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Static analysis of a thick ring on a unilateral elastic foundation

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ABSTRACT

A thick ring on a unilateral elastic foundation can be used to model important applications such as non-pneumatic tires or bushing bearings. This paper presents a reduced-order compensation scheme for computing the static deformation response of a thick ring supported by a unilateral elastic foundation to an arbitrary applied force. The ring considered is an orthotropic and extensible ring that can be treated as a Timoshenko beam. The elastic foundation is a two-parameter foundation with a linear torsional stiffness but a unilateral radial stiffness whose value vanishes when compressed or tensioned. The paper first derives the deformation response for the linear foundation case for which Fourier expansion techniques can be applied to obtain an analytical solution. Then, the nonlinear unilateral foundation problem is solved via an iterative compensation scheme that identifies regions with vanishing radial stiffness and applies a compensation force to the linear foundation model to counteract the excessive foundation forces that would not be there with the unilateral foundation. This scheme avoids the need for solving the complex set of nonlinear differential equations and gives a computationally efficient tool for rapidly analyzing and designing such systems. Representative results are compared with Finite Element Analysis (FEA) results to illustrate the validity of the proposed approach.

1. Introduction

The flexible Ring on Elastic Foundation (REF) model [1,2] is a classical one that has been studied for decades. This is because of its broad and important applications such as automotive tires [3–5], railway wheels and gears [6], and others [7].

Different criteria can be used to classify the focus of existing works that analyze REFs. The simplest categories are the treatment of the static deformation problem [7,8] vs. the dynamic problem as free vibration [6] or forced vibration [2,9]. Considering the ring mechanics, the ring has been treated as a tensioned string that has direct tensile strain but no bending stiffness [10]; as an Euler–Bernoulli beam or thin ring whose plane section remains plane and always normal to the neutral axis of the ring after deformation [7,5,9]; or as a Timoshenko beam or thick ring [11] which takes the shear deformation into account by assuming that the normal of a plane section is subject to rotation. Further distinctions exist between extensional and inextensional rings. [12,1] studied the vibration problem for both a rotating thin ring and a thick ring, and pointed out that the inextensional assumptions in thick ring theories are improper because extensional coupling effects are as important as shearing effects especially for a rotating ring.

As an important component of the REF model, the treatment of the elastic foundation can be used as another criterion to classify the existing research. Numerous works, including all of the ones cited above, assume a linear and uniformly distributed stiffness for the whole elastic foundation, independent of location and deformation state. The distributed stiffness can be modeled with one parameter [7,10] where the foundation has a stiffness only in the radial direction; or with two parameters [2,9] involving both radial and torsional stiffness values; or even more parameters [6], where in addition to the radial and torsional stiffness, a stiffness associated with the distortion of the foundation due to in-plane rotation of a cross-section of the ring is included.

Although much has been gained from linear and uniform elastic foundation assumptions, not all REF problems have a perfect linear elastic foundation with uniformly distributed stiffness. Examples for nonuniform distribution include planetary gearing where tooth meshes for the ring and planets are not equally spaced [13] and tires with non-uniformity [14]. For these type of problems, [13] studied the free vibration of rings on a general elastic foundation, whose stiffness distribution can be variable circumferentially in the radial, tangential, or inclined orientation, and gave the closed-form expression for natural frequencies and vibration modes. [14] studied the natural frequencies and mode

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shapes of rings supported by a number of radial spring elements with a constant radial stiffness using modal expansion and receptance method. However, in both works, the distributions of the stiffnesses for the elastic foundations still did not change with the deformation of the ring resting on them.

A more complicated problem was invoked by considering deformation-dependent elastic foundations, such as those with unilateral stiffness whose values vanish when compressed or tensioned. The difficulty in solving this group of problems is in the fact that the compressed or tensioned region is not known in advance. It depends on the loading and consequently on the deformation. An example for the application is the non-pneumatic tire presented in [15], whose structure is a deformable shear ring supported by collapsible spokes which offer stiffness only in tension [16]. Another example is a bushing bearing, whose external sleeve can be modeled as a ring on a tensionless foundation and the internal sleeve as a ring on a collapsible foundation. Not too many works exist that deal with such rings on unilateral foundations. [17] worked on the forced response of a thin and inextensible ring on a tensionless two-parameter foundation under a time varying in-plane load. To solve the unilateral problem, an auxiliary function is used in the coefficients of the equations to track and reflect the status of the foundation. This auxiliary function makes the differential equations of the system nonlinear and difficult to solve. Furthermore, the tangential displacement of the ring is obtained from the inextensible assumption, which cannot be adopted in a more general case, such as that with extensible Timoshenko ring. [18] studied the static deformation and the contact pressure of a non-pneumatic tire resting on collapsible spokes, when it contacts against a rigid plane ground. The governing differential equations were derived only for the thick ring modeled via Timoshenko beam theory by treating the supporting force by collapsible spokes as radial distributed force which vanishes in collapsed spoke regions. The ring was divided into three regions according to the post-deformation status of the spokes (tensioned or buckled spokes) and contact status with the ground (contact region or free region). Closed-form expressions for the deformation and contact stress are given in terms of angular bounds of these three regions, which then need to be solved numerically. However, the method has limitations in two aspects: (1) For more complex loading cases, such as that with multiple forces applied at multiple locations, the number of regions into which the ring must be divided grows with the number of the load regions. Multiple unknown angular bounds would then need to be solved numerically. (2) It is difficult to extend this method to practical dynamic cases.

The present paper studies the deformation of a thick ring on an elastic two-parameter foundation where the radial stiffness is unilateral. The deformation response to an arbitrary in-plane force is considered. The ring is modeled as an orthotropic and extensible circular Timoshenko beam. As the first step, the linear foundation problem is solved analytically using Fourier expansion techniques for both the radial and tangential directions. It is then shown that the linear foundation case includes an excessive foundation force compared with that of the unilateral foundation. An iterative compensation scheme is then set up to both find the region of vanishing radial stiffness with the unilateral foundation and that of the required compensation force to counteract the excessive force predicted via the linear foundation. The method is an intuitive and efficient alternative to numerically solving the coupled and complex system of nonlinear differential equations for a flexible ring on a unilateral foundation. In addition, compared with discretization-based numerical methods such as nonlinear FEA, the proposed scheme avoids the time-consuming modeling and meshing work, which makes it attractive specially for rapid parameter studies at the design stage. Compared with the method in [18], the method proposed in this paper is capable of handling arbitrary force distributions and directions, without increasing complexity. Furthermore, since the proposed scheme is Fourier expansion-based, it can be easily extended to the dynamic cases (both forced response and dynamic contact) as we illustrate in our other work [19].

