A Novel Preconditioning Technique and Comparison of Three Formulations for Hybrid FEM/MoM Methods

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***Abstract* – Hybrid FEM/MoM methods combine the finite element method (FEM) and the method of moments (MoM) to model inhomogeneous unbounded problems. These two methods are coupled by enforcing field continuity on the boundary that separates the FEM and MoM regions. There are three ways of formulating hybrid FEM/MoM methods: outward-looking formulations, inward-looking formulations and combined formulations. In this paper, the three formulations are compared in terms of computer-resource requirements and stability for four sample problem geometries. A novel preconditioning technique is developed for the outward-looking formulation. This technique greatly improves the convergence rate of iterative solvers for the types of problems investigated in this study.**

***Index Terms*: FEM, MoM, EMC, sparse matrix, permutation, preconditioning, iterative solvers.**

I. INTRODUCTION

Hybrid FEM/MoM methods, which are also referred to as FE-BI, FE-MM, or FEM/BEM in the literature, combine the finite element method (FEM) and the method of moments (MoM) to model inhomogeneous unbounded problems. FEM is used to analyze the details of the structure and MoM is employed to terminate the FEM meshes and to provide an exact radiation boundary condition (RBC). These two methods are coupled by enforcing tangential-field continuity on the boundary separating the FEM and MoM regions. Hybrid FEM/MoM techniques were introduced in the early seventies by Silvester and Hsieh [1], and McDonald and Wexler [2] as attempts to apply FEM to model unbounded radiation problems. FEM/MoM was not widely used until the late eighties due to its large computational requirements. Yuan [3], and Jin and Volakis [4], [5] were among the first to apply FEM/MoM to 3D electromagnetic problems using vector basis functions. Angélini *et al*. [6], and Antilla and Alexopoulos [7] later applied FEM/MoM to 3D scattering in anisotropic media.

FEM/MoM has been used to analyze electromagnetic compatibility (EMC) problems since the mid-nineties. Ali *et al*. [8] employed FEM/MoM to analyze scattering and radiation from structures with attached wires. Shen and Kost [9] used FEM/MoM to analyze EMC problems in power cable systems. FEM/MoM has also been utilized to model thin shielding sheets and microstrip lines [10], [11]. Electronic devices with printed circuit boards (PCBs) are usually composed of many detailed structures: dielectrics, traces, cables, holes and vias. MoM is not well suited to model this kind of complex geometry efficiently. With a hybrid FEM/MoM technique, the details of a printed circuit board can be modeled using FEM and an exact radiation boundary can be provided using MoM to terminate the FEM meshes. When the structure has long cables, a FEM/MoM method is particularly efficient because the cables can be modeled by MoM without meshing the empty space around the cable.

There are three formulations for hybrid FEM/MoM methods [12]-[14]. The first formulation constructs an RBC using MoM and incorporates the RBC into the FEM equations. The second formulation derives an RBC from FEM and incorporates the RBC into the MoM equations. The third formulation combines the FEM and MoM matrix equations to form a large matrix equation and solves for all unknowns simultaneously. The first and second formulations are referred to as *outward-looking* and *inward-looking*, respectively, in [13], [14]. The last formulation is referred to as the *combined formulation* in this paper.

This paper compares the three formulations for hybrid FEM/MoM methods and presents a novel preconditioning technique that can be applied to outward-looking formulations. Section II describes the matrix equations generated by FEM/MoM. Section III introduces four sample problems used to compare the three formulations. In Section IV, preconditioning and permutation techniques are presented. Section V presents the outward-looking formulation and the new preconditioning technique. The inward-looking formulation is described in Section VI. Section VII presents the combined formulation. Section VIII compares the three formulations in terms of computer resource requirements. Finally, conclusions are drawn in Section IX.

II. MATRIX EQUATIONS GENERATED BY FEM/MoM

Full-wave hybrid FEM/MoM methods are well suited for solving problems that combine small complex structures and large radiating conductors. The original problem must be divided into an exterior equivalent problem and an interior equivalent problem. MoM is used to model the exterior equivalent problem and FEM is employed to analyze the interior equivalent problem. The two equivalent problems are related by enforcing the continuity of tangential fields on the boundary separating the FEM and MoM regions [14]-[16].

The electric-field integral-equation (EFIE) is generally used to describe the exterior equivalent problem [17],

$$\mathbf{E}^{inc}(\mathbf{r}) = \frac{1}{2}\mathbf{E}(\mathbf{r}) + \fint_S \left\{ \begin{array}{l} \mathbf{M}(\mathbf{r}')\times\nabla' G_0(\mathbf{r},\mathbf{r}') + j k_0 h_0 \mathbf{J}(\mathbf{r}') G_0(\mathbf{r},\mathbf{r}') \\ - j\frac{h_0}{k_0}\nabla'\bullet\mathbf{J}(\mathbf{r}')\nabla' G_0(\mathbf{r},\mathbf{r}') \end{array} \right\} dS' \quad (1)$$

where k_0 and h_0 are the wavenumber and the intrinsic wave impedance in free-space, and S is the surface enclosing the exterior equivalent problem. The integral term with a bar in Equation (1) denotes a principal-value integral. The singularity at $\mathbf{r}=\mathbf{r}'$ is excluded. The three-dimensional homogeneous Green's function is given by,

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4p|\mathbf{r}-\mathbf{r}'|}\ . \quad (2)$$

If S is a closed surface, the EFIE is not immune to false interior resonances [15], [17], [18]. If the interior resonances cause serious problems, the combined field formulation may be employed [12], [18].

Triangular basis functions (RWG functions) [19] may be employed to approximate surface fields. A Galerkin procedure can be used to test Equation (1). The resulting MoM matrix equation follows [8],

$$\begin{bmatrix} C_{hh} & C_{hc} \\ C_{ch} & C_{cc} \end{bmatrix}\begin{bmatrix} J_h \\ J_c \end{bmatrix} = \begin{bmatrix} D_{hd} & 0 \\ D_{cd} & 0 \end{bmatrix}\begin{bmatrix} E_d \\ 0 \end{bmatrix} - \begin{bmatrix} F_h \\ F_c \end{bmatrix} \quad (3)$$

where $\{J_h\}$ and $\{J_c\}$ are sets of unknowns for the electric current densities on the dielectric surface and perfect-electric-conductor (PEC) surface, respectively; $\{E_d\}$ is a set of unknowns for the electric field on the dielectric surface; C_{hh}, C_{hc}, C_{ch}, C_{cc}, D_{hd} and D_{cd} are dense coefficient matrices; F_h and F_c are source terms. The matrix formed by C_{hh}, C_{hc}, C_{ch} and C_{cc} in Equation (3) is called the MoM matrix or matrix $\boldsymbol{C}$ in this paper.

