_t) = \{ X : w_s l_2(X,P_s) = w_t l_2(X,P_t) \}$, that is, $L(P_s,P_t)$ is the set of points whose weighted distance to P_s equals the weighted distance to P_t . If the ratio $r = \frac{w_t}{w_s} = 1$, then $L(P_s,P_t)$ is a straight line, i.e., the perpendicular bisector of the line joining P_s and P_t. If $r \neq 1$, then $L(P_s,P_t)$ is a circle with radius $\frac{r\, l_2(P_s,P_t)}{|1 - r^2|}$, and center $\frac{P_s - r^2 P_t}{1 - r^2}$.

Figure 7 shows three points $P_1 = (0,0)$, $P_2 = (3,0)$, and $P_3 = (0,4)$ with weights $w_1 = 6$, $w_2 = 8$, $w_3 = 3$, and the sets $L(P_1, P_2)$, $L(P_1, P_3)$, and $L(P_2, P_3)$ intersecting at a common point. The common point is the solution to the minimax location problem on the points P_1, P_2, and P_3 with weighted Euclidean distance.

The following two results show how to determine the optimal solution for the 2 and 3 point weighted minimax location problems.

Result 1: For a weighted minimax location problem with two points P_s and P_t, the optimal solution X lies at the intersection of the line between P_s and P_t, and the set $L(P_s,P_t)$.

Result 2: For a weighted minimax location problem with three points P_s, P_t, and P_u, either the optimal solution is determined by one of the pair of points: P_s and P_t, or P_s and P_u, or P_t and P_u, or the optimal solution is determined by all three points in which case X lies at the intersection of $L(P_s,P_t)$, $L(P_s,P_u)$, and $L(P_t,P_u)$.

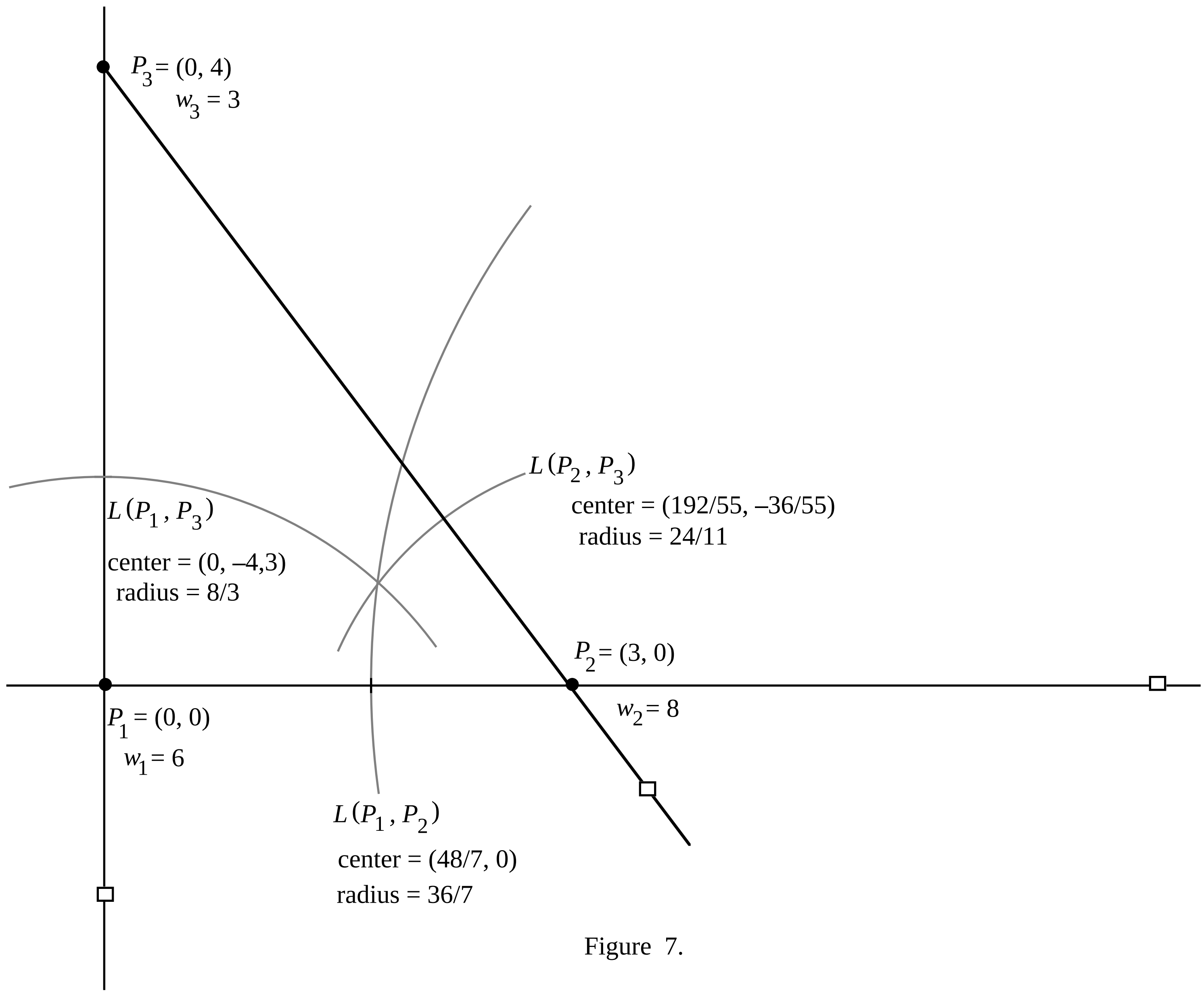
P3 = (0, 4)
w3 = 3
L (P1 , P3)
center = (0, –4,3)
radius = 8/3
L (P2 , P3)
center = (192/55, –36/55)
radius = 24/11
P2 = (3, 0)
w2 = 8
P1 = (0, 0)
w1 = 6
L (P1 , P2)
center = (48/7, 0)
radius = 36/7