The rest of the paper is organized as follows. Section 2 restates the problem and gives the governing equations. Section 3 gives the analytical solutions for the linear foundation problem and extends them to the unilateral case. Then, in Section 4, discussions of some example results are given and compared with FEA results. Conclusions and future work are given in Section 5.

2. Statement of problem and governing equations

Fig. 1 shows a schematic of the model for a thick ring on a two-parameter elastic foundation. The ring with thickness h is assumed to have a radius R at its centroid. The width perpendicular to the plane of the ring is b. The uniformly distributed radial and tangential stiffnesses are assumed to be K_r and K_θ, respectively. They have units of stiffness per radian. For a linear elastic foundation, the distributed radial stiffness K_r is constant. However, for the unilateral elastic foundation, the radial stiffness vanishes when the elastic foundation is tensioned or compressed. A polar coordinate system with origin located at the ring center is adopted. The center of the ring is fixed and friction is neglected.

The radial and tangential displacements at the centroid are assumed to be $u_r(R,\theta)$ and $u_\theta(R,\theta)$, respectively. Following [20], the cross-section of the ring is assumed to have a rotation $\phi(R,\theta)$ at the centroid at circumferential position θ and keeps its plane after deformation. Then, the radial and tangential displacements at an arbitrary point on the ring with radius r and circumferential position θ, $u_r(r,\theta)$ and $u_\theta(r,\theta)$, can be represented by

$$\begin{aligned} u_r(r,\theta) &= u_r(R,\theta) \\ u_\theta(r,\theta) &= u_\theta(R,\theta) + (r-R)\phi(R,\theta) \end{aligned} \tag{1}$$

The strain–displacement relationships in polar coordinates are [21]

$$\begin{aligned} \epsilon_{rr}(r,\theta) &= \frac{\partial}{\partial r}u_r(r,\theta) \\ \epsilon_{\theta\theta}(r,\theta) &= \frac{1}{r}\frac{\partial}{\partial\theta}u_\theta(r,\theta) + \frac{1}{r}u_r(r,\theta) \\ \gamma_{r\theta}(r,\theta) &= \frac{1}{r}\frac{\partial}{\partial\theta}u_r(r,\theta) + \frac{\partial}{\partial r}u_\theta(r,\theta) - \frac{1}{r}u_\theta(r,\theta) \end{aligned} \tag{2}$$

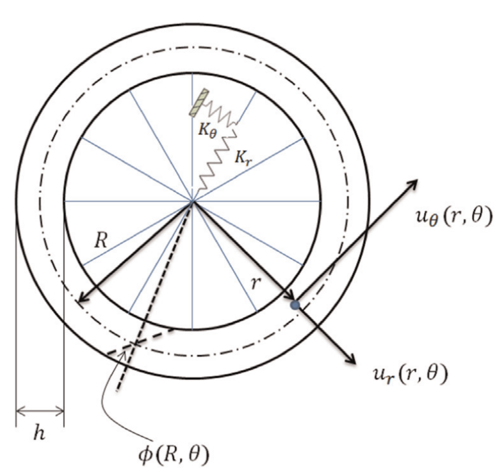

Fig. 1. Thick ring on a two-parameter elastic foundation.

where $\epsilon_{rr}(r, \theta)$, $\epsilon_{\theta\theta}(r, \theta)$, $\gamma_{r\theta}(r, \theta)$ are the radial, tangential and shear strains, respectively.

The ring is assumed to be orthotropic and homogeneous. We consider that the ring material axes coincide with the polar coordinate system adopted. The stress–strain relationships are [21]

$$\sigma_{rr}(r, \theta) = \frac{E_r\left(\nu_{\theta r}\, \epsilon_{\theta\theta}(r, \theta) + \epsilon_{rr}(r, \theta)\right)}{-\nu_{r\theta}\, \nu_{\theta r} + 1}$$
$$\sigma_{\theta\theta}(r, \theta) = \frac{E_\theta\left(\nu_{r\theta}\, \epsilon_{rr}(r, \theta) + \epsilon_{\theta\theta}(r, \theta)\right)}{-\nu_{r\theta}\, \nu_{\theta r} + 1}$$
$$\tau_{r\theta}(r, \theta) = G\gamma_{r\theta}(r, \theta) \tag{3}$$

where $\sigma_{rr}(r, \theta)$, $\sigma_{\theta\theta}(r, \theta)$ and $\tau_{r\theta}(r, \theta)$ are radial, tangential and shear stresses, respectively. E_r, E_θ and G are the elastic moduli in the radial and tangential directions and the shear modulus, respectively. ν is the Poisson's ratio.

The strain energy stored in the ring is given by

$$U_1 = \frac{b}{2}\int_{-\pi}^{\pi}\int_{R-h/2}^{R+h/2} (\sigma_{rr}\epsilon_{rr} + \sigma_{\theta\theta}\epsilon_{\theta\theta} + \tau_{r\theta}\gamma_{r\theta})\, r\, \mathrm{d}r\, \mathrm{d}\theta \tag{4}$$

The strain energy stored in the elastic foundation is given by

$$U_2 = \frac{1}{2}\int_{-\pi}^{\pi}\left(K_r\left(u_r\left(R - \frac{h}{2}, \theta\right)\right)^2 + K_\theta\left(u_\theta\left(R - \frac{h}{2}, \theta\right)\right)^2\right)\mathrm{d}\theta \tag{5}$$

Note here that the radial and tangential displacements for internal edge of the ring $u_r(R - h/2, \theta)$ and $u_\theta(R - h/2, \theta)$ couple the ring and the elastic foundation.

The work by the applied forces is obtained from

$$W = b\int_{-\pi}^{\pi}\left(q_r u_r\left(R + \frac{h}{2}, \theta\right) + q_\theta u_\theta\left(R + \frac{h}{2}, \theta\right)\right)\left(R + \frac{h}{2}\right)\mathrm{d}\theta \tag{6}$$

where $q_r = q_r(R + h/2, \theta)$ and $q_\theta = q_\theta(R + h/2, \theta)$ are distributed forces applied to the external edge of the ring (at the radial location $R + h/2$) in the radial and tangential directions, respectively. The units of q_r and q_θ are in *Force*/*Area*.