The interior equivalent problem is modeled using FEM. The goal is to solve the weak form of the vector wave equation as follows [14], [20]. (This equation can also be derived using a variational approach [16], [21].)

$$\int_{V_1}\left[\left(\frac{\nabla\times\mathbf{E}(\mathbf{r})}{j w m_0 m_r}\right)\bullet(\nabla\times\mathbf{w}(\mathbf{r})) + j w \in_0 \in_r \mathbf{E}(\mathbf{r})\bullet\mathbf{w}(\mathbf{r})\right] dV$$

$$= \int_{S_1}(\hat{\mathbf{n}}\times\mathbf{H}(\mathbf{r}))\bullet\mathbf{w}(\mathbf{r})\, dS - \int_{V_1}\mathbf{J}^{int}(\mathbf{r})\bullet\mathbf{w}(\mathbf{r})\, dV \quad (4)$$

where S_1 is the surface enclosing the interior equivalent problem, $\mathbf{w}(\mathbf{r})$ is the weighting function, and $\mathbf{J}^{int}$ is an impressed source. Vector tetrahedral elements [22] can be used to approximate the E field. A Galerkin procedure can be used to test Equation (4). The resulting FEM matrix equation follows [8],

$$\begin{bmatrix} A_{ii} & A_{id} \\ A_{di} & A_{dd} \end{bmatrix}\begin{bmatrix} E_i \\ E_d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_{dh} \end{bmatrix}\begin{bmatrix} 0 \\ J_h \end{bmatrix} + \begin{bmatrix} g_i \\ g_d \end{bmatrix} \quad (5)$$

where $\{E_i\}$ and $\{E_d\}$ are sets of unknowns for the electric field within the FEM volume and on the dielectric surface, respectively; $\{J_h\}$ is a set of unknowns for the electric current density on the dielectric surface; A_{ii}, A_{id}, A_{di}, A_{dd} and B_{dh} are sparse coefficient matrices; g_i and g_d are source terms. The matrix formed by A_{ii}, A_{id}, A_{di}, and A_{dd} in Equation (5) is called the FEM matrix or matrix $\boldsymbol{A}$ in this paper. Both the FEM and the MoM matrices are symmetric. Note that neither the FEM matrix equation nor the MoM matrix equation can be solved independently. They are coupled through the J_h and E_d terms.

One objective of this study is to determine which formulation works best for various problems. A coupling index, ρ, is defined in this paper as follows,

$$\rho = \frac{\text{Number of FEM unknowns}}{\text{Number of MoM unknowns}}\ . \quad (6)$$

The value of ρ is determined by the problem geometry and how it is meshed. As shown in later sections, the coupling index ρ can be used as a rough measure to determine which formulation is preferred for a given problem.

III. SAMPLE PROBLEMS

Four sample problems are presented to compare the outward-looking, inward-looking and combined formulations and to validate the preconditioning techniques discussed in later sections. Three of the problems include PCB structures, which are key elements of devices that are frequently modeled by EMC and signal integrity (SI) engineers. Each of these three problems has a thin rectangular shape and presents a unique challenge. The remaining problem has a spherical shape and provides a contrast to the PCB-like structures.

A. Problem 1: A PCB Power Bus Structure

The first problem is to model the input impedance of a PCB power bus structure. As shown in Figure 1, the board dimensions are 5 cm × 5 cm × 1.1 mm. The top and bottom planes are PECs. The dielectric between the PEC layers has a relative dielectric constant of 4.5. A source is placed in the middle of the board between the planes. The MoM boundary is chosen to coincide with the physical boundary of the board. The E fields tangential to the top and bottom planes are zero, thus no E-field unknowns are assigned on the two planes and the number of FEM unknowns is small. Table 1 summarizes the discretization of this problem and the other problems presented in this section.

B. Problem 2: Scattering from a Dielectric Sphere

The second problem is to model the scattering fields from a dielectric sphere. As illustrated in Figure 2, the radius of the sphere is 0.15λ. The relative dielectric constant of the sphere material is 4.5. The incident wave travels along the z-axis. The polarization of the E field is along the x-axis. The goal is to model the far fields. The discretization of this problem is summarized in Table 1.

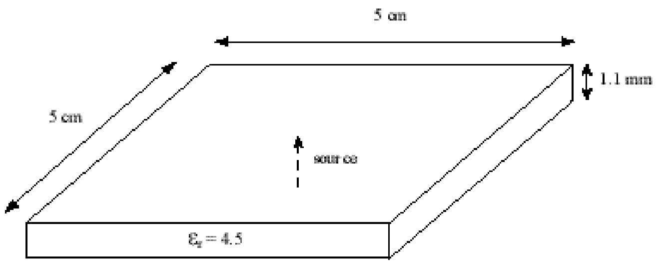

Figure 1. A PCB power bus structure.

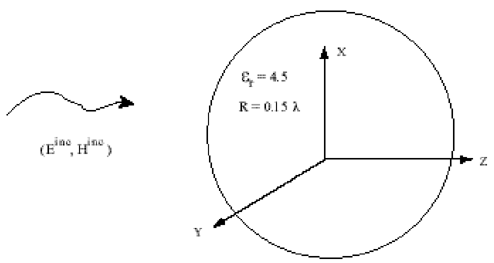

Figure 2. Scattering from a dielectric sphere.

C. Problem 3: A Gapped Power Bus Structure

The third problem is to model a gapped power bus structure. As shown in Figure 3, the board dimensions are 152.4 mm $\times$ 101.6 mm $\times$ 2.39 mm. The board has a solid PEC plane on the bottom and a gapped PEC plane on the top. The dielectric between the top and bottom planes has a relative permittivity of 4.5. The gap is 5.1 mm wide and located in the center of the top plane. The discretization of this problem is summarized in Table 1. This board is much larger than the board in Problem 1. A fine mesh is used in the vicinity of the gap. To reduce the number of MoM elements, the MoM boundary is placed 9.56 mm above the gap, resulting in a large number of FEM unknowns.

D. Problem 4: A Microstrip Line

The fourth problem is to model the behavior of a microstrip line. The board dimensions are 5 cm $\times$ 5 cm $\times$ 1.1 mm as shown in Figure 4. The bottom is a solid PEC plane. The trace placed on the top plane is 3 cm long and 0.5 mm wide. The dielectric has a relative permittivity of 4.5. The goal of this problem is to determine the input impedance of the microstrip line at one end when the other end is terminated by a resistor. The discretization of this problem is summarized in Table 1. To reduce the number of boundary elements, the MoM boundary is placed 3.3 mm above the microstrip line. A fine FEM mesh is required near the vicinity of the microstrip line as shown in Figure 5. As a result, this problem has a large coupling index.

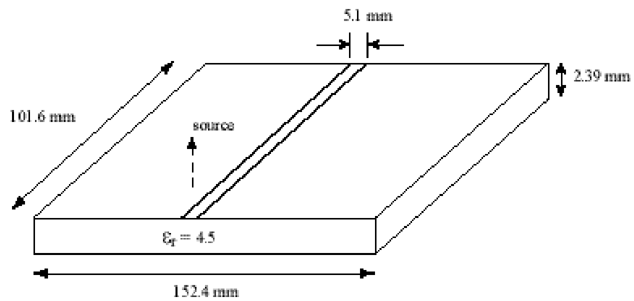

Figure 3. Configuration of a gapped power bus structure.

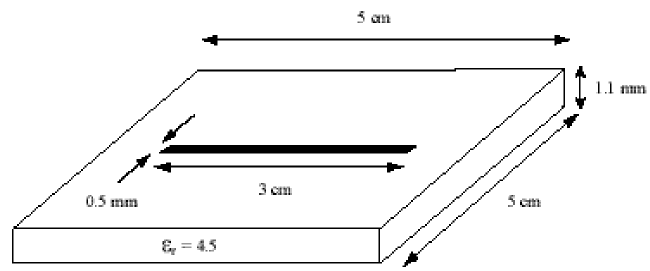

Figure 4. A microstrip line configuration.