Figure 7.

An algorithm for the weighted minimax Euclidean distance problem is given as follows:

1. Choose any two points P_s and P_t. Solve the weighted minimax location problem with P_s and P_t for X and $z = w_s l_2(X,P_s)$ using Result 1.
2. If $w_i\, l_2(X,P_i) \leq z$ for all P_i, stop. Else select a point P_u such that $w_u\, l_2(X,P_u) > z$ and go to 3.
3. Solve the weighted minimax location problem with P_s, P_t, and P_u, for X and z using Result 2.
4. If X and z are determined by two points, call them P_s and P_t and go to 2.
5. Else, X and z are determined by three points. If $w_i\, l_2(X,P_i) \leq z$ for all P_i, stop. Otherwise choose P_v such that $w_v\, l_2(X,P_v) > z$.
6. Using P_s, P_t, P_u and P_v, choose all combinations of two points and solve for X and z using Result 1, and choose all combinations of three points and solve for X and z usint Result 2. If X and z are determined by two points, call them P_s and P_t and go to 2. If X and z are determined by three points, call them P_s, P_t, and P_u and go to 5.

This is a finite algorithm, however in the worst case, the minimax problem must be solved on 4 points $C(n, 4)$ times. Elzinga and Hearn give heuristic improvements and alternate methods.

Drezner and Wesolowsky (1980) give a similar algorithm for the weighted minimax location problem with l_p distances

General Results for minimax location

The following general results are from Francis (1967). The first result gives a lower bound on the objective function value.

Property 2-1: Define $b_{ij} = \frac{w_i w_j}{w_i + w_j} d(P_i,P_j)$ and $b_{st} = \max_{1 \le i < j \le n} b_{ij}$. Then $b_{st} \le f(X^*)$.

Proof: $b_{st} = \frac{w_s w_t}{w_s + w_t} d(P_s,P_t) \le \frac{w_t}{w_s + w_t} w_s d(P_s,X^*) + \frac{w_s}{w_s + w_t} w_t d(X^*,P_t)$

$$\le \frac{w_t}{w_s + w_t} f(X^*) + \frac{w_s}{w_s + w_t} f(X^*) = f(X^*).$$

Corrollary 2-1: The function f equals the lower bound b_{st} at a point X if and only if

(1) $d(P_s,P_t) = d(P_s,X^*) + d(X^*,P_t)$,

(2) $w_s d(P_s,X^*) = w_t d(X^*,P_t)$, and

(3) $w_i d(X^*,P_i) \le b_{st}$ for $i = 1, \ldots, m \quad i \ne s, t$.

Corrollary 2-2: Given the lower bound b_{st}, define $X = \frac{w_s}{(w_s + w_t)} P_s + \frac{w_t}{(w_s + w_t)} P_t$.

If $w_i d(X,P_i) \le b_{st}$ for $i = 1, \ldots, m \quad i \ne s, t$, then X minimizes f.

