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Q-Factor and Resonance in the Time and Frequency Domain

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Abstract

This report provides a brief overview of resonance concepts. It shows that the Q-factor of a damped oscillation can be closely approximated by counting the number of visible oscillations as viewed on a typical oscilloscope screen.

1. Introduction

Sometimes, when making measurements of a signal with an oscilloscope, oscillations are observed that may or may not correspond to peaks in the radiated emissions spectrum. It is helpful to be able to recognize the correlation between a damped sinusoid in the time domain and a resonant peak in the frequency domain. This report reviews some basic resonance concepts and shows that the Q-factor of a resonance can be closely approximated by counting the number of visible oscillations on a typical oscilloscope screen.

2. Q-Factor in the Frequency Domain

The Q-factor or quality factor is defined as the ratio of energy stored to the energy dissipated per cycle for under-damped systems. It can be mathematically approximated as [1],

$$Q = \frac{f_r}{\Delta f} = \frac{\omega_r}{\Delta \omega} \tag{1}$$

where f_r is the resonant frequency, Δf is the half-power (or 3-dB) bandwidth as shown in Fig. 1, ω_r is the angular resonant frequency, and $\Delta \omega$ is the angular bandwidth ($\Delta \omega = 2\pi \Delta f$).
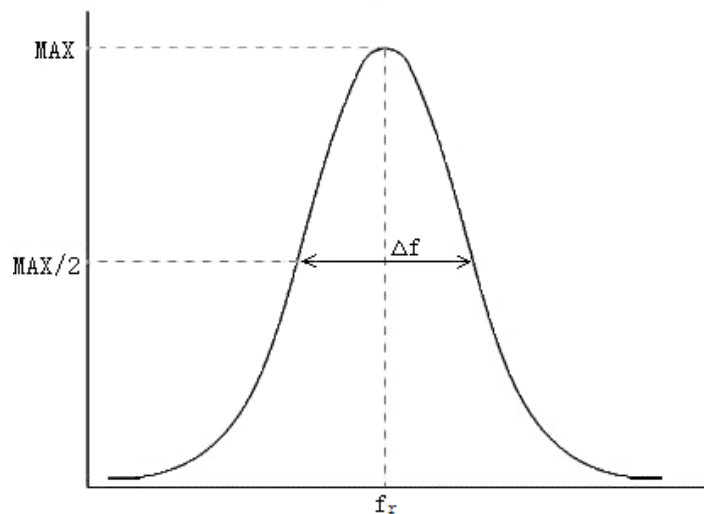

Fig. 1. Frequency bandwidth at half of the peak power.
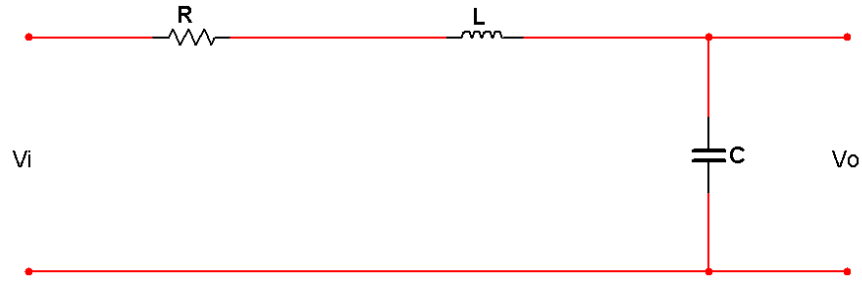
3. The Under-Damped RLC Circuit

Fig. 2. A series RLC circuit.

Figure 2 shows a series RLC circuit. If it is an under-damped system, for a unit impulse input, assuming zero initial energy is stored in the circuit, the output will be [2],

$$v_0 = e^{-\alpha \cdot t} \cdot \sin(\omega_d t) \tag{2}$$

where α is the natural exponential decay rate of the impulse response of the RLC circuit and ω_d is the damped angular resonant frequency. We also have [2,3],

$$\alpha = \frac{R}{2L} \tag{3}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{4}$$