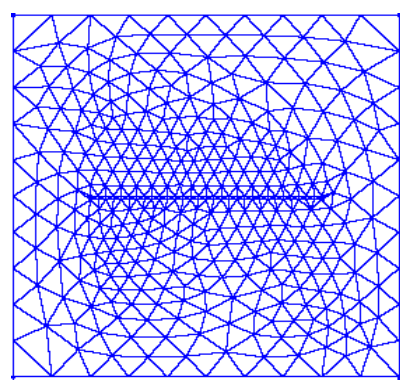

Figure 5. The FEM mesh in the plane of the trace.

Table 1. Summary of the discretization of the four sample problems

	# of FEM unknowns		# of MoM unknowns		Total # of unknowns	Coupling index ρ
	E_i	E_d	J_h	J_c		
Problem 1	402	80	80	575	1,137	0.74
Problem 2	699	612	612	0	1,923	2.14
Problem 3	4,521	1,223	1,223	454	7,421	3.43
Problem 4	2,277	360	360	136	3,133	5.32

IV. TECHNIQUES FOR SOLVING SPARSE MATRIX EQUATIONS

A. Preconditioning

Iterative solvers are widely used to solve large sparse matrix equations of the form,

$$Mx = b \tag{7}$$

where M is a square matrix and b and x are column vectors. b is the source vector and x is the unknown vector. Equation (7) is also called a *linear system*.

To have a non-trivial solution, the matrix M must be non-singular (det(M)≠0). The convergence rate of iterative solvers depends mainly on the condition number of the matrix M, which is defined as [14],

$$K(M) = \sqrt{\frac{l_{\max}}{l_{\min}}} \tag{8}$$

where $l_{\min}$ and $l_{\max}$ are the smallest and largest eigenvalues of the matrix $M^H M$, where M^H is the transpose conjugate of M. The condition number provides a measure of the spectral properties of a matrix. The identity matrix has a condition number of 1.0. A singular matrix has a condition number of infinity. A matrix with a large condition number is nearly singular, and is called *ill-conditioned*. An ill-conditioned linear system is very sensitive to small changes in the matrix. Iterative solvers may not converge smoothly, or may even diverge when applied to ill-conditioned systems.

The coefficient matrices generated by FEM and MoM usually have very large condition numbers. It may be difficult to apply iterative solvers to the original FEM and MoM matrix equations. However, a linear system can be transformed into another linear system so that the new system has the same solution as the original one, but has better spectral properties. For instance, both sides of Equation (7) can be multiplied by a square matrix P^{-1},

$$P^{-1}Mx = P^{-1}b \tag{9}$$

where P has the following properties,

(A) $K(P^{-1}M) << K(M)$

(B) $\det(P^{-1}M) \neq 0$

(C) It is inexpensive to solve $Px = b$.

Such a matrix P is called a *preconditioner*. This technique is called *preconditioning*. Condition (A) assures favorable spectral properties for the new linear system. Condition (B) guarantees that the new system, Equation (9), has the same non-trivial solution as Equation (7). Condition (C) is essential to ensure the efficiency of preconditioned iterative solvers. In preconditioned iterative algorithms, it is not necessary to solve P^{-1} explicitly. Instead, a linear system of the form $Px = b$ is solved at each step.

If the preconditioner P is chosen to be M, $P^{-1}M$ becomes an identity matrix. However, finding M^{-1} is generally more difficult than solving Equation (7). It is more practical to find a preconditioner P that is an approximation of M, and satisfies all three conditions. There is a trade-off between the cost of constructing and applying the preconditioner, and the gain in the convergence rate [23].

LU factorization and incomplete LU (ILU) factorization are commonly used to construct preconditioners. LU factorization decomposes a matrix M into a lower triangular matrix L and an upper triangular matrix U, which satisfy,

$$M = LU\,. \tag{10}$$

ILU factorization ([23], [24]), decomposes matrix M into a lower triangular matrix L and an upper triangular matrix U so that the residue matrix $R = M\text{-}LU$ is subject to certain constraints, such as levels of fill-in, or drop tolerance.

B. Permutation

Because the FEM matrix, $\boldsymbol{A}$, is sparse, LU factorization may generate a lot of *fill-in elements*, which refer to matrix entries that are zero in the matrix $\boldsymbol{A}$ but are non-zero in the L and U matrices [24]. *Permutation* is a technique that can be used to reduce the number of fill-ins in LU factorization by reordering the matrix. Generally, a symmetric permutation on matrix M is defined as follows [24],

$$M_P = P M P^T \tag{11}$$

where M_P is the new matrix after permutation and P is the permutation matrix. P is a unitary matrix [24], which satisfies,

$$P^{-1} = P^T\ . \tag{12}$$

Figure 6 illustrates the sparsity pattern of the original FEM matrix in Problem 1. The number of unknowns in the FEM matrix is 482. A fully populated matrix has 482×482 = 232,324 entries. Figure 6 shows only 3,772 non-zero entries. The percentage of non-zero elements is 1.6%, indicating that the FEM matrix is highly sparse. Figure 7 illustrates the sparsity patterns of the L and U matrices after applying LU factorization to the FEM matrix in Problem 1. The data in Figure 7 was generated using MATLAB® [25]. The L matrix obtained by MATLAB is a "psychologically lower triangular matrix" (i.e. a product of lower triangular and permutation matrices) [26]. This explains why the L matrix is not a strictly lower triangular matrix. The total number of non-zero entries in L and U is 34,640 + 35,379 = 70,019. The total number of fill-ins is 70,019-3772 = 66,247.

The *reverse Cuthill-McKee algorithm* can be used to minimize the bandwidth of a matrix [16], [27]. Bandwidth reduction techniques are useful because they save both storage and operation counts in LU factorization. Figure 8 shows the sparsity pattern of the FEM matrix in Figure 6 after performing a symmetric reverse Cuthill-McKee permutation. Figure 9 illustrates the sparsity patterns of the

L and *U* matrices after the permutation. The number of fill-ins is 10,457+12,457 – 3,772 = 19,142. Compared with Figure 7, the number of fill-ins has been reduced by 71%.

The *minimum degree permutation* is a complicated and powerful technique that has many advantages over other permutation techniques [16], [26]. One widely used implementation was proposed by George and Liu [28]. This technique reduces fill-ins during Gaussian elimination based on graph theory [16], [29]. In this study, the authors used the symmetric minimum degree permutation provided by MATLAB®. Figure 10 shows the sparsity pattern of the FEM matrix in Figure 6 after performing the symmetric minimum degree permutation. Figure 11 illustrates the sparsity patterns of the *L* and *U* matrices after performing the symmetric minimum degree permutation. The number of fill-ins is 7,901+9,628 – 3,772 = 13,757. Compared with Figure 7, the number of fill-ins has been reduced by 79%.