Proof: X is a convex combination of P_s and P_t so that

$$d(P_s,P_t) = d(P_s,X^*) + d(X^*,P_t).$$

Also, $w_s d(P_s,X^*) = \frac{w_t}{w_s + w_t} w_s d(P_s,P_t) = b_{st}$, and likewise

$$w_t d(X^*,P_t) = b_{st}.$$

Property 2-2: If X^* minimizes f, then there are at least two given points P_i and P_j such that

$$f(X^*) = w_i d(X,P_i) = w_j d(X,P_j).$$

For the Euclidean distance minimax location problem with all $w_i = 1$, observe that the lower bound is not necessarily tight. Consider the given points $P_1 = (5, 0)$, $P_2 = (-3, 4)$, and $P_3 = (-3, -4)$. Then $b_{12} = b_{13} = \frac{1}{2} l_2(P_1,P_2) = \frac{1}{2} l_2(P_1,P_3) = 2\sqrt{5}$ and $b_{23} = \frac{1}{2} l_2(P_2,P_3) = 4$, so that $b_{st} = 2\sqrt{5}$. However, the optimal location is $X^* = (0, 0)$ with $f(X^*) = 5 > b_{st}$.

One facility minimax location with rectangular distance: 1 / P / w_i / l_1 / max

Given distinct points $P_i = (a_i, b_i)$ in the plane, and positive weights w_i for $i = 1, \ldots, n$. The problem is to find a point $X = (x, y)$ that minimizes the maximum weighted rectangular distance from X to the given points. Recall that $l_1(X,P_i) = |x - a_i| + |y - b_i|$.

Let $f(X) = \max_{1 \le i \le n} w_i \, l_1(X,P_i)$. The problem is to minimize $f(X)$, i.e.,

$$\min \max_{1 \le i \le n} w_i \, l_1(X,P_i).$$

The set of points of equal distance z from a given point P_i in R^2 is a "diamond" as shown in Figure 8.

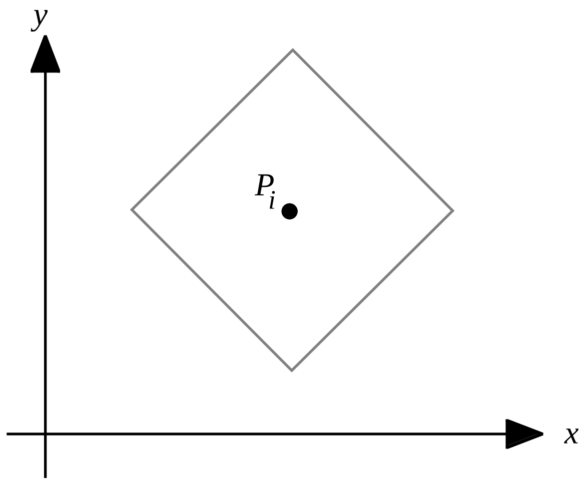

Figure 8.

The approach is to transform the problem with rectangular distances into an equivalent problem using l_∞ distances where $l_\infty(X,P_i) = \max\{\,|x - a_i|, |y - b_i|\,\}$. The set of points of equal l_∞ distance from a given point P_i in R^2 is a square as shown in Figure 9.

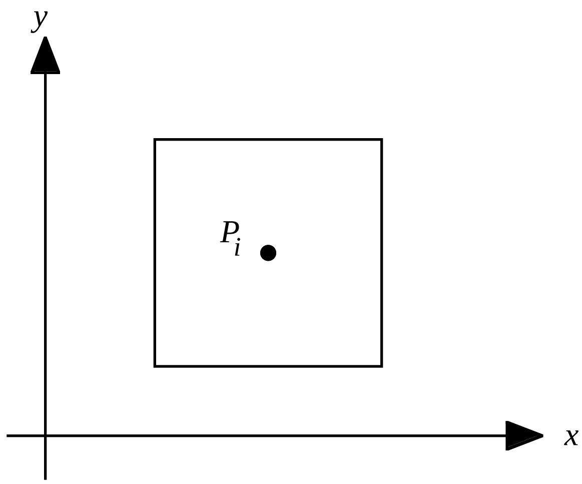

Figure 9.

Consider a transformation T that rotates the coordinate axes clockwise through 45 degrees. The transformation T is given by the nonsingular matrix $T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Property 2-3: $l_1(X,P_i) = \sqrt{2}\; l_\infty\big(T(X),T(P_i)\big)$.

Proof: $\sqrt{2}\; l_\infty\big(T(X),T(P_i)\big) = \sqrt{2}\; \max\{ \frac{1}{\sqrt{2}} \,|\, x+y-a-b \,|, \frac{1}{\sqrt{2}} \,|\, -x+y+a-b \,| \}$

$= \max\{ (x+y-a-b),\; (-x-y+a+b),\; (-x+y+a-b),\; (x-y-a+b) \}$

$= \max\{ (x-a+y-b),\; (x-a-y+b),\; (-x+a+y-b),\; (-x+a-y+b) \}$

$= |\, x-a \,| + |\, y-b \,| = l_1(X, P_i)$.

