# A Numerical Investigation of Interior Resonances in the Hybrid FEM/MoM Method

Yun Ji, Hao Wang, and Todd H. Hubing

***Abstract*—The interior resonance problem that can occur when using a hybrid finite-element method/method of moments (FEM/MoM) method to model electromagnetic scattering problems is investigated. Calculations of the bistatic radar cross section of a coated dielectric sphere are presented using different formulations, solution approaches, and solvers. The solutions using the electric-field integral equation have significant errors near an interior resonance frequency. When the combined-field integral equation is employed, satisfactory solutions can be obtained that do not depend on the particular solution approach or solver.**

***Index Terms*—CFIE, electromagnetic scattering, EFIE, hybrid FEM/MoM, interior resonance, internal resonance, MFIE.**

## I. Introduction

The hybrid finite-element method/method of moments (FEM/MoM), (also referred to as FE-BE or FE-MM) can be used to analyze the scattering from an inhomogeneous body by applying FEM to model the field inside the scatterer and using a surface integral equation to provide a radiation boundary condition (RBC) to terminate the FEM mesh [1]–[4]. One of the limitations of surface integral equation methods is that they are sometimes prone to errors at certain frequencies corresponding to the resonant frequencies of the defined closed surface [5]. This is called the *interior resonance* or *internal resonance* problem. FEM/MoM is not immune to the interior resonance problem if the integral equation used is either the electric-field integral equation (EFIE) or the magnetic-field integral equation (MFIE) [3], [4]. This communication investigates the numerical solution accuracy near interior resonance frequencies when the combined field integral equation (CFIE) is used in the FEM/MoM.

## II. The Hybrid FEM/MoM Formulation

The interior equivalent problem in the hybrid FEM/MoM is modeled using the FEM. A Galerkin procedure can be used to test the weak form of the vector wave equation resulting in an FEM matrix equation as follows [6]:

$$\begin{bmatrix} A_{ii} & A_{is} \\ A_{si} & A_{ss} \end{bmatrix} \begin{bmatrix} E_i \\ E_s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_{ss} \end{bmatrix} \begin{bmatrix} 0 \\ J_s \end{bmatrix} + \begin{bmatrix} g_i \\ g_s \end{bmatrix}. \tag{1}$$

The exterior equivalent problem can be analyzed using the EFIE, MFIE, or CFIE. The CFIE is a linear combination of the EFIE and MFIE

$$\alpha \mathrm{LHS(EFIE)} + (1-\alpha)\eta_0 \mathrm{LHS(MFIE)} = \alpha \mathrm{RHS(EFIE)} + (1-\alpha)\eta_0 \mathrm{RHS(MFIE)} \tag{2}$$

where $\mathrm{LHS}(\bullet)$ and $\mathrm{RHS}(\bullet)$ denote the left-hand side and right-hand side of an equation and $\alpha$ is a real-value parameter in the range $0 < \alpha > 1$. The triangular patch basis function $\mathbf{f}(\mathbf{r})$ (RWG basis function) and the tetrahedral basis function $\mathbf{w}(\mathbf{r})$ used in this study are related by

$$\mathbf{w}(\mathbf{r}) = \hat{n} \times \mathbf{f}(\mathbf{r}). \tag{3}$$

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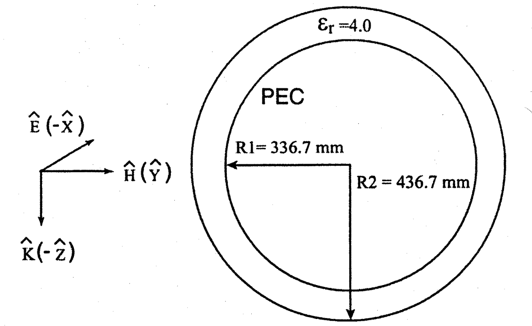


Fig. 1. A coated sphere excited by a plane wave.

This suggests $\mathbf{f}(\mathbf{r})$ and $\mathbf{w}(\mathbf{r})$ are completely compatible on the surface [7]. The resulting MoM matrix equation is as follows:

$$[C]\,[J_s] = [D]\,[E_s] - \left[F^i\right]. \tag{4}$$

Equations (1) and (4) form a coupled and determined system where $J_s$, $E_s$, and $E_i$ are unknowns. Three different matrix solution approaches can be used to solve the coupled system [6], [8]. *The combined matrix solution approach* forms a large determined matrix equation by combining (1) with (4) directly

$$\begin{bmatrix} A_{ii} & A_{is} & 0 \\ A_{si} & A_{ss} & -B_{ss} \\ 0 & -D & C \end{bmatrix} \begin{bmatrix} E_i \\ E_s \\ J_s \end{bmatrix} = \begin{bmatrix} g_i \\ g_s \\ -F^i \end{bmatrix}. \tag{5}$$