Invoking the principle of virtual work [22]:

$$\delta(U_1 + U_2) = \delta W \tag{7}$$

After substitution of (1)–(6) into (7) and some manipulations according to Euler–Lagrange equation, the governing equations for the present thick ring on an elastic foundation problem can be derived. Here, for brevity, $\nu_{r\theta}$ and $\nu_{\theta r}$ are set to zero for the governing equations given as (8). However, the proposed approach will hold for the general case where these are not zero:

$$-\frac{GA}{b}\frac{\partial}{\partial\theta}\phi - \frac{GA}{Rb}\frac{\partial^2}{\partial\theta^2}u_r + \left(\frac{EA_\theta}{Rb} + \frac{GA}{Rb}\right)\frac{\partial}{\partial\theta}u_\theta$$
$$+ \left(\frac{EA_\theta}{Rb} + \frac{K_r}{b}\right)u_r = \left(R + \frac{h}{2}\right)q_r$$
$$\left(-\frac{GA}{b} - \frac{1}{2}\frac{K_\theta\, h}{b}\right)\phi - \frac{EA_\theta}{Rb}\frac{\partial^2}{\partial\theta^2}u_\theta + \left(-\frac{EA_\theta}{Rb} - \frac{GA}{Rb}\right)\frac{\partial}{\partial\theta}u_r$$
$$+ \left(\frac{GA}{Rb} + \frac{K_\theta}{b}\right)u_\theta = \left(R + \frac{h}{2}\right)q_\theta$$
$$\left(\frac{GA\, R}{b} + \frac{1}{4}\frac{K_\theta\, h^2}{b}\right)\phi - \frac{EI_\theta}{Rb}\frac{\partial^2}{\partial\theta^2}\phi + \frac{GA}{b}\frac{\partial}{\partial\theta}u_r$$
$$+ \left(-\frac{GA}{b} - \frac{1}{2}\frac{K_\theta\, h}{b}\right)u_\theta = \frac{1}{2}\left(R + \frac{h}{2}\right)hq_\theta \tag{8}$$

where the following notations are used:

$$u_r = u_r(R, \theta)$$
$$u_\theta = u_\theta(R, \theta)$$
$$\phi = \phi(R, \theta)$$
$$q_r = q_r\left(R + \frac{h}{2}, \theta\right)$$
$$q_\theta = q_\theta\left(R + \frac{h}{2}, \theta\right) \tag{9}$$

and

$$EA_\theta = E_\theta \times A$$
$$GA = G \times A$$
$$EI_\theta = E_\theta \times I \tag{10}$$

where $A = b \times h$ is the cross-sectional area of the ring and $I = (1/12)bh^3$ is the area moment of inertia of the cross-section. The following approximations are used in the manipulations to obtain the governing equations above, considering that $R \gg h$:

$$\int_{R-h/2}^{R+h/2}\frac{1}{r}\,\mathrm{d}r \doteq \frac{1}{R}\int_{R-h/2}^{R+h/2}\mathrm{d}r = \frac{h}{R}$$
$$\int_{R-h/2}^{R+h/2}\frac{(r-R)}{r}\,\mathrm{d}r \doteq \frac{1}{R}\int_{R-h/2}^{R+h/2}(r-R)\,\mathrm{d}r = 0$$
$$\int_{R-h/2}^{R+h/2}\frac{(r-R)^2}{r}\,\mathrm{d}r \doteq \frac{1}{R}\int_{R-h/2}^{R+h/2}(r-R)^2\,\mathrm{d}r = \frac{1}{12}\frac{h^3}{R} \tag{11}$$

3. Method of solution

In this section, the governing equations (8) are first solved analytically by considering a linear foundation with constant circumferential distribution of the foundation stiffnesses. The approach we use is to first get Fourier series expansion of the applied arbitrary force, and then the governing equations are solved harmonic-wise in both the radial and tangential directions. The solution for the total displacements will then be the superposition of the harmonic contributions in both directions. Then the solutions are extended to the unilateral foundation case via the iterative compensation scheme proposed in this work.

3.1. Solution for linear elastic foundation

An arbitrary circumferentially distributed (with respect to θ) or concentrated force $F(\theta)$ can be decomposed into its radial and tangential components as

$$\vec{F}(\theta) = F_r(\theta)\vec{r} + F_\theta(\theta)\vec{\theta} \tag{12}$$

where $F_r(\theta)$ and $F_\theta(\theta)$ are force components in the radial and tangential directions, respectively. Using Fourier expansion on $[-\pi, \pi]$, these components can be approximated by

$$F_r(\theta) = \sum_{n=-N}^{N} Q_{n,r,c}\cos(n(\theta - \theta_0)) + \sum_{n=-N}^{N} Q_{n,r,s}\sin(n(\theta - \theta_0))$$
$$F_\theta(\theta) = \sum_{n=-N}^{N} Q_{n,\theta,c}\cos(n(\theta - \theta_0)) + \sum_{n=-N}^{N} Q_{n,\theta,s}\sin(n(\theta - \theta_0)) \tag{13}$$

where N is the cut-off harmonic number, $Q_{n,r,c}$, $Q_{n,r,s}$, $Q_{n,\theta,c}$ and $Q_{n,\theta,s}$ are corresponding coefficients of the nth harmonic force for radial, tangential and cosine, sine components. The subscript r or θ indicates that the coefficient is for radial or tangential direction, respectively, while c or s represents cosine or sine component, respectively. θ_0 is used to define the rotation of the local cylindrical coordinate system with respect to the global Cartesian coordinate system. In this paper, θ_0 is always set to 0, which indicates $\theta = 0$ corresponds to the bottom point of the ring.