Property 2-4: X is an optimal solution to $\min \max_{1 \le i \le n} w_i\, l_1(X,P_i)$ with objective function value z if and only if $T(X)$ is an optimal solution to $\min \max_{1 \le i \le n} w_i\, l_\infty\big(T(X),T(P_i)\big)$ with objective function value $\sqrt{2}\; z$.

The approach is to solve the problem $\min \max_{1 \le i \le n} w_i\, l_\infty\big(T(X),T(P_i)\big)$.

Let $T(X) = X' = (x', y')$ and $T(P_i) = P'_i = (a'_i, b'_i)$.

Then the problem may be written as

$$\min \max_{1 \le i \le n} w_i \, (\max \{ |x' - a'_i|, |y' - b'_i| \})$$

or $$\min \max \{ \max_{1 \le i \le n} w_i |x' - a'_i|, \max_{1 \le i \le n} w_i |y' - b'_i| \}$$

from which two subproblems may be defined in the variables x' and y' respectively.

P(x'): $\min \max_{1 \le i \le n} w_i |x' - a'_i|$, and P($y'$): $\min \max_{1 \le i \le n} w_i |y' - b'_i|$

<u>Property 2-5:</u> If x' is an optimal solution to subproblem P(x') with objective function value z'_x and if y' is an optimal solution to subproblem P(y') with objective function value z'_y, then $X' = (x', y')$ is an optimal solution to $\min \max_{1 \le i \le n} w_i \, l_\infty\big(T(X), T(P_i)\big)$ with objective function value $\max \, (z'_x, z'_y)$.

How to solve P(x'): Write the equivalent constrained problem:

$$\begin{aligned} &\min \; z'_x \\ &\text{s.t.} \quad |x - a'_i| \le \frac{z'_x}{w_i} \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

and the equivalent linear programming problem:

$$\begin{aligned} &\min \; z'_x \\ &\text{s.t.} \quad x - a'_i \le \frac{z'_x}{w_i} \quad \text{for } i = 1, \ldots, n. \\ &\quad\;\; - x + a'_i \le \frac{z'_x}{w_i} \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

From Property 2-1, a lower bound is given by

$$\max_{1 \le i < j \le n} \frac{w_i w_j}{w_i + w_j} | a'_i - a'_j | = \frac{w_s w_t}{w_s + w_t} | a'_s - a'_t | \text{ for some } s \text{ and } t.$$

Using properties of the linear program, this lower bound may be shown to be tight for the minimax location problem P(x'). Thus the optimal solution has

$$z'_x = \frac{w_s w_t}{w_s + w_t} \mid a'_s - a'_t \mid \quad \text{and} \quad x' = \frac{w_s a'_s + w_t a'_t}{w_s + w_t}.$$

For the subproblem P(y'): $\min \max_{1 \le i \le n} w_i \mid y' - b'_i \mid$ a lower bound is given by

$$\max_{1 \le i < j \le n} \frac{w_i w_j}{w_i + w_j} \mid b'_i - b'_j \mid = \frac{w_p w_q}{w_p + w_q} \mid b'_p - b'_q \mid \text{ for some } p \text{ and } q,$$

which is tight. Thus the optimal solution has

$$z'_y = \frac{w_p w_q}{w_p + w_q} \mid b'_p - b'_q \mid \quad \text{and} \quad y' = \frac{w_p b'_p + w_q b'_q}{w_p + w_q}.$$

Then an optimal solution to the problem: $\min \max_{1 \le i \le n} w_i\, l_\infty(X', P'_i)$
is $X' = (x', y')$ and $z' = \max(z'_x, z'_y)$.

If $z'_x = z'_y$, then $X' = (x', y')$ is the unique solution.
If $z'_x > z'_y$, then all (x', y') such that $w_i \mid y' - b'_i \mid \le z'_x$ are alternative optimal solutions.
The inequality is equivalent to $b'_i - z'_x / w_i \le y' \le b'_i + z'_x / w_i$ for all $i = 1, \ldots, n$,
which is equivalent to $\max_i \{ b'_i - z'_x / w_i \} \le y' \le \min_i \{ b'_i + z'_x / w_i \}$.
This shows that the alternative solutions are given by an interval in y'.
If $z'_y > z'_x$, then all (x', y') such that $w_i \mid x' - a'_i \mid \le z'_y$ are alternate optimal solutions,
which can be expressed as an interval in x' similar to the above.