*The inward-looking matrix solution approach* derives $E_s$ from (1) and incorporates it into (4). *The outward-looking matrix solution approach* derives $J_s$ from (4) and substitutes it into (1)

$$\begin{bmatrix} A_{ii} & A_{is} \\ A_{si} & A_{ss} - B_{ss}C^{-1}D \end{bmatrix} \begin{bmatrix} E_i \\ E_s \end{bmatrix} = \begin{bmatrix} g_i \\ g_s - B_{ss}C^{-1}F^i \end{bmatrix}. \tag{6}$$

The preconditioning technique reported in [8], referred to as partial LU (PLU) preconditioning in this communication, was used to improve the convergence rate and accuracy of iterative solutions for (6). The default used in this communication is the outward-looking approach, the bi-conjugate gradient stabilized (BiCGSTAB) iterative solver [9], and the PLU preconditioning.

## III. Numerical Results and Analysis

A coated dielectric sphere shown in Fig. 1 is excited by a plane wave traveling along the $-\hat{z}$ direction. The sphere has a perfect electric conductor (PEC) core with a radius of 336.7 mm, a dielectric coating with a radius of 436.7 mm, and a dielectric constant of 4.0. The analytical radar-cross-section (RCS) results for this geometry can be obtained using the Mie series [10]. The first interior resonance frequency is 300 MHz. Two different implementations of integral equations, the TE (an EFIE type) and TENH (a CFIE type) formulations [4], were used in the hybrid FEM/MoM solutions.

Fig. 2 plots the mean error[1] of the bistatic RCS results obtained using the two formulations from 290 MHz to 310 MHz at 1.0-MHz intervals. Two different types of matrix equation solvers were used to test the

[1]The error is with respect to the Mie solutions and averaged over observing angle.

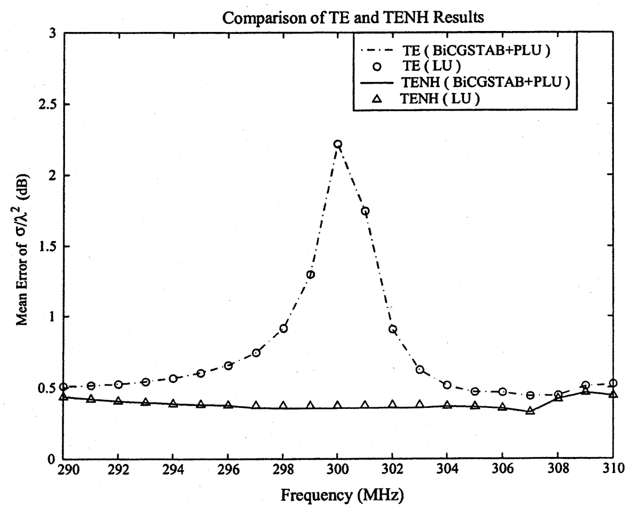


Fig. 2. Mean error of the bistatic RCS results using TE and TENH (Outward-looking, 7282 unknowns).

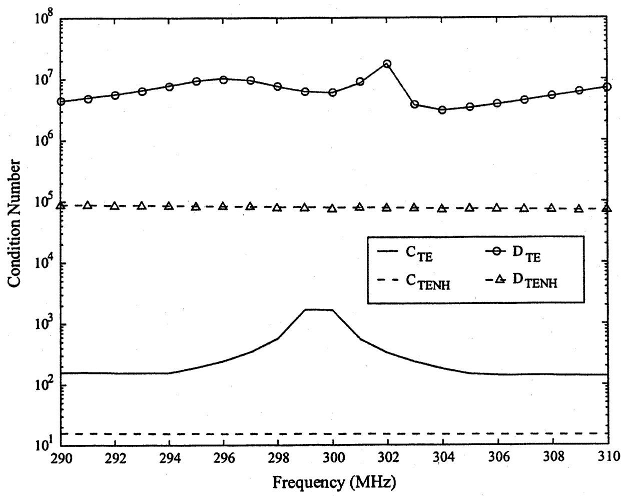


Fig. 3. Comparison of TE and TENH matrix condition numbers.