Considering only the nth harmonic force, the force per unit area applied on the external edge of the ring (where the radial location is $R + h/2$) can be written for the radial and tangential directions independently, where each direction consists of cosine and sine components:

$$q_{nr}\left(R+\frac{h}{2},\theta\right)=q_{n,r,c}\left(R+\frac{h}{2},\theta\right)+q_{n,r,s}\left(R+\frac{h}{2},\theta\right)$$
$$q_{n\theta}\left(R+\frac{h}{2},\theta\right)=q_{n,\theta,c}\left(R+\frac{h}{2},\theta\right)+q_{n,\theta,s}\left(R+\frac{h}{2},\theta\right) \tag{14}$$

The cosine or sine component in each direction can be expressed in terms of the width of the ring, the radial coordinate and the corresponding force component:

$$q_{n,r,c}\left(R+\frac{h}{2},\theta\right)=\frac{1}{b\left(R+\frac{h}{2}\right)}Q_{n,r,c}\cos(n(\theta-\theta_0))$$
$$q_{n,r,s}\left(R+\frac{h}{2},\theta\right)=\frac{1}{b\left(R+\frac{h}{2}\right)}Q_{n,r,s}\sin(n(\theta-\theta_0))$$
$$q_{n,\theta,c}\left(R+\frac{h}{2},\theta\right)=\frac{1}{b\left(R+\frac{h}{2}\right)}Q_{n,\theta,c}\cos(n(\theta-\theta_0))$$
$$q_{n,\theta,s}\left(R+\frac{h}{2},\theta\right)=\frac{1}{b\left(R+\frac{h}{2}\right)}Q_{n,\theta,s}\sin(n(\theta-\theta_0)) \tag{15}$$

The four components of the nth harmonic force in (15) can be solved separately and then superposed to get the deformation for the nth harmonic. In order to solve for the response to the first force component in (15), which is the cosine part in the radial direction, q_r and q_θ in (8) can be replaced with q_{nr} and $q_{n\theta}$:

$$q_{nr}=q_{n,r,c}\left(R+\frac{h}{2},\theta\right)=\frac{1}{b\left(R+\frac{h}{2}\right)}Q_{n,r,c}\cos(n(\theta-\theta_0))$$
$$q_{n\theta}=0 \tag{16}$$

Noting that the right hand side of the first equation in (8) has a cosine function, combined with the order of differentiation to the variables, it is easy to obtain the assumed ansatz:

$$u_{r,n,r,c}(R,\theta)=Cur_{r,c}(n)\cos(n(\theta-\theta_0))$$
$$u_{\theta,n,r,c}(R,\theta)=Cu\theta_{r,c}(n)\sin(n(\theta-\theta_0))$$
$$\phi_{n,r,c}(R,\theta)=C\phi_{r,c}(n)\sin(n(\theta-\theta_0)) \tag{17}$$

where $u_{r,n,r,c}(R,\theta)$, $u_{\theta,n,r,c}(R,\theta)$ are displacement solutions in radial and tangential direction for the centroid and $\phi_{n,r,c}(R,\theta)$ is the cross-section rotation. The subscripts n, r, c indicate that they are the solution components due to the cosine part of nth harmonic force in the radial direction. $Cur_{r,c}$, $Cu\theta_{r,c}$ and $C\phi_{r,c}$ are modal coefficients to be solved for. They are indexed by the harmonic (circumferential mode) number n. Substituting (16) and (17) into (8) and noting that the coefficients of $\cos(n(\theta-\theta_0))$ or $\sin(n(\theta-\theta_0))$ in all three equations in (8) should be equal, one obtains 3 equations with 3 unknown coefficients $Cur_{r,c}$, $Cu\theta_{r,c}$ and $C\phi_{r,c}$. These can be solved explicitly and the results are given in the Appendix. The responses to the other three components of the nth harmonic force in (15) can be solved for in a similar way.

The solutions for the sine component in the radial direction are given by

$$u_{r,n,r,s}(R,\theta)=Cur_{r,s}(n)\sin(n(\theta-\theta_0))$$
$$u_{\theta,n,r,s}(R,\theta)=Cu\theta_{r,s}(n)\cos(n(\theta-\theta_0))$$
$$\phi_{n,r,s}(R,\theta)=C\phi_{r,s}(n)\cos(n(\theta-\theta_0)) \tag{18}$$

And the solutions for the cosine and sine components in the tangential direction are given by

$$u_{r,n,\theta,c}(R,\theta)=Cur_{\theta,c}(n)\sin(n(\theta-\theta_0))$$
$$u_{\theta,n,\theta,c}(R,\theta)=Cu\theta_{\theta,c}(n)\cos(n(\theta-\theta_0))$$
$$\phi_{n,\theta,c}(R,\theta)=C\phi_{\theta,c}(n)\cos(n(\theta-\theta_0))$$
$$u_{r,n,\theta,s}(R,\theta)=Cur_{\theta,s}(n)\cos(n(\theta-\theta_0))$$
$$u_{\theta,n,\theta,s}(R,\theta)=Cu\theta_{\theta,s}(n)\sin(n(\theta-\theta_0))$$
$$\phi_{n,\theta,s}(R,\theta)=C\phi_{\theta,s}(n)\sin(n(\theta-\theta_0)) \tag{19}$$

The corresponding modal coefficients are given in the Appendix.

The final displacement solutions for the governing equations in (8) are then written as follows:

$$u_r(R,\theta)=\sum_{n=-N}^{N}\left[u_{r,n,r,c}(R,\theta)+u_{r,n,r,s}(R,\theta)+u_{r,n,\theta,c}(R,\theta)+u_{r,n,\theta,s}(R,\theta)\right]$$
$$u_\theta(R,\theta)=\sum_{n=-N}^{N}\left[u_{\theta,n,r,c}(R,\theta)+u_{\theta,n,r,s}(R,\theta)+u_{\theta,n,\theta,c}(R,\theta)+u_{\theta,n,\theta,s}(R,\theta)\right]$$
$$\phi(R,\theta)=\sum_{n=-N}^{N}\left[\phi_{n,r,c}(R,\theta)+\phi_{n,r,s}(R,\theta)+\phi_{n,\theta,c}(R,\theta)+\phi_{n,\theta,s}(R,\theta)\right] \tag{20}$$

3.2. Solution for unilateral elastic foundation

The radial displacement solution $u_r(R,\theta)$ obtained from (20) considers support by the foundation whether the displacement is positive and negative. With a collapsible foundation, there is no support in regions where $u_r(R,\theta)$ is negative. In this region, an excessive force is included by the linear foundation assumption. The amount of this excessive force $F_e(\theta)$ is proportional to the extent of negative deformation:

$$F_e(\theta)=\begin{cases}u_r(R,\theta)Kr & \text{where } u_r(R,\theta)<0\\ 0, & \text{where } u_r(R,\theta)\geq 0\end{cases} \tag{21}$$

For tensionless foundation, a similar excessive force exists in regions where $u_r(R,\theta)$ is positive. But the remainder of the analysis will be the same as the case of the collapsible foundation.