Figure 10 shows an example with three points and all weights equal 1. The optimal solution is determined by P_1 and P_2. Alternative solutions are indicated by the vertical line segment adjacent to X.

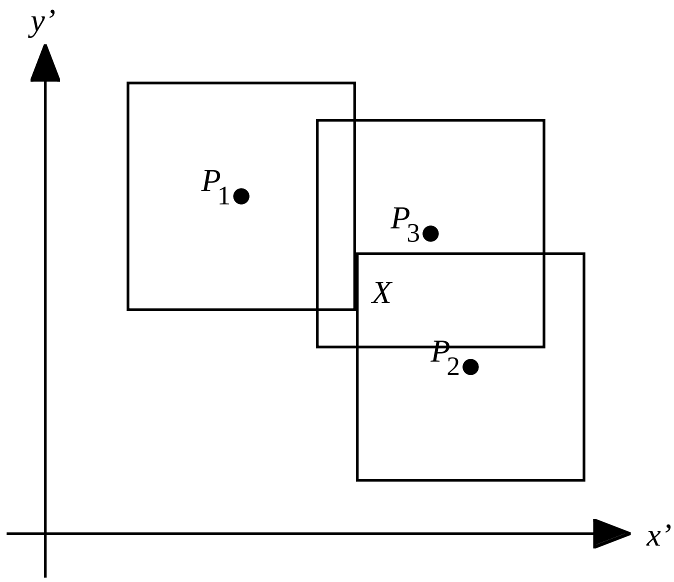

Figure 10.

Finally, optimal solutions to $\min \max_{1 \le i \le n} w_i \, l_1(X,P_i)$ are given by $X = T^{-1}(X')$ and $z = \frac{1}{\sqrt{2}} z'$.

An alternative approach for the rectangular distance minimax location problem is given as follows. Property 2-1 shows that the expression

$$\max_{1 \le i < j \le n} \frac{w_i w_j}{w_i + w_j} l_1(P_i,P_j) = \frac{w_s w_t}{w_s + w_t} l_1(P_s,P_t) = b_{st} \text{ for some } s \text{ and } t$$

is a lower bound for the objective function value. This bound may be shown to be tight, and one optimal solution X is given by $X = \frac{w_s P_s + w_t P_t}{w_s + w_t}$. Alternative solutions are given by the set $\{ X : w_i \, l_1(X, P_i) \le b_{st} \}$. Explicit expressions may be given that determine and interval of alternative solutions.

Multifacility minimax location with rectangular distance: $M / P / w_i / l_1 / \max$

The problem is to locate several new facilities with respect to a given set of existing facilities and with respect to other new facilities, so as to minimize the maximum weighted distance between pairs of new facilities or between pairs of new and existing facilities.
Let $P_i = (a_i, b_i)$ $i = 1, \ldots, n$ be given points in R^n. Let X_j, $j = 1, \ldots, m$ denote the m new facilities to be located.

Let w_{ji} be a nonnegative weight associated with the distance between each X_j and P_i for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let v_{jk} be a nonnegative weight associated with the distance between each X_j and X_k for $1 \leq j < k \leq m$. Then the multifaciliity minimax location problem with rectangular distance can be stated as:

$$\min_{X_1 \ldots X_m} \quad \max \left\{ \max_{1 \leq j < k \leq m} v_{jk}\, l_1(X_j, X_k), \ \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} w_{ji}\, l_1(X, P_i) \right\}.$$

Thus each of the m new facilities is to be located with respect to the n existing facilities and also with respect to the other new facilities. The location of X_j may depend on the location of some point X_k because of the terms involving v_{jk}.

New facility locations X_j and X_k are said to be *linked* if v_{jk} is positive and *not linked* if v_{jk} is zero. It is assumed that each new facility location X_j is linked with at least one other new facility location, otherwise the location of X_j could be determined independently of the other new facility locations by considering a separate problem.

New facility location X_j and existing facility location P_i are said to be l*inked* if w_{ji} is positive and *not linked* if w_{ji} is zero. If a new facility X_j is not linked to any existing facility, then it must be linked to some new facility that is linked to some existing facility. Otherwise, the set of all new facilities that are not linked to any existing facility can be located at a common point anywhere. Henceforth, we assume the multifacility location problem is well formulated with respect to facilities being linked to one another. These assumptions imply that there exist an optimal solution. For the convenience of the presentation, we assume all the w_{ji} and all the v_{jk} are positive.