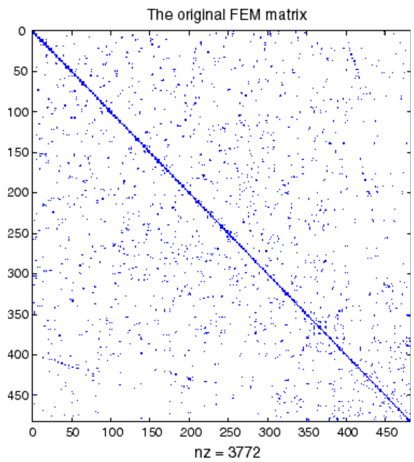

Figure 6. Sparsity pattern of Problem 1 FEM matrix ("nz" is # of non-zero entries).

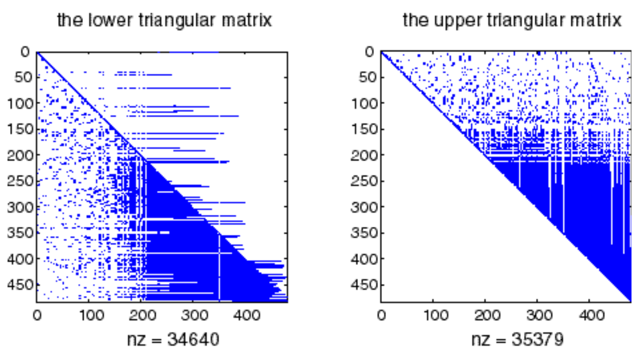

Figure 7. Sparsity pattern of the Problem 1 *L* and *U* matrices after LU factorization

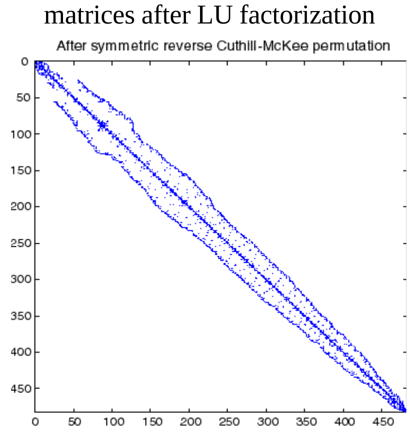

Figure 8. Sparsity pattern of the Problem 1 FEM matrix after symmetric reverse Cuthill-McKee permutation.

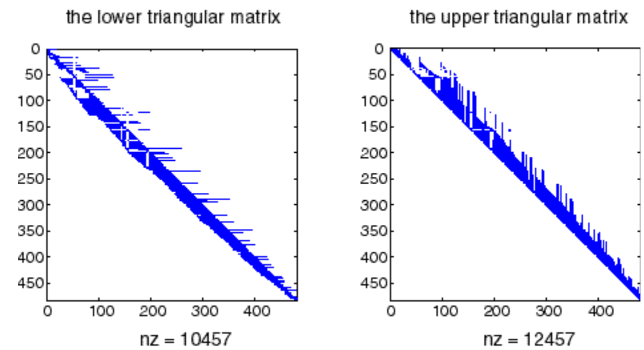

Figure 9. Sparsity pattern of the Problem 1 *L* and *U* matrices after symmetric reverse Cuthill-McKee permutation.

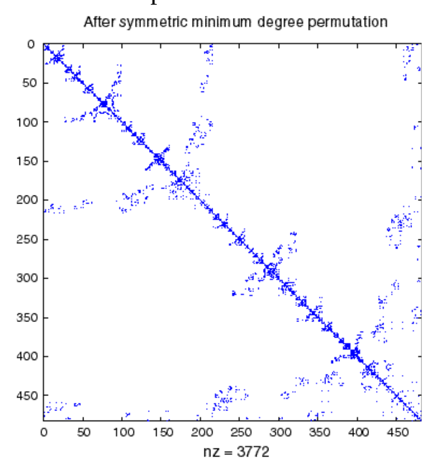

Figure 10. Sparsity pattern of the Problem 1 FEM matrix after symmetric minimum degree permutation.

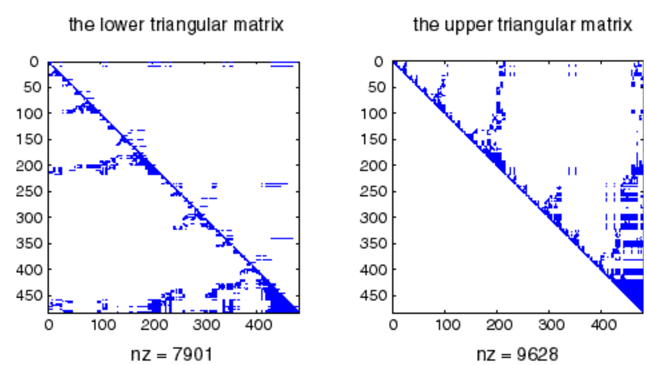

Figure 11. Sparsity pattern of the Problem 1 *L* and *U* matrices after symmetric minimum degree permutation.

V. THE OUTWARD-LOOKING FORMULATION AND A NOVEL PRECONDITIONING TECHNIQUE

The outward-looking formulation uses the coefficients of the electric field expansion in the interior equivalent problem, E_i and E_d in Equation (5), as the primary unknowns in the final matrix equation. This formulation has been employed by Paulsen *et al.* [31], Jin and Volakis [32], and Ramahi and Mittra [33].

From Equation (3), the following equations can be derived,

$$C_{ch} J_h + C_{cc} J_c = D_{cd} E_d - F_c$$

$$\Rightarrow \quad J_c = C_{cc}^{-1}(D_{cd} E_d - C_{ch} J_h - F_c) \tag{13}$$

$$C_{hh} J_h + C_{hc} J_c = D_{hd} E_d - F_h \quad . \tag{14}$$

Substituting Equation (13) into Equation (14) gives,

$$\left(C_{hh} - C_{hc} C_{cc}^{-1} C_{ch}\right) J_{hh}$$
$$= (D_{hd} - C_{hc} C_{cc}^{-1} D_{cd}) E_d + C_{hc} C_{cc}^{-1} F_c - F \ . \tag{15}$$

To save computation time and memory, the following intermediate terms are introduced,

$$N_{hc} \equiv C_{hc} C_{cc}^{-1} \tag{16}$$

$$C'_{hh} \equiv \left(C_{hh} - N_{hc} C_{ch}\right)^{-1} \tag{17}$$

$$D'_{hd} \equiv D_{hd} - N_{hc} D_{cd} \tag{18}$$

$$K_h \equiv N_{hc} F_c - F_h \ . \tag{19}$$

Equation (15) can now be written as,

$$J_h = C'_{hh}(D'_{hd} E_d + K_h) \ . \tag{20}$$

Substituting Equation (20) into Equation (5) gives,

$$\left([A] + \begin{bmatrix} 0 & 0 \\ 0 & -B_{dh} C'_{hh} D'_{hd} \end{bmatrix} \right) \begin{bmatrix} E_i \\ E_d \end{bmatrix}$$
$$= \begin{bmatrix} g_i \\ g_d \end{bmatrix} + \begin{bmatrix} 0 \\ B_{dh} C'_{hh} K_h \end{bmatrix} \tag{21}$$

where the matrix A is the FEM matrix. Matrix A_c, A', and vector b are defined as follows,

$$A_c \equiv \begin{bmatrix} 0 & 0 \\ 0 & -B_{dh} C'_{hh} D'_{hd} \end{bmatrix} \tag{22}$$

$$A' \equiv A + A_c \tag{23}$$

$$b \equiv \begin{bmatrix} g_i \\ g_d \end{bmatrix} + \begin{bmatrix} 0 \\ B_{dh} C'_{hh} K_h \end{bmatrix} \tag{24}$$

$$x \equiv \begin{bmatrix} E_i \\ E_d \end{bmatrix} \ . \tag{25}$$

Equation (21) now becomes,

$$A' x = b \ . \tag{26}$$

Equation (26) is a fully determined system and is the final matrix equation to solve. Note that the order of this linear system is the same as the order of the original FEM matrix. The Bi-Conjugate Gradient Stabilized (BiCGSTAB) method [23], [24], can be used to solve Equation (26). Although BiCGSTAB requires less memory than direct solvers such as the Gaussian elimination method, it may have difficulty converging, or may even diverge. Preconditioning techniques can be utilized to improve the efficiency and accuracy of BiCGSTAB. LU factorization can be employed to construct preconditioners.