In order to counteract the excessive force and obtain the deformation for the unilateral foundation case, a compensation force $F_{cp}(\theta)$ which has the same magnitude as the excessive force can be applied to the linear elastic foundation:

$$F_{cp}(\theta)=F_e(\theta) \tag{22}$$

F_{cp} can be expanded into Fourier series on $[-\pi,\pi]$ using the same harmonic numbers as $F(\theta)$:

$$F_{cp}(\theta)=\sum_{n=-N}^{N}F_{cp,n}(\theta)=\sum_{n=-N}^{N}\left[H_{n,r,c}\cos(n(\theta-\theta_0))+H_{n,r,s}\sin(n(\theta-\theta_0))\right] \tag{23}$$

Because the unilateral property of the foundation only exists in the radial direction, this compensation force will only have radial components.

The compensation force is then applied to the external edge of the foundation, at internal ring radius $R-h/2$. The distributed compensation force per unit area in both the radial and tangential directions is given by

$$q_{cp,n,r,c}\left(R-\frac{h}{2},\theta\right)=\frac{1}{b\left(R-\frac{h}{2}\right)}H_{n,r,c}\cos(n(\theta-\theta_0))$$
$$q_{cp,n,r,s}\left(R-\frac{h}{2},\theta\right)=\frac{1}{b\left(R-\frac{h}{2}\right)}H_{n,r,s}\sin(n(\theta-\theta_0))$$
$$q_{cp,n,\theta,c}\left(R-\frac{h}{2},\theta\right)=0$$
$$q_{cp,n,\theta,s}\left(R-\frac{h}{2},\theta\right)=0 \tag{24}$$

Substitution of (24) into (8) for q_{nr} and $q_{n\theta}$ leads to the governing equations for the compensation displacements driven by the compensation force. Note that the external ring radius $R+h/2$ in the right hand sides of (8) needs to be replaced by the internal ring radius $R-h/2$. The governing equations can then be solved using the same approach as above as they are still linear differential equations. In so doing, the nonlinear unilateral foundation problem is approximated by applying a compensation force to the linear foundation model instead of directly solving the system of coupled nonlinear differential equations for the problem.

However, instead of solving the linear differential equations again with the compensation force, the following observations can be used to get the compensation displacements to further reduce the computational load. Since the radius on the right hand sides of (8) (updated) will be canceled out by those in the denominators of (24), the displacement solutions are only affected by the magnitude of the forces regardless of how they are applied on the external edge of the foundations or the outside of the ring. So, based on (17) and (18), the cosine and sine components of the compensation displacements caused by the compensation force can be obtained directly by

$$u_{r,cp,n,r,c}(R,\theta)=\frac{H_{n,r,c}}{Q_{n,r,c}}Cur_{r,c}(n)\cos(n(\theta-\theta_0))$$
$$u_{\theta,cp,n,r,c}(R,\theta)=\frac{H_{n,r,c}}{Q_{n,r,c}}Cu\theta_{r,c}(n)\sin(n(\theta-\theta_0))$$
$$\phi_{cp,n,r,c}(R,\theta)=\frac{H_{n,r,c}}{Q_{n,r,c}}C\phi_{r,c}(n)\sin(n(\theta-\theta_0))$$
$$u_{r,cp,n,r,s}(R,\theta)=\frac{H_{n,r,s}}{Q_{n,r,s}}Cur_{r,s}(n)\sin(n(\theta-\theta_0))$$
$$u_{\theta,cp,n,r,s}(R,\theta)=\frac{H_{n,r,s}}{Q_{n,r,s}}Cu\theta_{r,s}(n)\cos(n(\theta-\theta_0))$$
$$\phi_{cp,n,r,s}(R,\theta)=\frac{H_{n,r,s}}{Q_{n,r,s}}C\phi_{r,s}(n)\cos(n(\theta-\theta_0)) \tag{25}$$

By using these simple algebraic equations, even the linear differential equations need to be solved only once.

Correspondingly, the compensation displacements are obtained from

$$u_{r,cp}(R,\theta)=\sum_{n=-N}^{N}\left[u_{r,cp,n,r,c}(R,\theta)+u_{r,cp,n,r,s}(R,\theta)\right]$$
$$u_{\theta,cp}(R,\theta)=\sum_{n=-N}^{N}\left[u_{\theta,cp,n,r,c}(R,\theta)+u_{\theta,cp,n,r,s}(R,\theta)\right]$$
$$\phi_{cp}(R,\theta)=\sum_{n=-N}^{N}\left[\phi_{cp,n,r,c}(R,\theta)+\phi_{cp,n,r,s}(R,\theta)\right] \tag{26}$$

The new total displacements are the summation of the original solutions (20) and the compensation displacements (26):

$$\hat{u}_r(R,\theta)=u_r(R,\theta)+u_{r,cp}(R,\theta)$$
$$\hat{u}_{\theta,cp}(R,\theta)=u_\theta(R,\theta)+u_{\theta,cp}(R,\theta)$$
$$\hat{\phi}_{cp}(R,\theta)=\phi(R,\theta)+\phi_{cp}(R,\theta) \tag{27}$$

Due to the change of radial displacement from $u_r(R,\theta)$ to $\hat{u}_r(R,\theta)$, the excessive force in (21) needs to be updated as well. Thus, the procedure from (21) to (27) needs to be repeated, thereby setting up an iterative scheme. This scheme is considered to converge when values of the compensated displacements in (27) do not change anymore with further iteration steps. The converged displacements in (27) can then be taken as the solution for the unilateral foundation case.

4. Results and discussion

In this section, two examples are included to demonstrate the utility of the proposed method. In the first example, a radial point force with both positive and negative magnitudes is considered and the results obtained using the iterative compensation approach introduced above are compared with those obtained via FEA. In the second one, a more general distributed force, with both radial, tangential and sine, cosine components are applied and the solutions are shown.

