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External Meniscus on a Ribbon-Like Fiber

M. M. Alimov[a,*] and K. G. Kornev[b]

[a] *Kazan Federal University, Kazan, Russia*

[b] *Clemson University, Clemson, United States*

**e-mail: mars.alimov@kpfu.ru*

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Abstract—We consider the problem of configuration of the external meniscus formed by the capillary rise of a liquid during immersion of a ribbon-like fiber into the liquid. In the case when the fiber thickness is small, a new analytical solution of the problem is constructed using the asymptotic approach. In this solution, at the sharp edges of the fiber, the liquid level has a jump for all values of the contact angle different from $\pi/2$.

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INTRODUCTION

When a fiber of arbitrary profile is immersed in a liquid, the liquid free surface deforms forming an external meniscus. The height of liquid rise and the meniscus form depend considerably on both the fiber profile and wetting properties of the fiber surface [1–3]. The menisci generated by the fiber with a round profile have been studied sufficiently well [4–8]. There are also a number of works [6, 9–12] on the numerical or asymptotic analysis of menisci formed by the fiber with a smooth profile slightly different from the round one. At the same time, there is no complete investigation of the menisci on the fibers with a nonsmooth profile. In particular, the meniscus on a fiber of elliptic cross-section has been analyzed by finite difference methods [11]. The case of the ellipse being close to degenerate, when the ratio of semiaxes is nearly zero, may characterize the meniscus behavior on the fiber with a nonsmooth profile. However, the limiting ratio of ellipse semiaxes for which the calculations were performed [11] is equal to 0.5 which is very distant from the degenerate case.

The asymptotic approach was suggested in [13] for the analysis of the external meniscus configuration formed by a fiber of arbitrary profile in the case of small profile size compared to the capillary length. Attempts to apply this approach to the analysis of the meniscus configuration formed by a ribbon-like fiber revealed that the liquid level on the fiber surface cannot be a smooth function everywhere [14]. This result is confirmed by independent investigations of the existence of the smooth external meniscus for the fibers with a nonsmooth profile [15, 16]. Moreover, analysis performed in [14] illustrated the following: for all values of contact angle γ different from $\pi/2$, the liquid level should jump at the sharp edges of the fiber, and the value of this jump is unknown in advance. This important qualitative conclusion was verified by separate experiments [14]; it allowed us to understand the structure of the problem solution; however, the solution itself was not obtained.

Below, to develop the asymptotic approach of papers [13, 14], we construct analytical solutions to the problem of configuration of the external meniscus on the ribbon-like fiber.

1. PROBLEM FORMULATION

Let us consider immersion of the ribbon-like fiber into a liquid and the formation of an external meniscus Σ with contact line Γ_c at the boundary of three continua: air, liquid, and fiber material (see Fig. 1a). We assume that the normal cross section of the fiber is a rectangle whose thickness may be neglected when compared to the width. Taking the results of analysis performed earlier [14] into consideration, we assume the existence of the liquid level jump at the fiber sharp edges; but a zero jump is not excluded when the points C^A and C^B coincide with the points E^A and E^B.

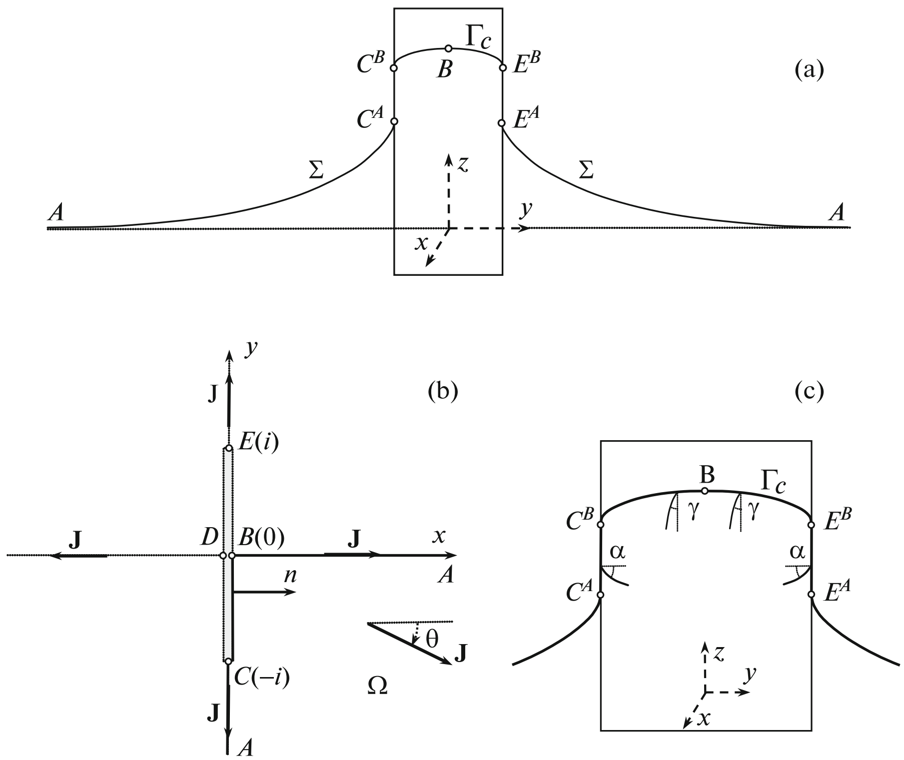

Fig. 1.