where the angular natural frequency [2],

$$\omega_n = 1 / \sqrt{LC} \tag{5}$$

and the damping ratio [3],

$$\zeta = \alpha / \omega_n = \frac{R}{2} \sqrt{\frac{C}{L}} \tag{6}$$

If we substitute $R = 0.6 \ \Omega$, $C = 5$ nF, $L = 0.2$ μH into (2 - 6), v_0 can be plotted in the time domain as shown in Fig. 3. The amplitude of the ringing is decaying at a rate of $e^{-1.5 \times 10^6 t}$ and the number of the rings, N, before it decays to 5% of its initial value is 10. Note that N can have a value as low as 0.5 if the magnitude of the ringing only exceeds the 5% boundary once in a cycle.
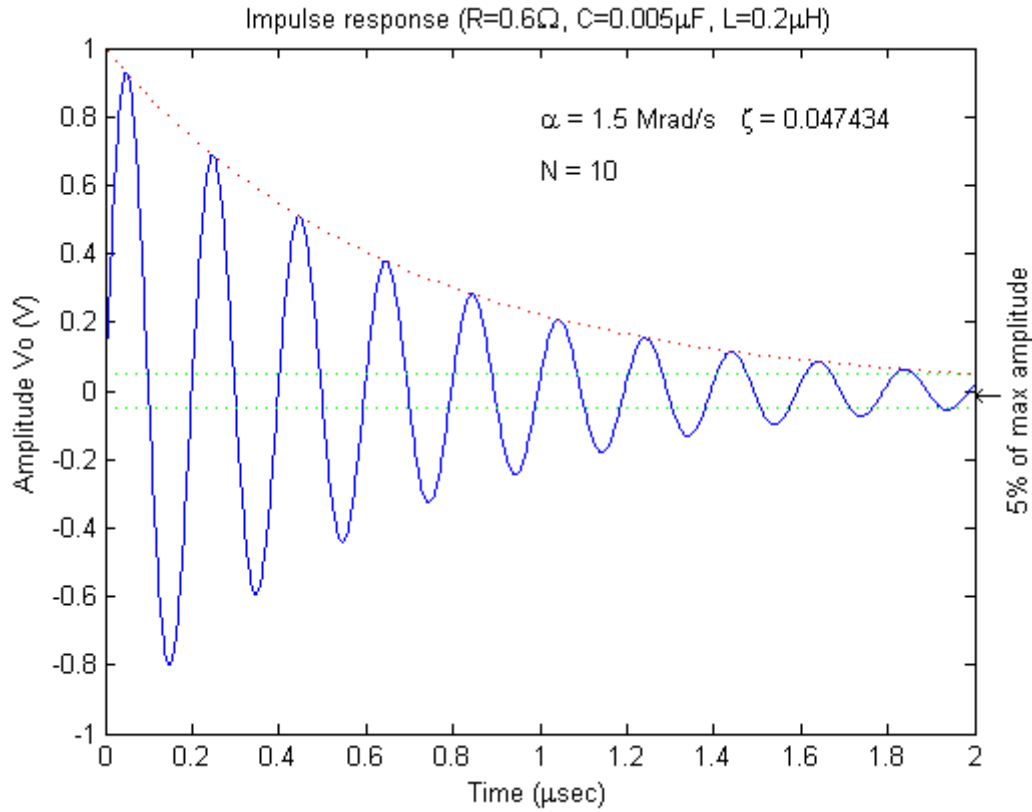
Fig. 3. Impulse response of a series RLC circuit in time domain.

Next, we want to express the output voltage in frequency domain. We know the transfer function of the RLC circuit can be expressed in the Laplace domain as [3],

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{7}$$

By replacing s by $j\omega$ and taking the magnitude, we get,

$$|H(j\omega)| = \frac{\omega_n^2}{\left|(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2\right|} = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}. \tag{8}$$

From (8), we can plot the impulse response in the frequency domain as shown in Fig. 4. In this figure, the vertical scale is in decibels in order to better represent a typical spectrum analyzer screen. The angular 3-dB bandwidth is 3 Mrad/s. It is approximately twice the decay rate, α, in Fig. 3. The Q factor is 10.5, which is approximately the number of the rings observed in Fig. 3. This relationship between the Q-factor and the number of observable rings is mathematically explored in Sections 4 and 5 of this report.
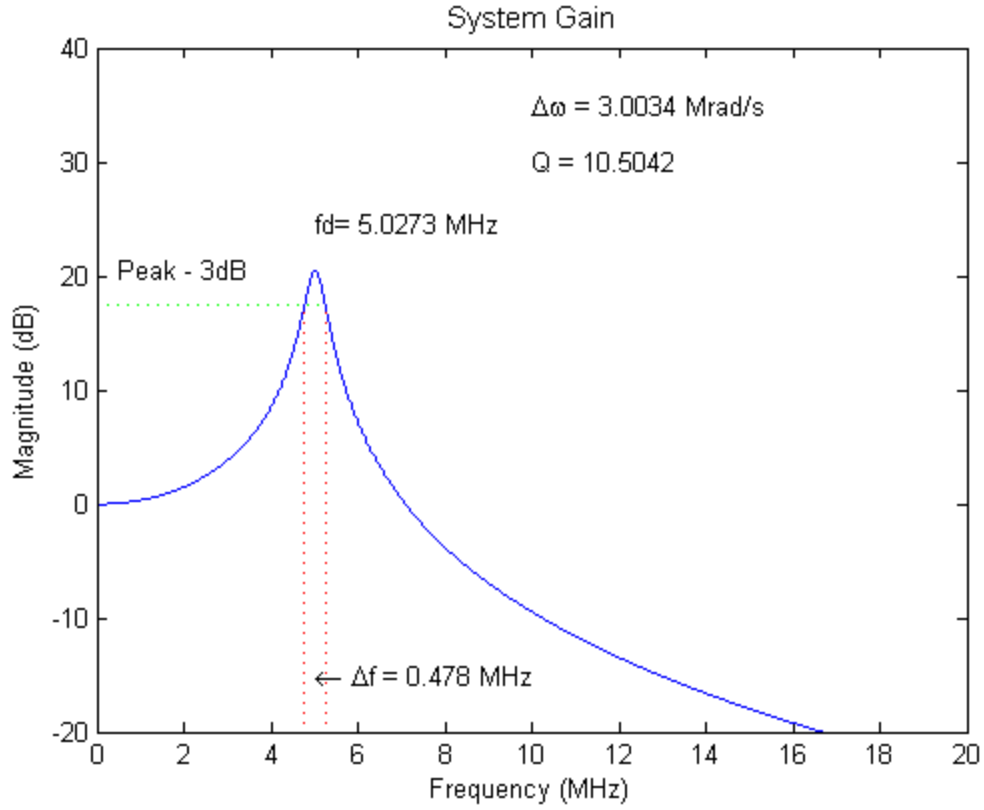
Fig. 4. System gain of a RLC circuit in frequency domain.