The transformation T is applied to the multifacility problem to obtain the following equivalent problem:

$$\min_{X'_1 \ldots X'_m} \max \left\{ \max_{1 \le j < k \le m} v_{jk}\, l_\infty(X'_j, X'_k),\ \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} w_{ji}\, l_\infty(X', P'_i) \right\}.$$

Thus the one dimensional multifacility minimax location problems in x' and y' may be considered independently. The subproblem in x' is written as:

$$\min_{x'_1 \ldots x'_m} \max \left\{ \max_{1 \le j < k \le m} v_{jk}\, | x'_j - x'_k |,\ \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} w_{ji}\, | x' - a'_i | \right\}.$$

Each one dimensional subproblem may be formulated as a linear programming problem. For convenience, a dual variable is written adjacent to each constraint.

		dual variables
$\min\ z'$		
s.t. $x'_j - x'_k + \dfrac{z'}{v_{jk}} \ge 0$	$1 \le j < k \le m$	f_{jk}
$-x'_j + x'_k + \dfrac{z'}{v_{jk}} \ge 0$	$1 \le j < k \le m$	f_{kj}
$x'_j + \dfrac{z'}{w_{ji}} \ge a'_i$	$1 \le j \le m,\ 1 \le i \le n$	f_{ijt}
$-x'_j + \dfrac{z'}{w_{ji}} \ge -a'_i$	$1 \le j \le m,\ 1 \le i \le n$	f_{sij}

Then the dual is written as follows:

$$\max \quad \sum_{j=1}^{m} \sum_{i=1}^{n} a'_i f_{ijt} - \sum_{j=1}^{m} \sum_{i=1}^{n} a'_i f_{sij} \tag{1}$$

$$\text{s.t.} \quad \sum_{k=1\ k \ne j}^{m} f_{jk} - \sum_{k=1\ k \ne j}^{m} f_{kj} + \sum_{i=1}^{n} f_{ijt} - \sum_{i=1}^{n} f_{sij} = 0 \qquad 1 \le j \le m, \tag{2}$$

$$\sum_{j=1}^{m} \sum_{k>j}^{m} f_{jk}/v_{jk} + \sum_{j=1}^{m} \sum_{k>j}^{m} f_{kj}/v_{jk} + \sum_{j=1}^{m} \sum_{i=1}^{n} f_{ijt}/w_{ji} + \sum_{j=1}^{m} \sum_{i=1}^{n} f_{sji}/w_{ji} = 1 \tag{3}$$

all variables nonnegative.

Add the two redundant constraints and the variable v:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f_{sji} = v \tag{4}$$

$$-\sum_{j=1}^{m} \sum_{i=1}^{n} f_{ijt} = -v. \tag{5}$$

Constraints (1), (2), (4), and (5) define a network flow problem. The set of nodes is $\{s, 1, . . . , m, t\}$. The nodes $1, . . . , m$ constitute a complete network with directed arcs (j, k) with flow f_{jk} and cost 0 for all $j \neq k$. There are n parallel arcs from node s to each node j with flow f_{sij}, and cost $-a'_i$. There are n parallel arcs from node each node j to node t with flow f_{ijt}, and cost a'_i. There is an arc (t, s) with flow v.

Constraints (2) are conservation of flow constraints for nodes $j = 1, . . . , m$. Constraint (4) requires conservation of flow for node s, and constraint (5) requires conservation of flow for node t.

Constraint (3) multiplies the flow on each arc by a weight of either $1/v_{jk}$ or $1/w_{ji}$ and restricts the total weighted flow to equal 1.

The objective is to maximize the total cost of flow cycling through the network.

Figure 11 illustrates a network with $m = 3$ facilities to be located, and $n = 2$ existing faciliites. Adjacent to each arc is the arc cost.

Let S be a simple path from node s to node t. The conservation of flow constraints imply that there is a constant flow, of say f, on path S. Let $W(S)$ be the total of all arc weights on the path S, that is,

$$W(S) = \sum_{kj \in S} 1/v_{jk} + \sum_{ji \in S} 1/w_{ji} .$$

Constraint (3) implies that $W(S)f = 1$, so that $f = 1/W(S)$. Let $C(S)$ be the total of all arc cost on the path S, then the objective function value is $C(S)f = C(S)/W(S)$. Therefore, the objective is to find the path S of maximum cost to weight ratio $C(S)/W(S)$.