As shown in Figure 12, most of the non-zero elements are located in the bottom-right corner of matrix A'. Table 2 summarizes the number of non-zero entries in A, A', and their LU factorizations. It is inefficient to perform LU factorization on A' because the computer resources required for factorization may exceed those required for an iterative solution.

In Equation (23), the entries in the matrix A_c have much smaller values than those in the matrix A for each of the sample problems. It seems reasonable to construct preconditioners from the matrix A instead of the matrix A'. Furthermore, the matrix A is sparse and symmetric, so the symmetric minimum degree permutation can be applied to reduce fill-ins in the LU factorization,

$$A_P = P A P^T \tag{27}$$

where P is the permutation matrix and A_P is the new matrix after permutation. Next, an LU factorization can be applied to A_P to obtain a lower triangular matrix L and an upper triangular matrix U,

$$A_P = LU. \tag{28}$$

Multiplying both sides of Equation (26) by P and combining with Equation (12) gives,

$$PA'P^T Px = Pb \ . \tag{29}$$

The following new terms are defined,

$$A'' \equiv P A' P^T \tag{30}$$

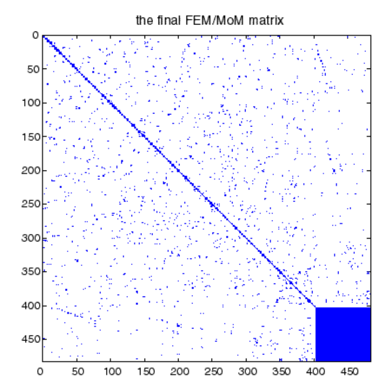

Figure 12. Sparsity pattern of Problem 1 A' in Equation (26).

$$y \equiv Px \quad \text{and} \quad b'' \equiv Pb \ . \tag{31}$$

Equation (29) becomes,

$$A''y = b'' \ . \tag{32}$$

Permutation does not change the condition number of a matrix,

$$K(A'') = K(A') \ . \tag{33}$$

Next, the preconditioners L and U can be applied to Equation (32),

$$(LU)^{-1} A'' y = (LU)^{-1} b'' \ . \tag{34}$$

Iterative solvers can be used to solve Equation (34). Note that it is not necessary to explicitly compute $(LU)^{-1}$ when using iterative solvers [23], [24]. After y is obtained, x can be calculated from Equation (31),

$$x = P^{-1} y = P^{T} y \ . \tag{35}$$

The technique discussed above was implemented using MATLAB®. Table 3 shows the condition number of A' in Equation (26) and $(LU)^{-1} A''$ in Equation (34) for all four sample problems. This preconditioning technique greatly reduced the condition number of the matrix A' and therefore improved the efficiency of the iterative solver.

Table 4 shows the solution times for each of the four problems using the un-preconditioned BiCGSTAB solver and the preconditioned BiCGSTAB solver. Only a small amount of time was spent constructing preconditioners. The preconditioning technique reduced the number of iterations by a factor ranging from 202 to 879, and achieved 15.9- to 149.6-fold improvements in the Equation (26) solution time. Table 5 examines the time spent on each step of the solution process for the four sample problems using the un-preconditioned solver and the preconditioned solver. For the first problem, there is not much difference between the un-preconditioned and the preconditioned solvers, because the time spent computing the matrix entries and on the coupling process is the dominant factor. For Problems 2, 3, and 4, the preconditioned solver yields 2.21-, 7.83- and 6.36- fold improvements, respectively. The bottom-right part of A' is dense as shown in Figure 12 and is scattered after A' is permuted as illustrated in Figure 13. This is not preferred because the locality of data in matrix A_c is destroyed and this has a negative effect on the efficiency of the iterative solver. BiCGSTAB only needs to compute the inner product between the matrix A' and the searching vector q. Because

$$A'q = Aq + A_c q \ , \tag{36}$$

it is not necessary to compute the matrix A' explicitly. The FEM matrix can be stored using the ITPACK format [16], and the bottom-right part of A_c can be stored in a two-dimensional array. Permutation is performed on the matrix A, vector q and $A_c q$ but the matrix A_c is not permuted. This storage scheme makes it unnecessary to keep track of the row and column information for every entry in A_c. Therefore, it uses much less computer memory than computing A' explicitly and storing A' as a sparse matrix.

Table 2. Non-zero elements in A, A′, and their LU factorizations

	nz(A)*	nz(A')	$\frac{\text{nz}(A)}{\text{nz}(A')}$ (%)	nz(L)+nz(U) $A = LU$ **	nz(L')+nz(U') $A' = L'U'$ ***
Problem 1	3,772	9,924	38%	17,175	33,488
Problem 2	17,745	389,229	4.6%	192,865	1,000,728
Problem 3	65,558	1,555,144	4.2%	983,322	2,962,187
Problem 4	36,135	163,829	22%	468,849	798, 028

* nz(A) refers to the number of non-zero elements in matrix A.
** After symmetric minimum degree permutation.
*** After symmetric Cuthill-McKee permutation.

Table 3. Outward-looking formulation condition numbers before and after preconditioning

	$K(A')$	$K((LU)^{-1} A'')$
Problem. 1	8.32×10^{6}	1.07
Problem. 2	4.27×10^{3}	18.7
Problem. 3	4.27×10^{7}	N/A*
Problem. 4	5.56×10^{7}	813

* Data not available due to excessive memory requirement.

Table 4. Solution times for Equation (26) using the un-preconditioned and preconditioned BiCGSTAB solvers (The drop tolerance for the BiCGSTAB solver is 1.0×10^{-3}.)

	LU Factorization (sec)	Iteration			Total (sec)	Improvement (fold)
		Number	Converged (Yes/No)	Time (sec)		
Problem 1 (orig.*)	N/A	202	Yes	2.03	2.03	
Problem 1 (new**)	0.03	1	Yes	0.09	0.12	15.9
Problem 2 (orig.*)	N/A	715	Yes	206.10	206.10	
Problem 2 (new**)	1.63	2	Yes	1.92	3.55	58.1
Problem 3 (orig.*)	N/A	5,096	Yes	6,037.90	6,037.9	
Problem 3 (new**)	12.06	9	Yes	28.03	40.09	149.6
Problem 4 (orig.*)	N/A	2,637	No	386.77	386.77	
Problem 4 (new**)	5.19	3	Yes	2.91	8.10	46.7

* "orig." refers to the un-preconditioned BiCGSTAB solver.
** "new" refers to the preconditioned BiCGSTAB solver.