4. Angular bandwidth and Oscillation Decay Rate

From Fig. 4, we see that when $\omega = \omega_d$, the gain reaches its maximum value and so does v_0. Substituting ω_d into (8), we get,

$$|H(j\omega)|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} . \tag{9}$$

To find the 3-dB frequencies, we set Equation (8) equal to $1/\sqrt{2}$ times the maximum,

$$\left|H(j\omega)\right| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta\sqrt{1-\zeta^2}} . \tag{10}$$

Rearranging (10) results in,

$$\omega^4 + \omega^2(4\zeta^2\omega_n^2 - 2\omega_n^2) + \omega_n^4(1 - 8\zeta^2 + 8\zeta^4) = 0 \tag{11}$$

To simplify the analysis, we let,

$$b = 4\zeta^2\omega_n^2 - 2\omega_n^2 \tag{12}$$

$$c = \omega_n^4(1 - 8\zeta^2 + 8\zeta^4) . \tag{13}$$

Then (11) can be solved as,

$$\omega_1^2 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \tag{14a}$$

$$\omega_2^2 = \frac{-b - \sqrt{b^2 - 4c}}{2}. \tag{14b}$$

Since ω_1 and ω_2 are both positive, from (14) we have,

$$\omega_1 = \sqrt{(-b + \sqrt{4c})/2}, \tag{15a}$$

$$\omega_2 = \sqrt{(-b - \sqrt{4c})/2}. \tag{15b}$$

From (14), (15), we get,

$$\Delta\omega = \omega_1 - \omega_2 = \sqrt{\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2} = \sqrt{-b - 2\sqrt{c}}. \tag{16}$$

For small ζ $(\zeta < 0.2)$, which is the case for high-Q under-damped systems, the term $8\zeta^4$ is much smaller than the term $1 - 8\zeta^2$ in (13). Thus (13) can be simplified to,

$$c = \omega_n^4 (1 - 8\zeta^2 + 16\zeta^4) = [\omega_n^2 (1 - 4\zeta^2)]^2. \tag{17}$$

Substitution of (12) and (17) into (16) yields,

$$\Delta\omega = \sqrt{-4\zeta^2\omega_n^2 + 2\omega_n^2 - 2\omega_n^2(1 - 4\zeta^2)} = 2\zeta\omega_n. \tag{18}$$

Then after substituting (6) into (18), we obtain,

$$\Delta\omega = 2\alpha. \tag{19}$$

Equation (19) explains why the 3-dB angular bandwidth in Fig. 4 is approximately twice the decay rate, α, in Fig. 3.

5. Q-Factor and Number of Rings

From Equation (2), we know the time t after which the ringing decays to 5% of its initial value satisfies,

$$e^{-\alpha \cdot t} = 0.05. \tag{20}$$

Since Q is approximated in Equation (1) as $Q = \omega_d / \Delta\omega$, by substituting (19) in, we get,

$$Q = \pi f_d / \alpha \tag{21}$$

where f_d is the damped resonant frequency.

Then by substituting (21) it into (20), we obtain,

$$e^{-\frac{\pi f_d}{Q} t} = 0.05, \text{ or,} \tag{22a}$$

$$t = -\frac{ln0.05 \cdot Q}{\pi f_d}. \tag{22b}$$

The number of rings in time t is, $N = t/T = t f_d$. Substituting that into (22), we get,

$$N = -\frac{ln0.05}{\pi} Q = 0.954 Q \text{, or,} \qquad (23a)$$

$$N \cong Q \qquad (23b)$$

Equation (23) can also be applied to other forms of under-damped systems. It directly relates the number of rings that can be easily observed in the time domain to the Q-factor, which can be measured in the frequency domain. It gives us an intuitive idea of what should we expect in the time domain if we see a peak in the frequency domain and vice versa.