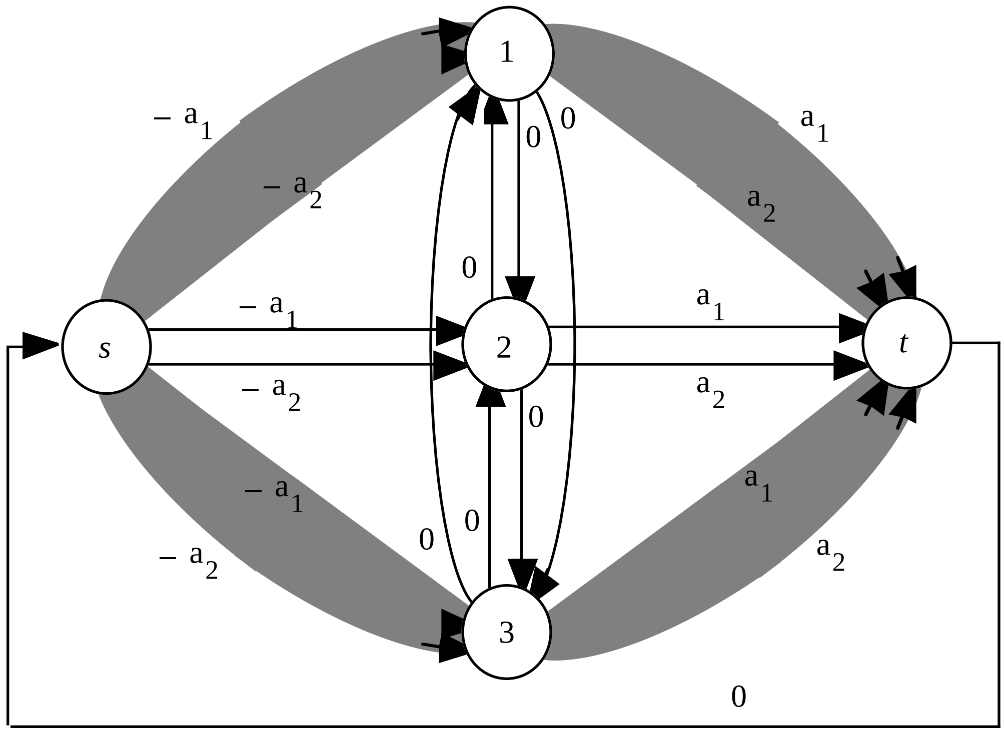

Figure 11.

The problem is solved by forming the Lagrangian with respect to constraint (3). Letting λ be the Lagrange multiplier for constraint (3), the Lagrangian problem adjusts the arc costs by subtracting λ times the arc weight from the arc cost of each arc, and asks for a simple path of maximum total adjusted arc cost. For a simple path S from s to t, the objective function of the Lagrangian gives the ratio $C(S)/W(S)$ over the path S.

Finding the simple path of maximum cost to weight ratio is accomplished as follows:
Set $\lambda_1 = 0$, and initiate with $i = 1$. Adjust the arc costs of the Lagrangian by λ_i and find the maximum cost simple path, S_i from s to t. Set $\lambda_{i+1} = C(S_i)/W(S_i)$, and continue until there is no improvement. It may be shown that $\lambda_{i+1} > \lambda_i$ at each step until termination, and that the algorighm stops after a finite number of iterations. See Dearing and Francis (1974).

The dual variable values (node labels) from the network solution give the optimal values of the locations: x'_j $j = 1, \ldots, n$, and the optimal objective function value $\lambda^* = z'_x$.

The subproblem in y'_j is solved in a similar fashion for y'_j $j = 1, \ldots, n$, and the optimal objective function value z'_y.

Then the solution to the transformed problem is $X'_j = (x'_j, y'_j)$ for $j = 1, \ldots, n$, and the optimal objective function value $z' = \max(z'_x, z'_y)$.
The solution to the original problem is obtained by the inverse transformation T^{-1}.

Alternatively, the constrained multifacility problem in R^2 with l_1 distances may be formulated directly as a linear programming problem with $4n + 4m$ constraints and $2m + 1$ variables. This is well within the capacity of modern LP solvers.

Multifacility minimax location with Euclidean distance: $M\ /\ P\ /\ w_i\ /\ l_2\ /\ \max$

The problem is to locate several new facilities with respect to a given set of existing facilities and with respect to other new facilities, so as to minimize the maximum weighted distance between pairs of new facilities or between pairs of new and existing facilities. Let $P_i = (a_i, b_i)$ $i = 1, \ldots, n$ be given points in R^n. Let X_j, $j = 1, \ldots, m$ denote the m new facilities to be located.

Let w_{ji} be a nonnegative weight associated with the distance between each X_j and P_i for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let v_{jk} be a nonnegative weight associated with the distance between each X_j and X_k for $1 \le j < k \le m$. Then the multifaciliity minimax location problem with Euclidean distance can be stated as:

$$\min_{X_1 \ldots X_m} \max \left\{ \max_{1 \le j < k \le m} v_{jk}\, l_2(X_j, X_k),\ \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} w_{ji}\, l_2(X, P_i) \right\}.$$

Thus each of the m new facilities is to be located with respect to the n existing facilities and also with respect to the other new facilities. The location of X_j may depend on the location of some point X_k because of the terms involving v_{jk}.