Table 5. Time required to solve the four problems

	Compute matrix entries (sec)	Coupling Equations (13) - (21) (sec)	Original		Preconditioned		Improvement (%)
			Solving Eq. (17) (sec)	Total (sec)	Solving Eq. (29) (sec)	Total (sec)	
Problem 1	46.00	20.76	2.03	68.79	0.12	66.88	3%
Problem 2	48.00	40.23	206.10	294.22	3.55	91.78	221%
Problem 3	287.20	438.60	6,037.90	6,763.7	40.10	765.90	783%
Problem 4	40.12	11.33	386.77	438.22	8.10	59.55	636%

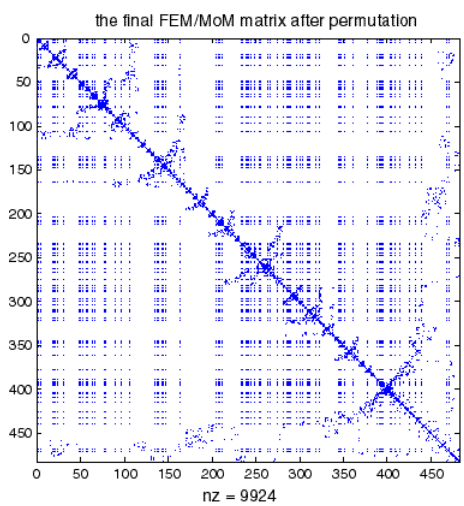

Figure 13. Sparsity pattern of Problem 1 A'' in Equation (32) after minimum degree permutation.

VI. THE INWARD-LOOKING FORMULATION

The inward-looking formulation chooses the coefficients of the equivalent surface current expansion in the exterior equivalent problem (J_h and J_c in Equation (3)) as the primary unknowns in the final matrix equation. This formulation has been implemented by Jin and Liepa [34], Yuan *et al.* [35], and Cangellaris and Lee [36].

From Equation (5), the following derivation can be made,

$$A_{ii}E_i + A_{id}E_d = g_i \Rightarrow E_i = A_{ii}^{-1}(g_i - A_{id}E_d) \tag{37}$$

$$A_{di}E_i + A_{dd}E_d = B_{dh}J_h + g_d \Rightarrow$$

$$(A_{dd} - A_{di}A_{ii}^{-1}A_{id})E_d = B_{dh}J_h + (g_d - A_{di}A_{ii}^{-1}g_i)\,. \tag{38}$$

To save computation time and memory, the following intermediate terms are introduced,

$$M_{dd} \equiv (A_{dd} - A_{di}A_{ii}^{-1}A_{id})^{-1} \tag{39}$$

$$N_{dh} \equiv M_{dd}B_{dh} \tag{40}$$

$$K_d \equiv M_{dd}(g_d - A_{di}A_{ii}^{-1}g_i)\,. \tag{41}$$

Equation (38) can be rewritten as,

$$E_d = N_{dh}J_h + K_d\,. \tag{42}$$

Substituting Equation (38) into Equation (3) gives the final matrix equation,

$$\begin{bmatrix} C_{hh} - D_{hd}N_{dh} & C_{hc} \\ C_{ch} - D_{cd}N_{dh} & C_{cc} \end{bmatrix} \begin{bmatrix} J_h \\ J_c \end{bmatrix} = \begin{bmatrix} D_{hd}K_d - F_h \\ D_{cd}K_d - F_c \end{bmatrix}. \tag{43}$$

Note that the order of this equation is the same as that of the MoM matrix. The inward-looking formulation inverts one sparse matrix A_{ii}, and one dense matrix $(A_{dd} - A_{di}A_{ii}^{-1}A_{id})$. Because the matrix in Equation (43) is dense, the Gaussian elimination method is used to solve the final matrix equation.

The outward-looking formulation is better suited to problems with a large number of FEM unknowns and fewer MoM unknowns. The inward-looking formulation is preferred for problems with a higher percentage of MoM unknowns. Of the four problems presented here, only Problem 1 has more MoM unknowns than FEM unknowns. As shown in Table 6, the inward-looking formulation is faster than the outward-looking formulation at solving Problem 1. However, the inward-looking formulation is not the best choice for the other three problems.

VII. THE COMBINED FORMULATION

The outward-looking and inward-looking formulations are computationally expensive because they invert two matrices. An alternative is to combine Equation (3) and Equation (5) to form the final matrix as follows,

$$\begin{bmatrix} A_{ii} & A_{id} & 0 & 0 \\ A_{di} & A_{dd} & -B_{dh} & 0 \\ 0 & -D_{hd} & C_{hh} & C_{hc} \\ 0 & -D_{cd} & C_{ch} & C_{cc} \end{bmatrix} \begin{bmatrix} E_i \\ E_d \\ J_h \\ J_c \end{bmatrix} = \begin{bmatrix} g_i \\ g_d \\ -F_h \\ -F_c \end{bmatrix} \qquad (44)$$

and solve for all unknowns simultaneously [14]. This is referred to as the *combined formulation* in this paper. It has become more popular recently and has been employed by Sheng *et al.* [18], Jankoviæ *et al.* [37], and Shen *et al.* [38].

The combined formulation does not require any matrix inversions. However, it generates a larger matrix equation. The order of the final matrix is equal to the sum of the orders of the FEM and MoM matrices. As shown in Table 7, the matrix in the combined formulation has a much larger condition number than the final matrix in the outward-looking formulation. Due to these large condition numbers, it can be very difficult to generate preconditioners using LU factorization or other preconditioning techniques. Consequently, iterative methods may not work well, especially when the MoM part is large. Table 8 lists the normalized residue of the solutions to Equation (44) for the four sample problems using the Bi-Conjugate Gradient (BiCG), BiCGSTAB and Generalized Minimal Residual (GMRES) methods [23]. None of them reaches the designated drop tolerance of 1.0×10^{-3}. Problem 2, which has a different geometry (a sphere) from the other three PCB problems, has a much smaller condition number and two of the iterative solvers converge to acceptable residues. This may explain why the authors in [18], [37] did not report convergence problems for the combined formulation. Shen *et al.* [38] showed that the ILU factorization with different fill-in levels worked very well for their applications. However, the problems presented in [38] have a large number of FEM unknowns (>16,000) and very few MoM unknowns (<200). The four sample problems presented in this paper have a higher percentage of MoM unknowns because the MoM boundary is applied closer to the object being modeled. For the four sample problems presented here, the ILU factorization technique fails to converge.

The Gaussian elimination method can also be used to solve Equation (44). However, a large number of fill-ins are generated during Gaussian elimination. To reduce the number of fill-ins, $\{E_i\}$ in Equation (44) can be permuted. However, permuting $\{E_d, J_d, J_c\}$ in Equation (44) destroys the data locality of the matrix C and D and therefore is not preferred.