6. Critically Damped Systems

By definition, a system with a Q factor of 0.5 is said to be critically damped. Fig. 5 shows the time domain impulse response of a critically damped RLC circuit and its FFT in the frequency domain. Note that Equation (23) still holds for this special case (i.e. N = Q = 0.5).
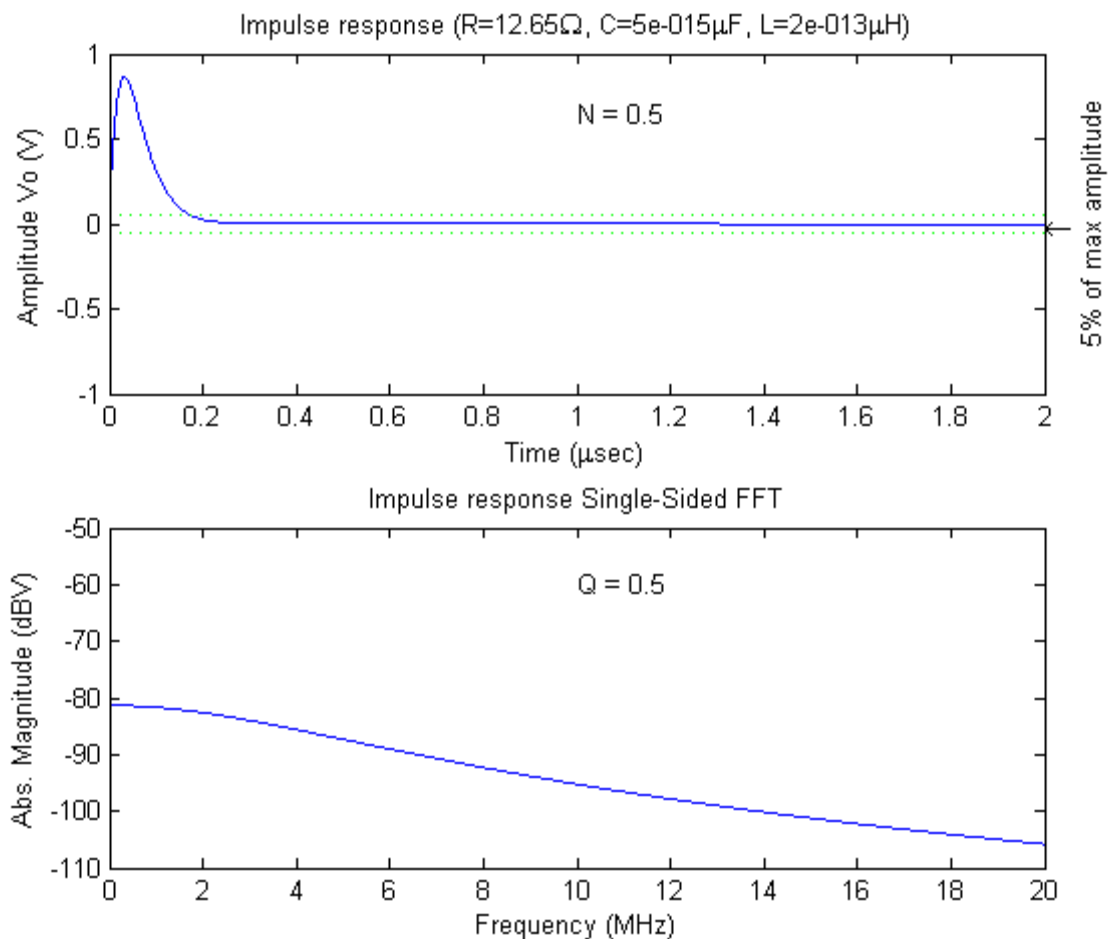
Fig. 5. Impulse response of a critical-damped RLC circuit and its FFT.