4.1. Response to a concentrated force

As a basic comparison, we consider a radial concentrated force F_c applied at the bottom of the ring. A Gaussian function representation allows one to model different degrees of "concentration" of this force:

$$f_c(\theta)=Q\frac{1}{\sqrt{\pi\,\sigma^2}}e^{-(\theta-\theta_0)^2/\sigma^2} \tag{28}$$

where $Q=\pm 1000$ N is the magnitude of the force, σ is a parameter that determines how concentrated the applied force is. The smaller σ is, the more concentrated the force is. Fig. 2 shows the effect of σ to the distribution density of a unit force. It can be seen that with $\sigma\to 0$, $f_c(\theta_0)\to\infty$, and the force becomes an ideal concentrated force with 0 distribution width. The integration of the distribution density function around the ring always equals to the concentrated force:

$$\int_{-\pi}^{\pi} f_c(\theta)\,d\theta=F_c \tag{29}$$

$f_c(\theta)$ can be expanded into Fourier series in $[-\pi,\pi]$:

$$f_c(\theta)=\sum_{n=-N}^{N}Q_n\cos(n(\theta-\theta_0))=\frac{1}{2\pi}Q\sum_{n=-N}^{N}e^{-(1/4)n^2\sigma^2}\cos(n(\theta-\theta_0)) \tag{30}$$

that is

$$Q_n=\frac{1}{2\pi}Qe^{-(1/4)n^2\sigma^2} \tag{31}$$

The displacement responses due to this concentrated force are then solved by using the iterative scheme described above. A collapsible unilateral foundation is considered and compared with a linear foundation. Parameter values used for the results given below are listed in Table 1. In this case, negligible tangential stiffness is considered for the comparison with FEA results.

A 2-D FE model of the ring on a unilateral foundation is built using the commercial finite element software Abaqus/Standard.

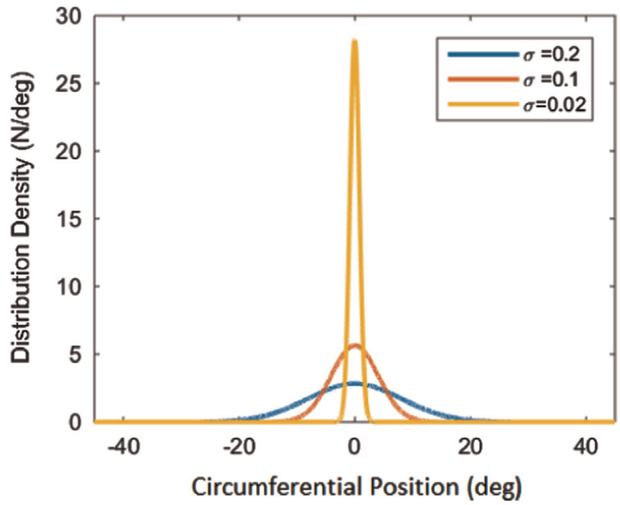

Fig. 2. Effect of σ to the distribution density of a unit force.

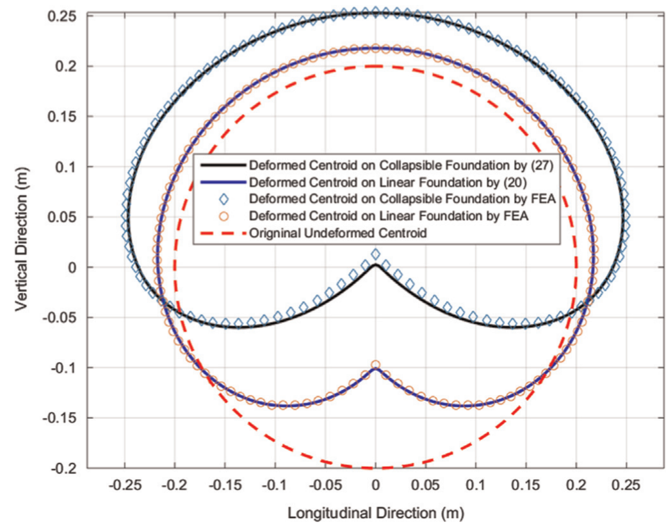

Fig. 3. Comparison of deformed centroid: collapsible foundation vs. linear foundation.

Table 1
Value of parameters.

Parameters	Definition	Value	Unit
R	Radius of ring centroid	0.2	m
h	Ring thickness	0.02	m
b	Ring width	0.06	m
E_θ	Extensional modulus of the ring	1e10	Pa
G	Shear modulus of the ring	4e6	Pa
K_r	Radial stiffness of the foundation	1e5	N/m per radian
K_θ	Radial stiffness of the foundation	1	N/m per radian
k_s	Stiffness of discretized spring element	5.236e3	N/m
F_c	Magnitude of concentrated force	±1000	N
σ	Distribution factor of Gaussian function	0.02	NA

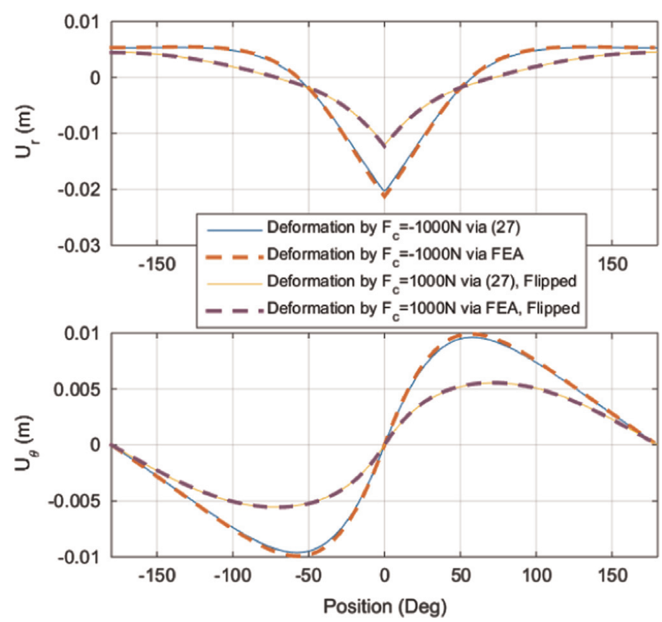

Fig. 4. Comparison of deformed centroid: response by positive and negative force.