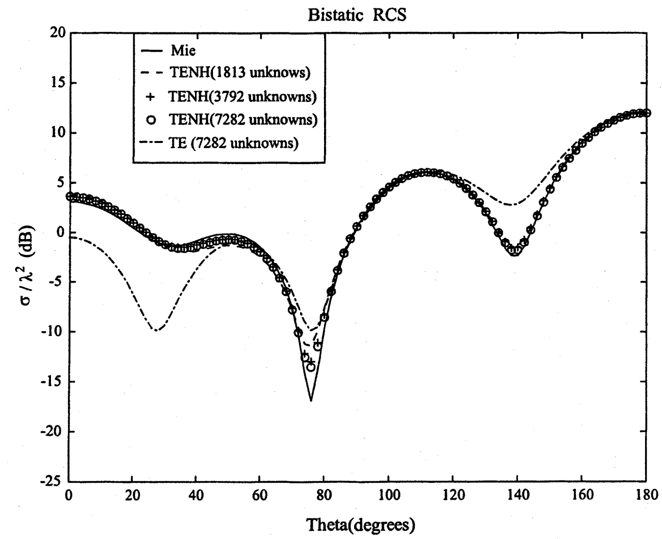


Fig. 4. The bistatic RCS with different meshes: TE and TENH ($\alpha = 0.5$), 300 MHz.

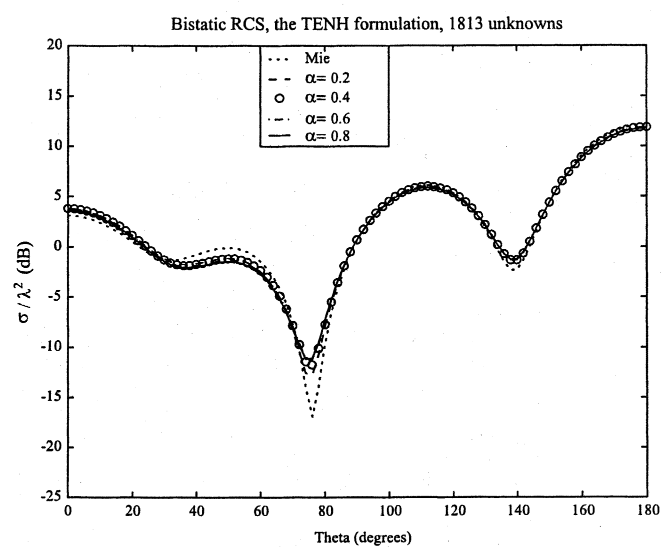


Fig. 5. The bistatic RCS at 300 MHz: 1813 unknowns, TENH with $\alpha = 0.2 \sim 0.8$.

solution sensitivity. One was an LU decomposition solver, which is a direct solver. The other was an iterative solver that used BiCGSTAB and PLU. Significant errors were present in the TE results. The maximum numerical error occurred at 300 MHz as expected. The TENH formulation did not exhibit significant error near the interior resonance frequency. The solutions were not sensitive to the two solvers used here.

Fig. 3 shows a comparison of the TE and TENH matrix condition numbers. The TENH formulation has much smaller condition numbers than the TE formulation does therefore it is less susceptible to numerical error. The relationship between matrix condition numbers and solution accuracy near interior resonances was reported in [5]. In order to evaluate the impact of the number of unknowns on the accuracy of the TENH formulation, three different meshes were used to analyze the geometry.

Fig. 4 shows the RCS results at 300 MHz obtained using the three different meshes. The TENH formulation generates satisfactory results when the number of unknowns is 1813 (seven elements/wavelength[2] ), 3792 unknowns (eight elements/wavelength 2) and 7282 unknowns (10 elements/wavelength 2), respectively. No significant errors were observed in the TENH results obtained using different meshes. The impact of $\alpha$ in the CFIE was also investigated.

[2]The wavelength is calculated inside the dielectric sphere.

Fig. 5 shows that values of $\alpha$ from 0.2 to 0.8 all lead to satisfactory results. The TENH results show some variations but are not very sensitive to the choice of $\alpha$.

Fig. 6 plots the bistatic RCS of the coated sphere at 300 MHz obtained using the Mie series and TENH formulation with the combined matrix solution approach applied to solve the final matrix equation, i.e., (5). Two different types of solvers were used to test the sensitivity of the results. One was an LU decomposition solver. The other used BiCGSTAB and the incomplete LU decomposition (ILU) preconditioning technique [9]. The TENH formulation generated satisfactory results in both cases.