7. Examples

Figs. 6 to 13 are some examples of the impulse responses of series RLC circuits with various *R*, *L*, and *C* values. By comparing the time- and frequency-domain plots, we can see that $N \approx Q$, where *N* is the number of observed rings before the oscillations essentially disappear (i.e. fall below 5% of the initial peak value).
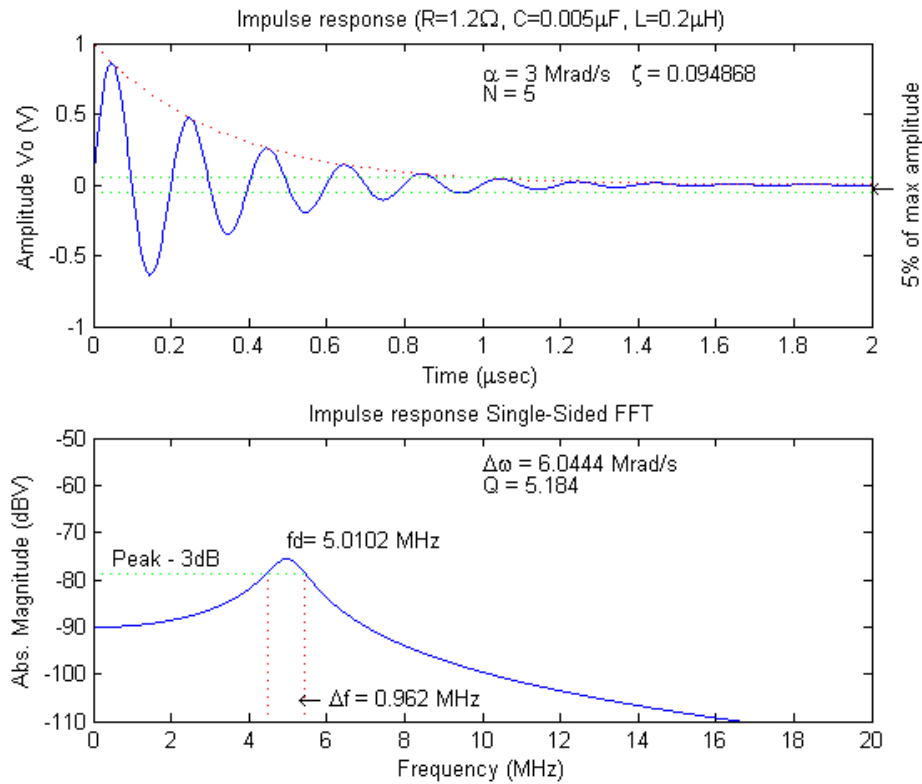

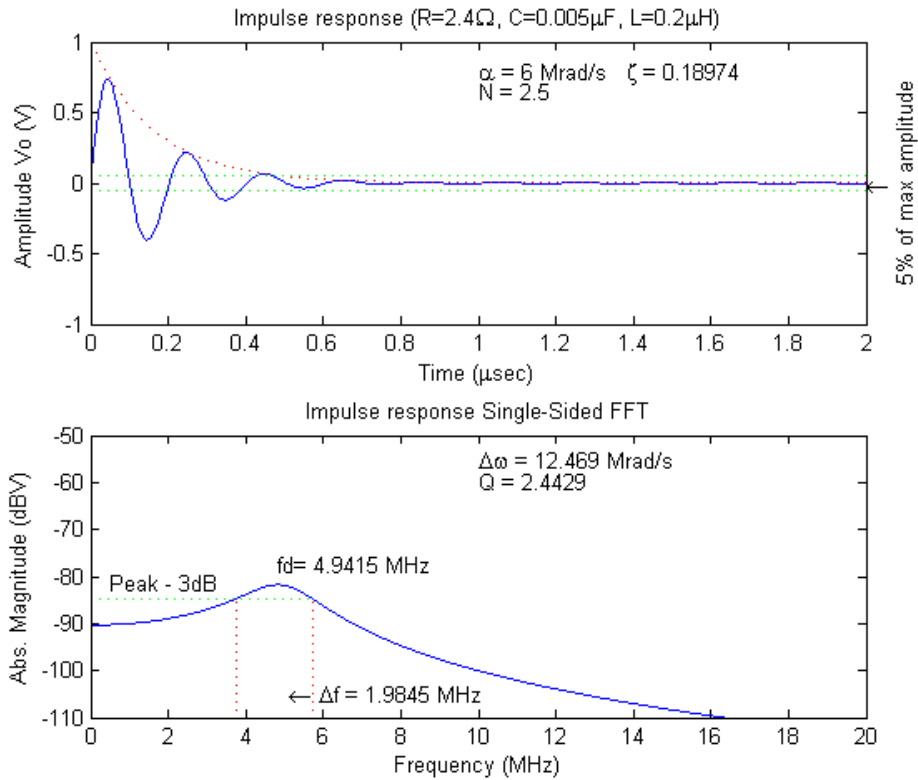Fig. 6. Impulse response of an RLC circuit with R, L, and C values of 1.2 Ω, 0.005 µF and 0.2 µH.