The thick ring is meshed using the second-order quadrilateral plane stress element with reduced integration, *CPS8R*, with 6 layers of elements in the radial direction and 120 divisions in the circumferential direction. In the examples, only radial stiffness is taken into account for the foundation, which is modeled using 120 evenly distributed and discretized spring elements *SpringA*. The property of these spring elements can be set to linear or nonlinear to represent the linear or unilateral elastic foundation. The stiffness of every linear spring element (or the effective non-vanished stiffness for the nonlinear spring element representing unilateral foundation) is calculated by

$$k_s = \frac{2\pi K_r}{N_s} \tag{32}$$

where $N_s = 120$ is the number of the spring elements.

Fig. 3 shows the deformation of the ring centroid for both collapsible and linear foundation with $F_c = -1000$ N (pointing to the ring center). The deformations are amplified by 10 times for clarity. It can be seen that the solutions obtained by the proposed iterative approach match the results by the FEA very well, for both the linear and unilateral foundation cases.

For the unilateral foundation case, it is expected that the magnitude of displacements will be different under a positive and negative force with same magnitude. Fig. 4 shows the deformation responses under $F_c = \pm 1000$ N for the ring on collapsible foundation. In this figure, the displacements by positive force are shown with flipped sign so that they can be easily compared with those by negative force. It can be seen that for the collapsible foundation case, a negative force will lead to a larger magnitude of u_r at the position where the force is applied because negative force will directly compress the local area of the foundation to make it collapse while the positive force will stretch it so the effective stiffness there is higher than that for a negative force.

4.2. Response to a more complex distributed force

A more complex distributed force is also considered as shown in Fig. 5. This arbitrarily assigned distributed force consists of both tangential and radial components, and the corresponding distribution density is plotted separately. Due to the asymmetry of the distribution, the Fourier expansion of this general distributed force will include both cosine and sine components. The same parameters as in Table 1 are used for the REF model except for the higher, more practical value of $K_\theta = 0.5e5$ N/m per radian. The deformation results solved by this general force are plotted in Fig. 6. Again the deformation is amplified by 10 times for clarity. In addition to the deformation of both linear and unilateral foundation cases, the identified collapsed regions can be seen in this

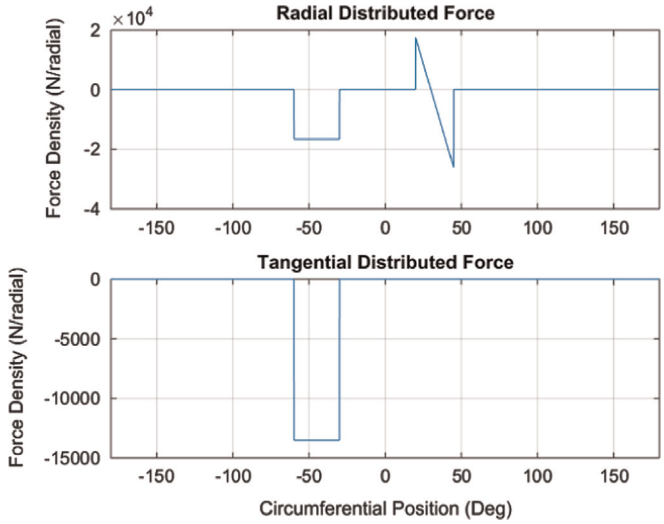

Fig. 5. Distribution density of a general distributed force.

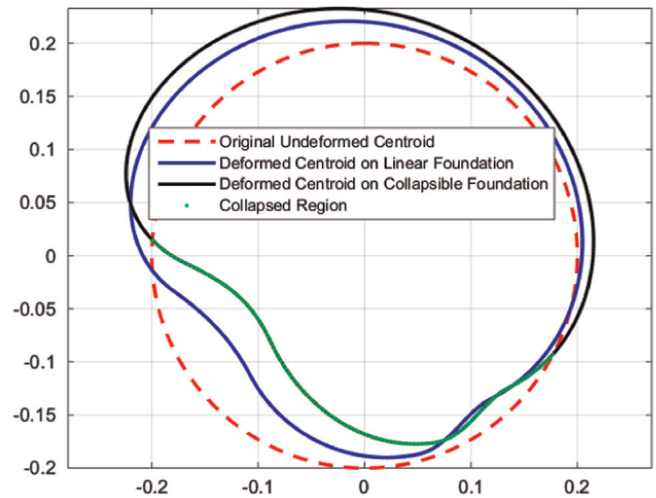

Fig. 6. Deformation of ring centroid due to the distributed force given in Fig. 5.

figure, corresponding roughly to the circumferential regions of the applied distributed force (more on the left than on the right).

Finally, we remark that with a cut-off harmonic number $N=50$, even for the complex applied force cases the iterative computations are completed within seconds on a modern PC. This makes the proposed methods suitable for parametric design studies where the different beam and foundation material properties, geometric dimensions, etc. can be quickly optimized.

5. Conclusion and discussion

In this paper, the analysis of the well-studied REF model has been extended to the unilateral foundation case. This paper focused on the static deformation problem of a Timoshenko ring resting on a two-parameter foundation with unilateral radial stiffness and subject to arbitrary in-plane force. For the case of a linear foundation, the governing equations have been derived and solved analytically for both the radial and tangential displacements as well as for the section rotation. It is then observed that for a unilateral foundation, the ring on the linear foundation can be treated as supported by a distributed excessive force to the linear foundation model. Then, the nonlinear unilateral foundation problem can be solved via the compensation of this excessive force, thereby setting up a simple iterative scheme. Comparison with FEA results for a concentrated radial force illustrated the validity and accuracy of the proposed approach. Then a more complex distributed force with both radial and tangential components is considered and the deformation as well as the determined collapsed region are shown.

This approach achieves the solution to the nonlinear unilateral foundation problem in a physically-motivated, fast and elegant way. The computation involves simple iterations based on linear analytical solutions. Direct numerical solution of the governing nonlinear differential equations is avoided. Compared with FEA, the proposed method achieves the nonlinear solution without time-consuming modeling and meshing work. Compared with the existing semi-analytical method in [18], the proposed method is capable of solving for the deformation response to an arbitrary complex distributed force in a unified way. Furthermore, in our continuing work, we shall show how the method can be easily extended to solve the static contact problem with arbitrary surface profiles. In a companion paper [19], the authors present the application of this iterative approach in the analysis of the dynamic forced response problem of a ring on a unilateral elastic foundation. Therein lies the major benefit of the iterative approach. It avoids the need for solving the complex, coupled and nonlinear differential equations for the unilateral elastic foundation. It can therefore be used to accelerate parametric design sensitivity studies and optimizations to select the proper ring and foundation materials and geometric properties for the envisaged application of the REF.