VIII. COMPARING THE THREE FORMULATIONS

Table 9 lists the time required using each of the three formulations to solve the sample problems. The outward-looking formulation inverts two dense matrices and performs a lot of matrix multiplication. However, this formulation is excellent when the coupling index ρ is large, mainly because the preconditioning technique presented in Section III greatly reduces the time spent solving the final matrix equation. The inward-looking formulation excels when the coupling index ρ is small. It performs poorly when ρ is large because the inverse of the sparse FEM matrix is dense and the coupling process is time-consuming. The combined formulation was not optimum for any of the sample problems although it worked reasonably well for solving Problem 1 and Problem 2.

Table 10 lists the computer memory requirements for each of the three formulations. The outward-looking formulation required the least amount of memory. One reason for this was that BiCGSTAB was used to solve the final equation and the FEM matrix was stored as a sparse matrix. Another reason was that the symmetric minimum degree permutation significantly reduced the number of fill-ins when constructing preconditioners. For the inward-looking formulation, the inverse of the FEM matrix and the matrix in Equation (43) were dense, so this formulation required more memory than the outward-looking formulation. The inward-looking formulation required less or more memory than the combined formulation, depending on the value of ρ. The combined formulation required much more memory than the outward-looking formulation because the Gaussian elimination method was used to solve the matrix equation. The exact amount of time and memory required to solve a problem depends on many factors such as the mesh quality, the order of $\{E_i, E_d, J_h, J_c\}$, and the convergence rate of iterative solvers. The coupling index ρ can be used as a rough measure to determine which formulation is preferred. Based on the four sample problems and the authors' experience, the outward-looking formulation is preferred when ρ>2.0; the inward-looking formulation is preferred when ρ<1.5. The combined formulation is not preferred due to its large memory requirement (when using a Gaussian elimination solver), and its poor convergence rate (when using an iterative solver). The combined formulation is acceptable when the problem is not memory-constrained.

Depending on the type of problems being solved, the three formulations may exhibit instability problems. As pointed out by Pearson *et al.* [13] and Peterson *et al.* [14], the inward-looking formulation is susceptible to uniqueness difficulties. As shown in Equation (37), A_{ii}^{-1} must be computed. A_{ii} is essentially the FEM matrix for a closed cavity that might be resonant, which means A_{ii}^{-1} does not exist or is nearly singular at resonant frequencies. However, typical EMC problems that model the high-frequency loss present in the problem geometries are not likely to exhibit this instability.

Table 6. Comparison between the outward-looking and inward-looking formulations

	Compute matrix entries (sec)	Outward-looking (preconditioned BiCGSTAB)			Inward-looking (Gaussian elimination)		
		Coupling Equations (13) - (21) (sec)	Solving Equation (32) (sec)	Total (sec)	Coupling Equations (37) - (43) (sec)	Solving Equation (43) (sec)	Total (sec)
Problem 1	46.00	20.76	0.12	66.88	1.63	11.19	58.82
Problem 2	48.00	40.23	3.55	91.78	46.83	8.80	103.60
Problem 4	40.12	11.33	8.10	59.55	174.92	4.89	219.90

* Problem 3 is not listed in this table because the inward-looking formulation requires excessive computer memory.

Table 7. The condition number for the outward-looking and combined formulations without preconditioning

	K (LHS*) (Outward-looking)	K (LHS**) (Combined)
Problem 1	8.32×10^{6}	4.38×10^{10}
Problem 2	2.87×10^{3}	2.71×10^{7}
Problem 3	4.27×10^{7}	3.81×10^{11}
Problem 4	5.56×10^{7}	1.78×10^{11}

* LHS refers to the matrix on the left-hand side of Equation (26).
** LHS refers to the matrix on the left-hand side of Equation (44).

Table 8. Solutions to Equation (44) using iterative solvers without preconditioning
(The drop tolerance was 1.0×10^{-3}; the maximum iteration number was set to be the size of the matrix equation.)

	Normalized least residue			
	Problem 1	Problem 2	Problem 3	Problem 4
BiCG	0.66	0.89	0.60	0.50
BiCGSTAB	0.34	0.0058	0.19	0.41
GMRES(5) *	0.31	0.0049	0.19	0.39

* GMRES restarted after every five search-directions.

Table 9. Time required by the three formulations

	ρ	Outward-looking* (sec)	Inward-looking (sec)	Combined (sec)
Problem 1	0.74	66.88	58.82	59.81
Problem 2	2.12	91.78	103.60	92.66
Problem 3	3.43	765.90	N/A**	N/A**
Problem 4	5.32	59.55	219.90	76.59

* The drop tolerance for the BiCGSTAB solver is 1.0×10^{-3}.
** The results are not available due to excessive memory requirements.

Table 10. Computer memory requirements of the three formulations

	ρ	Outward-looking (MBytes)	Inward-looking (MBytes)	Combined (MBytes)
Problem 1	0.74	7	17	34
Problem 2	2.12	23	42	70
Problem 3	3.43	107	N/A*	N/A*
Problem 4	5.32	11	126	36

* Data not available due to excessive memory requirements

The outward-looking and the combined formulations do not have the uniqueness problem associated with A_{ii}^{-1}. However, all three formulations may have uniqueness difficulties at interior resonant frequencies caused by the EFIE [15], [17], [18]. The exterior equivalent problem can be constructed in a manner (e.g. using a combined field formulation [6], [18]), to avoid the problem of interior resonance.

IX. CONCLUSIONS

This paper presents three formulations for the hybrid FEM/MoM method. The outward-looking formulation constructs an RBC using MoM and then substitutes the RBC into the FEM equations. Iterative solvers can be used to solve the final matrix equation efficiently. The authors have found that it is much faster and less memory intensive, to construct preconditioners based on LU factorization of the FEM matrix rather than the final matrix. The symmetric minimum degree permutation can reduce the number of fill-ins resulting in further memory reduction. The preconditioning technique presented greatly reduced the number of iterations required by the solver for the sample problems presented here. The outward-looking formulation is preferred when the coupling index ρ is larger than 2.0.

The inward-looking formulation derives an RBC using the FEM, then substitutes the RBC into the MoM equations. The Gaussian elimination method is generally used to solve the final matrix equation. The inward-looking formulation is preferred when the coupling index ρ is smaller than 1.5.

The combined formulation generates a large matrix equation directly without inverting any matrices, and solves for all unknowns simultaneously. For the types of problems studied here, it was difficult to apply iterative solvers to the resulting matrix equations due to their large condition numbers.

The choice of hybrid FEM/MoM formulation depends on the problem geometry and the way it is meshed. However, for the printed circuit board geometries investigated in this paper, the outward-looking formulation appears to be the most effective and most efficient approach.

REFERENCES

[1] P. Silvester and M. S. Hsieh, "Finite element solution of 2-dimensional exterior field problems," *Proc. Inst. Elect. Eng.*, vol. 43, pp. 1743-1747, Dec. 1971.

[2] B. H. McDonald and A. Wexler, "Finite element solution of unbounded field problems," *IEEE Trans. Microwave Theory Tech.*, vol. 20, pp. 841-847, Dec. 1972.

[3] X. Yuan, "Three dimensional electromagnetic scattering from inhomogeneous objects by the hybrid moment and finite element method," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1053-1058, Aug. 1990.

[4] J.-M. Jin and J. L. Volakis, "Electromagnetic scattering by and transmission through a three-dimensional slot in a thick conducting plane," *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 543-550, Apr. 1991.