Fig. 7. Impulse response of an RLC circuit with R, L, and C values of 2.4 Ω, 0.005 µF and 0.2 µH.
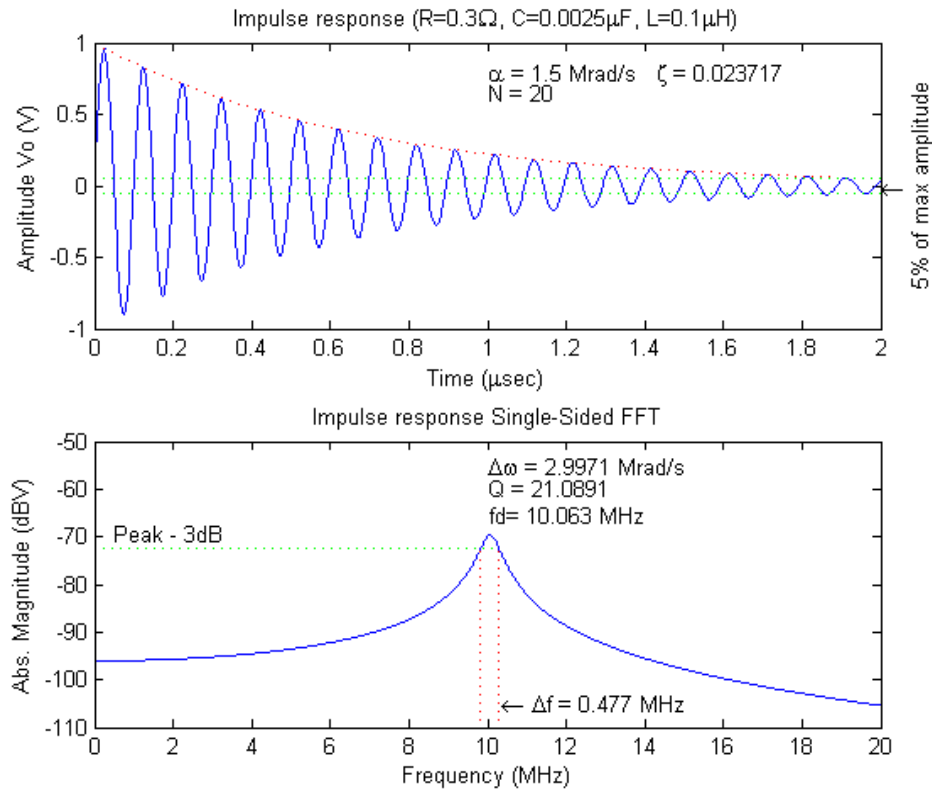
Fig. 8. Impulse response of an RLC circuit with R, L, and C values of 0.3 Ω, 0.0025 µF and 0.1 µH.
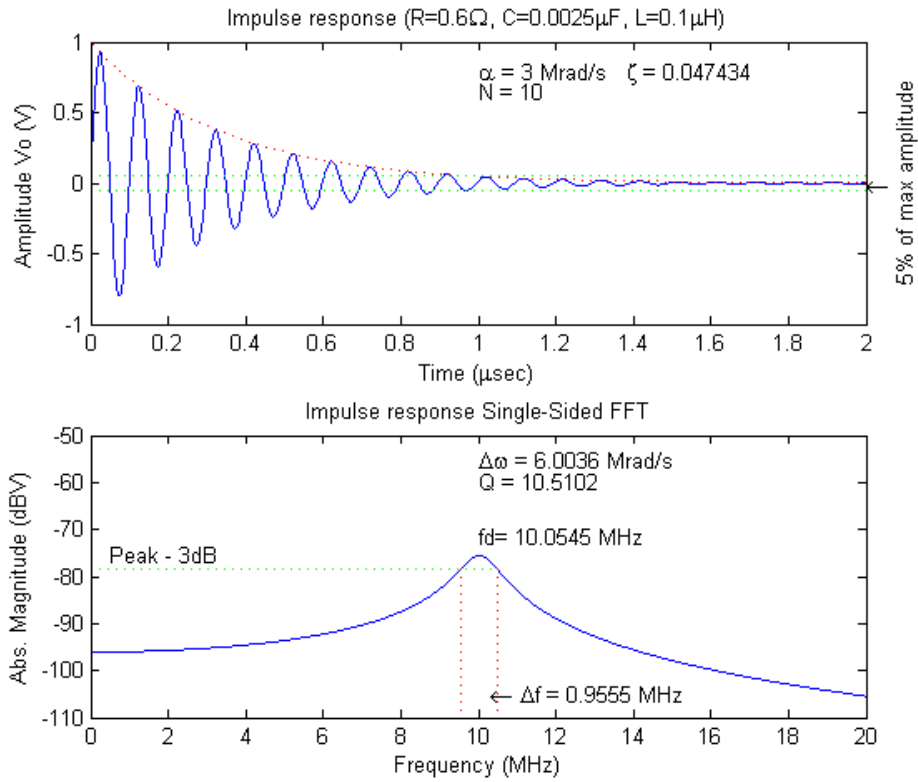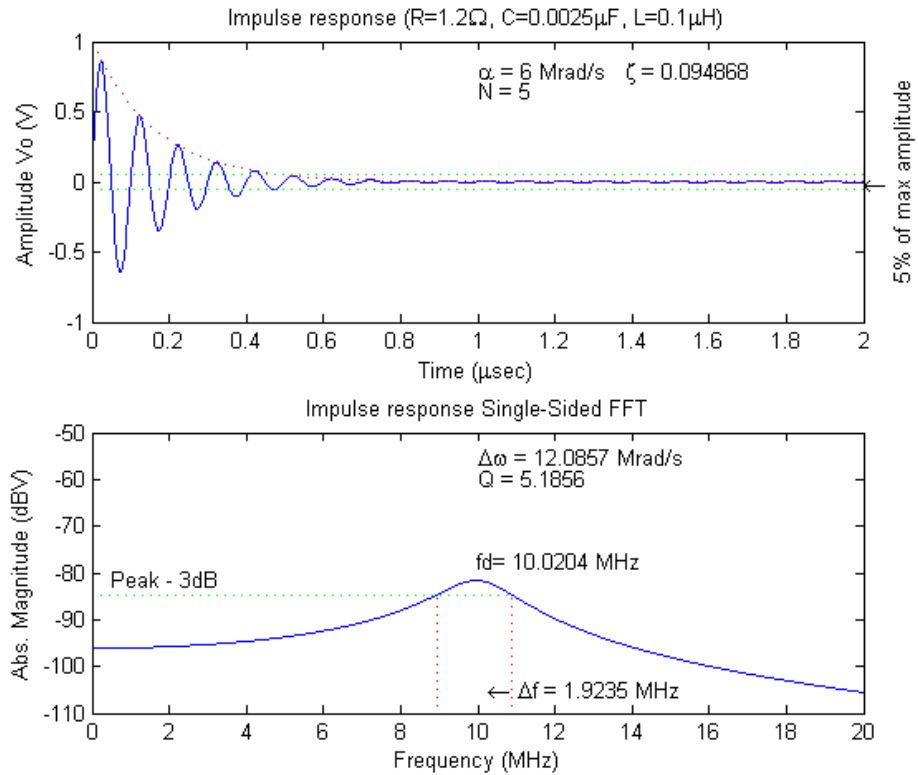
Fig. 9. Impulse response of an RLC circuit with R, L, and C values of 0.6 Ω, 0.0025 µF and 0.1 µH.