Appendix A. Coefficients for displacement solutions

Coefficients for solutions of radial direction components

$$\begin{aligned}
Cur_{r,c}(n) &= \frac{\left(Zrur4\,n^4 + Zrur2\,n^2 + Zrur0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,c}\\
Cu\theta_{r,c}(n) &= \frac{\left(Zrut3\,n^3 + Zrut1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,c}\\
C\phi_{r,c}(n) &= \frac{\left(Zr\phi3\,n^3 + Zr\phi1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,c}\\
Cur_{r,s}(n) &= \frac{\left(Zrur4\,n^4 + Zrur2\,n^2 + Zrur0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,s}\\
Cu\theta_{r,s}(n) &= -\frac{\left(Zrut3\,n^3 + Zrut1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,s}\\
C\phi_{r,s}(n) &= -\frac{\left(Zr\phi3\,n^3 + Zr\phi1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{rn,s}
\end{aligned} \tag{A.1}$$

Coefficients for solutions of tangential direction components

$$\begin{aligned}
Cur_{\theta,c}(n) &= \frac{\left(Z\theta ur3\,n^3 + Z\theta ur1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,c}\\
Cu\theta_{\theta,c}(n) &= \frac{\left(Z\theta ut4\,n^4 + Z\theta ut2\,n^2 + Z\theta ut0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,c}\\
C\phi_{\theta,c}(n) &= \frac{\left(Z\theta\phi4\,n^4 + Z\theta\phi2\,n^2 + Z\theta\phi0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,c}\\
Cur_{\theta,s}(n) &= -\frac{\left(Z\theta ur3\,n^3 + Z\theta ur1\,n\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,s}\\
Cu\theta_{\theta,s}(n) &= \frac{\left(Z\theta ut4\,n^4 + Z\theta ut2\,n^2 + Z\theta ut0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,s}\\
C\phi_{\theta,s}(n) &= \frac{\left(Z\theta ut4\,n^4 + Z\theta ut2\,n^2 + Z\theta ut0\right)}{ZD6\,n^6 + ZD4\,n^4 + ZD2\,n^2 + ZD0} Q_{\theta n,s}
\end{aligned} \tag{A.2}$$

where

$$
\begin{aligned}
ZD6 &= 4\,GA\,EA_\theta\,EI_\theta \\
ZD4 &= GA\,Rh^2EA_\theta\,K_\theta + 4\,GA\,REI_\theta\,K_\theta + 4\,K_r\,REA_\theta\,EI_\theta - 8\,GA\,EA_\theta EI_\theta \\
ZD2 &= K_r\,R^2h^2EA_\theta\,K_\theta + 4\,GA\,K_r\,R^3EA_\theta + 4\,GA\,R^2hEA_\theta\,K_\theta \\
&\quad - 2\,GA\,Rh^2EA_\theta\,K_\theta + 4\,K_r\,R^2EI_\theta\,K_\theta + 4\,GA\,K_r\,REI_\theta \\
&\quad + 4\,REA_\theta\,EI_\theta\,K_\theta + 4\,GA\,EA_\theta\,EI_\theta \\
ZD0 &= \left(4\,K_r\,R^4 - 4\,K_r\,R^3h + K_r\,R^2h^2 + 4\,R^3EA_\theta - 4\,R^2hEA_\theta + Rh^2EA_\theta\right) \\
Zrur4 &= 4\,REA_\theta\,EI_\theta \\
Zrur2 &= R\left(Rh^2EA_\theta\,K_\theta + 4\,GA\,R^2EA_\theta + 4\,REI_\theta\,K_\theta + 4\,GA\,EI_\theta\right) \\
Zrur0 &= R^2GA\,K_\theta\left(4\,R^2 - 4\,Rh + h^2\right) \\
Zrut3 &= \left(-4\,GA - 4\,EA_\theta\right)REI_\theta \\
Zrut1 &= \left(2\,GA\,R^2hK_\theta - GA\,Rh^2K_\theta - Rh^2EA_\theta\,K_\theta - 4\,GA\,R^2EA_\theta\right)R \\
Zr\phi 3 &= 4\,GA\,R^2EA_\theta \\
Zr\phi 1 &= 2\,R^2\left(2\,GA\,RK_\theta - GA\,hK_\theta - hEA_\theta\,K_\theta - 2\,GA\,EA_\theta\right) \\
Z\theta ur3 &= -2\,R\left(GA\,RhEA_\theta - 2\,GA\,EI_\theta - 2\,EA_\theta\,EI_\theta\right) \\
Z\theta ur1 &= -4\,GA\,R^3hK_\theta + 2\,GA\,R^2h^2K_\theta + 2\,R^2h^2EA_\theta\,K_\theta + 4\,GA\,R^3EA_\theta \\
&\quad + 2\,GA\,R^2hEA_\theta \\
Z\theta ut4 &= 4\,GA\,EI_\theta\,R \\
Z\theta ut2 &= 2\left(GA\,Rh^2K_\theta - GA\,RhEA_\theta + 2\,K_r\,REI_\theta + 2\,EA_\theta\,EI_\theta\right)R \\
Z\theta ut0 &= 2\,K_r\,R^3h^2K_\theta + 4\,GA\,K_r\,R^4 + 2\,GA\,K_r\,R^3h + 2\,R^2h^2EA_\theta\,K_\theta \\
&\quad + 4\,GA\,R^3EA_\theta + 2\,GA\,R^2hEA_\theta \\
Z\theta\phi 4 &= 2\,GA\,RhEA_\theta \\
Z\theta\phi 2 &= 2\left(2\,GA\,RhK_\theta + K_r\,RhEA_\theta - 2\,GA\,REA_\theta - 2\,GA\,hEA_\theta\right)R \\
Z\theta\phi 0 &= 4\,K_r\,R^3hK_\theta + 4\,GA\,K_r\,R^3 + 2\,GA\,K_r\,R^2h + 4\,R^2hEA_\theta\,K_\theta \\
&\quad + 4\,GA\,R^2EA_\theta + 2\,GA\,RhEA_\theta
\end{aligned}
\tag{A.3}
$$

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