Fig. 7 shows the comparison of matrix condition numbers of the outward and combined solution approaches. The condition number of (5), is a factor of $10^4$ higher than that of (6). For this case, the outward-looking approach is less sensitive to numerical errors. The outward-looking approach, inward-looking approach and the combined

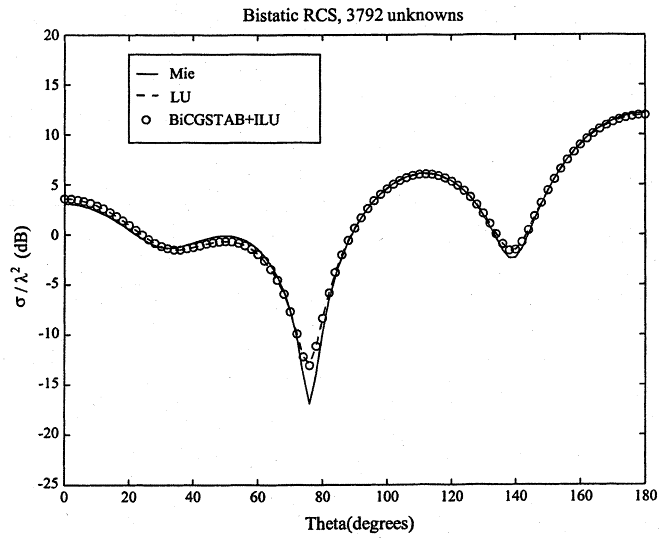


Fig. 6. The bistatic RCS at 300 MHz obtained using the Mie series, and TENH with the combined matrix solution approach.

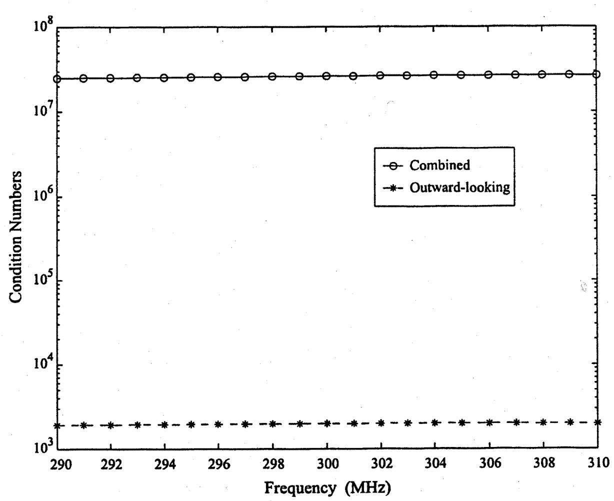


Fig. 7. The condition numbers of the final matrix equations generated by the combined and outward-looking approaches. (The TENH formulation, 3792 unknowns.)

approach each have their own advantages and disadvantages in terms of accuracy and computer resource requirements [8].

Another aspect of the interior resonance problem is how it affects the near field results. Fig. 8 illustrates a power bus structure used in printed circuit boards. An impressed source drives two PEC plates with negligible thickness. The source is modeled as a current filament [11]. The number of total unknowns is 7282. The first interior resonance frequency of the chosen MoM boundary is at 1772 MHz. Fig. 8 plots the relative error of the input impedance of the TE and TENH results with respect to a pure FEM solution, which does not have the interior resonance problem. The TE formulation yields near-field results that are incorrect, while the TENH formulation generates a satisfactory solution.

## IV. Summary

The interior resonance problem in the context of the three-dimensional (3-D) hybrid FEM/MoM has been investigated. Numerical results with different formulations, solution approaches, solvers, meshes

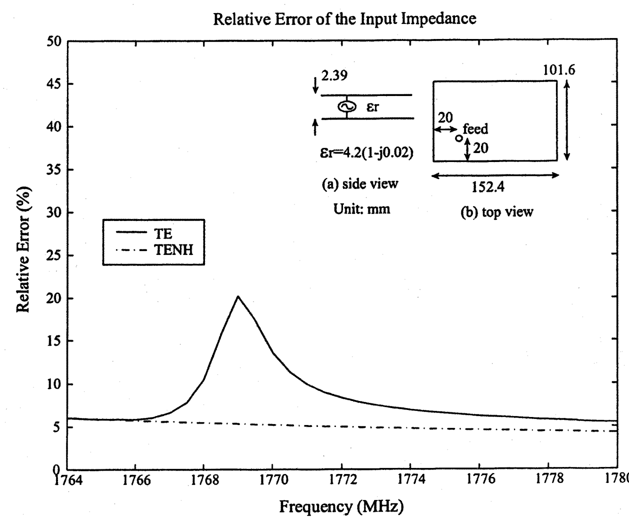


Fig. 8. Error in the input impedance results obtained using TE and TENH formulations near an interior resonance frequency.

and weight combinations have been presented. The EFIE formulation yields significant errors near interior resonance frequencies. The CFIE formulation generates satisfactory results.

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