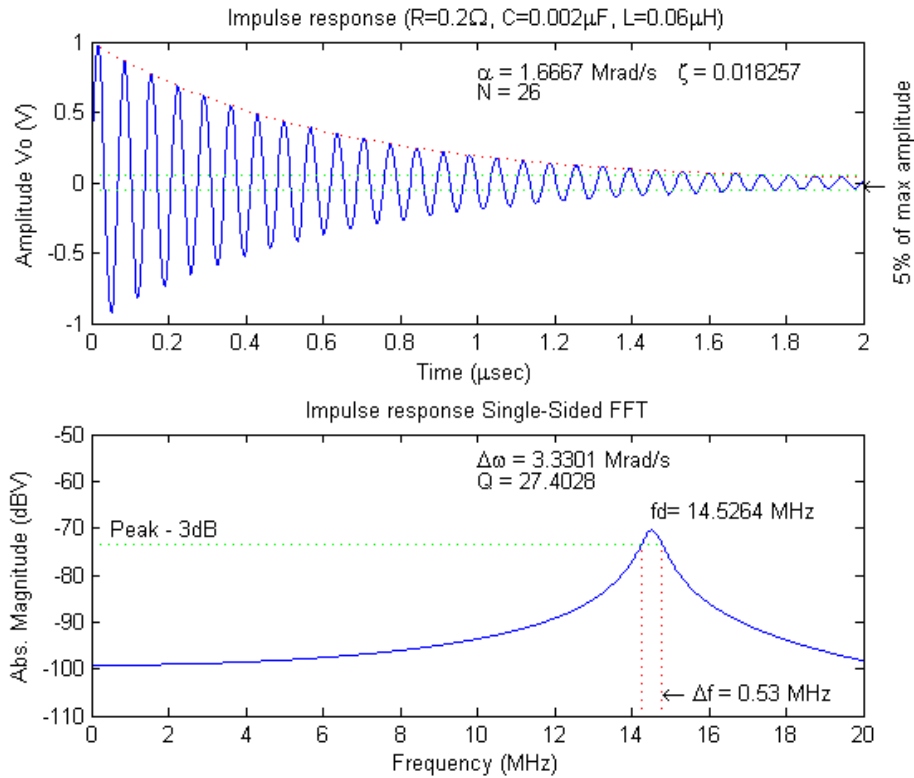Fig. 10. Impulse response of an RLC circuit with R, L, and C values of 1.2 Ω, 0.0025 µF and 0.1 µH.