We assume the multifacility location problem is well formulated with respect to facilities being linked to one another. These assumptions imply that there exist an optimal solution. Without loss of generality, we assume all the w_{ji} and all the v_{jk} are positive.

The problem is equivalent to the problem with weighted distances squared:

$$\min_{X_1 \ldots X_m} \max \left\{ \max_{1 \le j < k \le m} v^2_{jk}\, l_2(X_j, X_k)^2,\ \max_{\substack{1 \le i \le n \\ 1 \le j \le m}} w^2_{ji}\, l_2(X_j, P_i)^2 \right\}.$$

and to the constrained problem:

$$\begin{aligned} &\min\ z \\ &z \ge v^2_{jk}\, l_2(X_j, X_k)^2 && 1 \le j < k \le m \\ &z \ge w^2_{ji}\, l_2(X, P_i)^2 && 1 \le i \le n,\quad 1 \le j \le m \end{aligned}$$

As in the one facility model, write the Lagrangian of the constrained problem:

$$L(z, X, \lambda, \gamma) = z - \sum_{1\le<j\le m} \lambda_{,jk} \left(z - v^2_{jk}\, l_2(X_j, X_k)^2\right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij}\left(z - w^2_{ji}\, l_2(X_j,P_i)^2\right)$$

then the Lagrangian dual is: $\max_{\lambda\ \gamma} \min_{z\ X_j} L(z, X, \lambda, \gamma)$.

The necessary conditions are:

$$\frac{\partial}{\partial z} L(z, X, \lambda, \gamma) = 0 \text{ implies } 1 = \sum_{1\le<j\le m} \lambda_{,jk} + \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \qquad (1)$$

These conditions eliminate the terms involving z from the Lagrangian.

$$\frac{\partial}{\partial X} L(z, X, \lambda, \gamma) = 0 \text{ implies } \sum_{1\le<j\le m} \lambda_{,jk}\, v^2_{jk}\, (X_j - X_k) + \sum_{j=1}^{m} \gamma_{ij}\, w^2_{ji}\, (X_j - P_i) = 0 \qquad (2)$$

$$\frac{\partial}{\partial \lambda} L(z, X, \lambda, \gamma) = 0 \text{ implies } z \ge v^2_{jk}\, l_2(X_j, X_k)^2 \quad 1 \le j < k \le m \qquad (3)$$

$$\frac{\partial}{\partial \gamma} L(z, X, \lambda, \gamma) = 0 \text{ implies } z \ge w^2_{ji}\, l_2(X,P_i)^2 \quad 1\le i\le n, \quad 1\le j\le m \qquad (4)$$

Observe that equations (2) are a generalization of the necessary conditions for the squared Euclidean distance problem whose solution is the center of gravity. Given values of λ and γ, equations (2) may be solved for X by a system of equations. If λ and γ are the optimal values of the dual multipliers, then the solution X is the optimal location.

This leads to an iterative procedure for estimating λ and γ.

1. Set $t = 0$. Set initial values of $\lambda(t)_{jk}$ and $\gamma(t)_{ij}$ to 1.

2. Solve for $X(t)_j$ using equations (2) and $\lambda(t)_{jk}$ and $\gamma(t)_{ij}$.

3. Set $\lambda(t+1)_{jk} = \lambda(t)_{jk} w_{ji}\, l_2(X(t)_j, P_{\mathrm{i}})/U$ and $\gamma(t+1)_{ij} = \gamma(t)_{ij}\, v_{jk}\, l_2(X(t)_j, X(t)_k)/U$,

where $U = \sum_{1\le<j\le m} \lambda(t)_{jk}\, v_{jk}\, l_2(X(t)_j, X(t)_k) - \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma(t)_{ij}\, w_{ji}\, l_2(X(t)_j, P_i)$.

4. Find $z(t)$ as the objective function value using $X(t)_j$ $j = 1, \ldots, m$.
5. If a stopping criteria on $z(t)$ is met, stop, else, set $t = t + 1$, and go to 2.

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Problems

1. Suppose four given point are located at (0,0), (0,10), (5,0), (12,6) and all w_i are equal.
    a. Find the gravity solution.
    b. Use the gravity solution to intiate the iterative method and do 4 iterations.
    c. Verify the solution by the illustration on page 12.