[5] J.-M. Jin and J. L. Volakis, "A hybrid finite element method for scattering and radiation by microstrip patch antenna and arrays residing in a cavity," *IEEE Trans, Antennas Propagat.*, vol. 29, pp. 1598-1604, Nov. 1991.

[6] J. Angélini, C. Soize, and P. Soudais, "Hybrid numerical method for harmonic 3D Maxwell equations: Scattering by a mixed conducting and inhomogeneous anisotropic dielectric medium," *IEEE Trans. Antennas Propagat.*, vol. 41, pp. 66-76, Jan. 1993.

[7] G. E. Antilla and N. G. Alexopoulos, "Scattering from complex three-dimensional geometries by a curvilinear hybrid finite element-integral equation approach," *J. Opt. Soc. Amer. A*, vol. 11, no. 4, pp. 1445-1457, Apr. 1994.

[8] M. W. Ali, T. H. Hubing, and J. L. Drewniak, "A hybrid FEM/MOM technique for electromagnetic scattering and radiation from dielectric objects with attached wires," *IEEE Trans. Electromagnetic Compatibility*, vol. 39, pp. 304-314, Nov. 1997.

[9] J. Shen and A. Kost, "FE-BE method for EMC problems in power cable systems," *IEEE Trans. Magn.*, vol. 32, pp. 1493-1496, May 1996.

[10] A. Kost and H. Igarashi, "Different numerical methods for electromagnetic field computation with thin shielding sheets," *Proceedings of the IEEE 1997 International Symposium on Electromagnetic Compatibility*, pp. 248-253, Austin, Texas, Aug. 1997.

[11] Y. Ji, M. Ali, and T. H. Hubing, "EMC applications of the EMAP5 hybrid FEM/MoM code", *Proceedings of the IEEE 1998 International Symposium on Electromagnetic Compatibility*, vol. 1, pp. 177-181, Denver, Colorado, Aug. 1998.

[12] B.-S. Yang, A. W. Glisson, and P. M. Goggans, "Interior resonance problems associated with hybrid integral equation/partial differential equation methods," *Proceedings of the 1992 International IEEE Antennas and Propagation Society Symposium*, pp. 781-784, Chicago, Illinois, Jul. 1992.

[13] L. W. Pearson, A. F. Peterson, L. J. Bahrmasel, and R. A. Whitaker, "Inward-looking and outward-looking formulations for scattering from penetrable objects," *IEEE Trans, Antennas Propagat.*, vol. 40, pp. 714-720, Jun. 1992.

[14] A. F. Peterson, S. L. Ray, and R. Mittra, *Computational Methods for Electromagnetics*, New York: IEEE Press and Oxford University Press, 1997.

[15] J. L. Volakis, T. Özdemir, and J. Gong, "Hybrid finite-element methodologies for antennas and scattering", *IEEE Trans, Antennas Propagat.*, vol. 45, pp. 493-507, Mar. 1997.

[16] J. L. Volakis, A. Chatterjee, and L. C. Kempel, *Finite Element Method for Electromagnetics*, New York: IEEE Press and Oxford University Press, 1998.

[17] J. J. H. Wang, *Generalized Moment of Methods in Electromagnetics*, New York: John Wiley & Sons, 1990.

[18] X.-Q. Sheng, J.-M. Jin, J. Song, C.-C. Lu, and W.C. Chew, "On the formulation of hybrid finite-element and boundary-integral methods for 3-D scattering," *IEEE Trans, Antennas Propagat.*, vol. 46, pp. 303-311, Mar. 1998.

[19] S. M. Rao, D. R. Wilton, and A. W. Glisson, "Electromagnetic scattering by surfaces of arbitrary shape," *IEEE Trans. on Antennas and Propagat.*, vol. 30, pp. 409-418, May 1982.

[20] P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers*, Cambridge University Press, 1996.

[21] J.-M. Jin, *The Finite Element Method in Electromagnetics*, New York: John Wiley & Sons Inc, 1993.

[22] M. L. Barton and Z. J. Cendes, "New vector finite elements for three-dimensional magnetic field computation," *Journal of Applied Physics*, vol. 61, pp. 3919-3921, Apr. 1987.

[23] R. Barrett, M. Berry, T. F. Chan, F. Demmel, J. M. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. Van der Vorst, *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods*, Philadelphia: SIAM, 1994.

[24] Y. Saad, *Iterative Methods for Sparse Linear Systems*, Boston: PWS Publishing Company, 1996.

[25] MATLAB is a product of Mathworks, Inc, U.S.A..

[26] MATLAB manuals, Mathworks, Inc, U.S.A..

[27] E. Cuthill and J. McKee, "Reducing the bandwidth of sparse symmetric matrices," *Proceedings of the 24th National Conference of the ACM*, New Jersey: Brandon Systems Press, 1969.

[28] A. George and J. W. H. Liu, *Computer Solution of Large Sparse Positive Definite Systems*, New Jersey: Prentice-Hall, 1981.

[29] S. Pissanetzky, *Sparse Matrix Techniques*, New York: Academic Press, 1984.

[30] J. R. Gilber, C. Moler, and R. Schreiber, "Sparse matrices in MATLAB: Design and implementation," *SIAM J. Matrix Anal. Appl.*, vol. 13, pp. 333-356, Jan. 1992.

[31] K. D. Paulsen, D. R. Lynch, and J. W. Strohbehn, "Three-dimensional finite, boundary, and hybrid element solutions of the Maxwell equations for lossy dielectric media," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 682-693, Apr. 1988.

[32] J.-M. Jin and J. L. Volakis, "A finite element-boundary integral formulation for scattering by three-dimensional cavity-backed apertures," *IEEE Trans, Antennas Propagat.*, vol. 39, pp. 97-104, Jan. 1991.

[33] O. M. Ramahi and R. Mittra, "Finite element solution for a class of unbounded geometries," *IEEE Trans, Antennas Propagat.*, vol. 39, pp. 244-250, Feb. 1991.

[34] J.-M. Jin and V. V. Liepa, "A note on hybrid finite element method for solving scattering problems," *IEEE Trans, Antennas Propagat.*, vol. 36, pp. 1486-1490, Oct. 1988.

[35] X. Yuan, D. R. Lynch, and J .W. Strohbehn, "Coupling of finite element and moment methods for electromagnetic scattering from inhomogeneous objects," *IEEE Trans, Antennas Propagat.*, vol. 38, pp. 386-393, Mar. 1990.

[36] A. C. Cangellaris and R. Lee, "Finite element analysis of electromagnetic scattering from inhomogeneous cylinders at oblique incidence," *IEEE Trans, Antennas Propagat.*, vol. 39, pp. 645-650, May 1990.

[37] D. Jankoviæ, M. LaBelle, D. C. Chang, J. M. Dunn, and R. C. Booton, "A hybrid method for the solution of scattering from inhomogeneous dielectric cylinders of arbitrary shape," *IEEE Trans, Antennas Propagat.*, vol. 42, pp. 1215-1222, Sept. 1994.

[38] J. Shen, T. Hybler, and A. Kost, "Preconditioned iterative solvers for complex and unsymmetric systems of equations resulting from the hybrid FE-BE method," *IEEE Trans. Magn.*, vol. 33, pp. 1764-1767, Mar. 1997.