Fig. 11. Impulse response of an RLC circuit with R, L, and C values of 0.2 Ω, 0.002 µF and 0.06 µH.
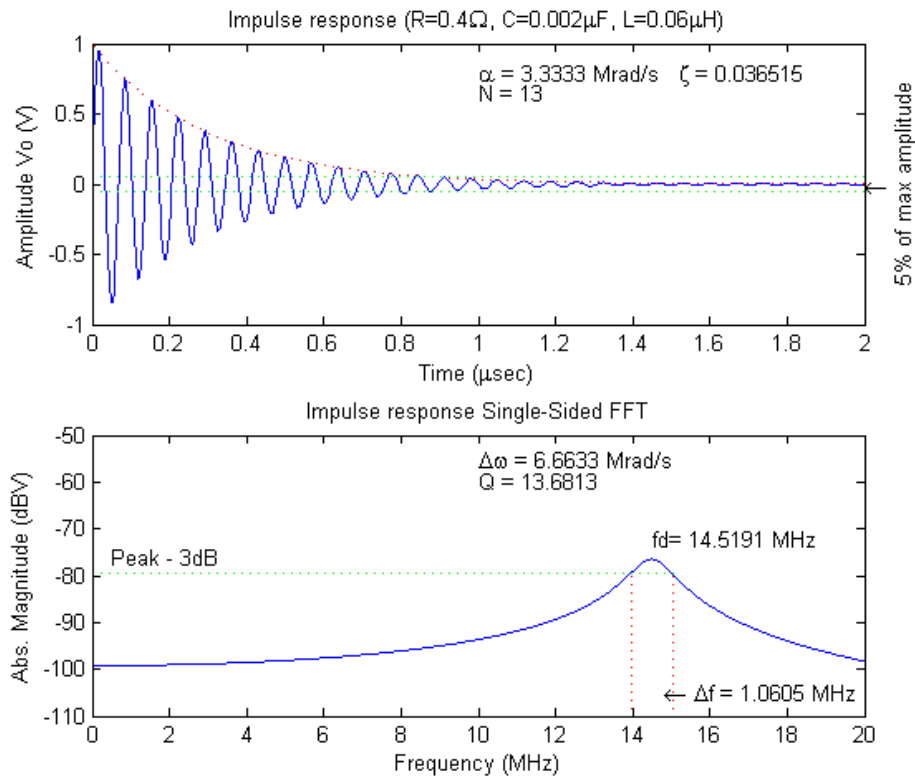

Fig. 12. Impulse response of an RLC circuit with R, L, and C values of 0.4 Ω, 0.002 µF and 0.06 µH.
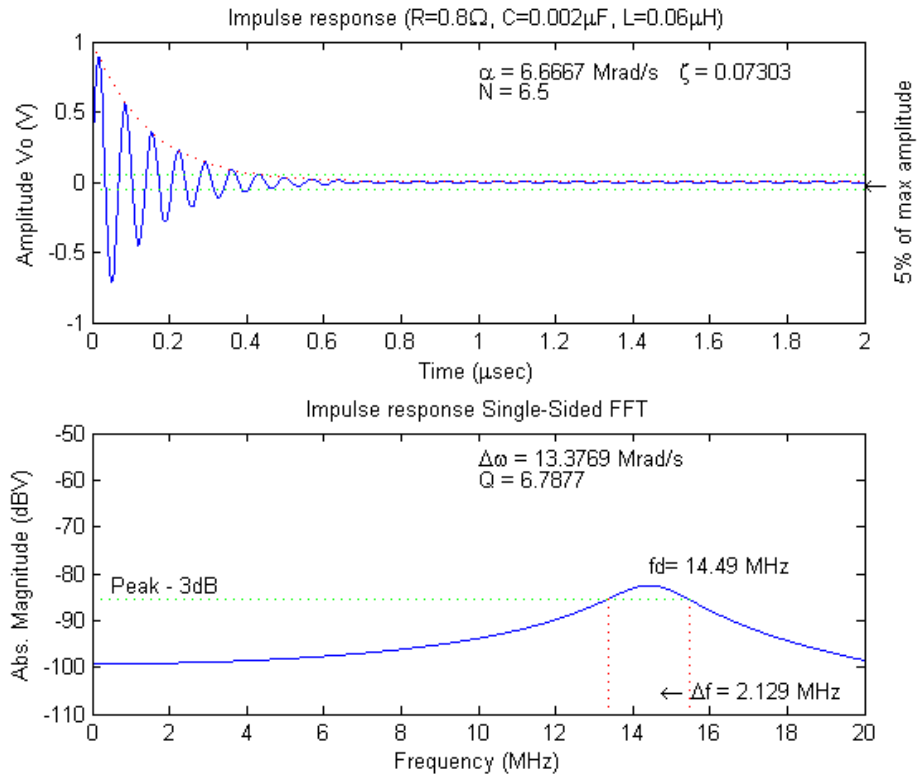
Fig. 13. Impulse response of an RLC circuit with R, L, and C values of 0.8 Ω, 0.002 µF and 0.06 µH.

References

[1]   Q-factor, Wikipedia, http://en.wikipedia.org/wiki/Q_factor.

[2]   James W. Nilsson, Susan A Riedel, "Electric Circuits" *Third Edition*, pp. 286-309, 2008.

[3]   Katsuhiko Ogata, "Modern Control Engineering" *Fourth Edition*, pp. 224-238, 2002.