We choose the half-width of the fiber as characteristic length L_m and introduce a dimensionless Cartesian coordinate system x, y, z such that z axis coincides with the fiber central axis, and the x axis is orthogonal to the fiber plane, whereas plane x, y touches the meniscus surface at infinity (at $r \to \infty$, where $r = \sqrt{x^2 + y^2}$). The configuration of meniscus surface Σ can be described by the function

$$z = h(x, y),$$

where $h(x, y)$ is the height of the liquid column at point (x, y) measured from the liquid level at infinity. We assume that function $h(x, y)$ is determined at all points of plane x, y including infinity and the boundary of cut $BCDEB$ (the fiber profile), except for points C and E (see Fig. 1b).

Note that we keep the original notation for the projections of the meniscus characteristic points onto the x, y plane except for points C and E. Entire regions of the meniscus boundary C^AC^B and E^AE^B are projected onto these points of the x, y plane.

In addition to the characteristic scale L_m related to the domain geometry, the problem possesses one more characteristic scale which is capillary length $L_c = \sqrt{\sigma\rho^{-1}g^{-1}}$, where σ is the surface tension, ρ is the liquid density, and g is the acceleration due to gravity. Consequently, the considered process of the capillary rise is characterized by the dimensionless complex (the Bond number) [17]

$$\varepsilon = L_m^2 / L_c^2.$$

The ribbon-like fiber has two planes of symmetry: the planes x, z and y, z. Meniscus surface Σ and function $h(x, y)$ have similar symmetry. Therefore, the domain of definition of the latter is the x, y plane with cut $BCDEB$; it admits selection of a symmetry element in the form of region $\Omega = ABCA$, where boundary segment BC corresponds to the fiber boundary. Therefore, the mathematical model of the capillary rise on the ribbon-like fiber may be written as the boundary value problem in the symmetry element [6]:

$$\Omega: \quad \nabla \cdot \left[\left(1 + |\nabla h|^2 \right)^{-1/2} \nabla h \right] - \varepsilon h = 0, \tag{1.1}$$

$$A: \quad h\Big|_{r\to\infty} \to 0, \quad AB: \quad h_y = 0, \quad AC^A: \quad h_x = 0, \tag{1.2}$$

$$BC^B: \quad \left(1 + |\nabla h|^2 \right)^{-1/2} \nabla h \cdot \mathbf{n} = -\cos\gamma. \tag{1.3}$$

Hereinafter, lower indices x and y denote the partial derivatives with respect to the corresponding variables, the gradient and divergence are two-dimensional operators in plane x, y, γ is the contact angle describing the liquid–air–fiber material triad, and $\mathbf{n}$ is the external normal to the fiber surface. According to Fig. 1b, we have

$$BC^B: \quad \mathbf{n} = (1, 0). \tag{1.4}$$

In segments AB and AC^A we specify conditions of the smooth symmetric extension of the meniscus, and in boundary BC^B we specify the Young–Laplace condition of the equilibrium of contact line Γ_c (see Fig. 1c) [1]. Although points C^A and C^B are not themselves determined in plane x, y, the boundary condition expressions in segments AC^A and BC^B are sufficiently correct. Indeed, the condition of the smooth symmetric extension of the meniscus through boundary AC^A is not related to meniscus boundary segment $(C^AC^B]$ whereas the Young–Laplace condition is not related to meniscus boundary segment $[C^AC^B)$.

Analysis of problems (1.1)–(1.4) shows that, instead of the complete interval of contact angle $\gamma \in [0, \pi]$, it is sufficient to study just interval $\gamma \in [0, \pi/2)$ to which solution $h(x, y) > 0$ corresponds. In fact, the case with $\gamma = \pi/2$ is uninteresting, because it corresponds to the trivial solution $h(x, y) \equiv 0$, whereas in the case with $\hat{\gamma}_+ \in (\pi/2, \pi]$ the solution to the problem $\hat{h}_+(x, y) < 0$ may be obtained by operation $\hat{h}_+(x, y) = -h(x, y)$, where $h(x, y)$ is the solution corresponding to contact angle $\gamma = \pi - \hat{\gamma}_+$, because $\cos\gamma = -\cos\hat{\gamma}_+$.

In the case of the thin fiber, when gravity forces are much less than capillary ones, the Bond number is small, $\varepsilon \ll 1$, and use of asymptotic methods becomes possible; it becomes possible to use the method of matched asymptotic expansions. It was illustrated in [13] that, in the vicinity of the fiber, inner asymptotic expansion $h^{(i)}(x, y)$ of function $h(x, y)$ in ε is true, and, in the vicinity of infinity, the outer asymptotic expansion $h^{(o)}(x, y)$ is true. The principal term of the outer expansion has a form independent of the fiber profile

$$h^{(o)}(x, y) = 2\pi^{-1} \cos\gamma K_0(r\varepsilon^{1/2}), \tag{1.5}$$

where K_0 is a Bessel function of the second kind. The principal term of inner asymptotic expansion $h^{(i)}(x, y)$ is determined by the solution to the boundary value problem [13]

$$\Omega: \quad \nabla \cdot \left[\left(1 + \left|\nabla h^{(i)}\right|^2 \right)^{-1/2} \nabla h^{(i)} \right] = 0, \tag{1.6}$$

$$A: \quad h^{(i)} = -2\pi^{-1} \cos\gamma \ln(2^{-1} r e^E \varepsilon^{1/2}) + O(r^{-1})\Big|_{r\to\infty}, \quad AB: \quad h_y^{(i)} = 0, \quad AC^A: \quad h_x^{(i)} = 0, \tag{1.7}$$

$$BC^B: \quad \left(1 + \left|\nabla h^{(i)}\right|^2 \right)^{-1/2} \nabla h^{(i)} \cdot \mathbf{n} = -\cos\gamma, \tag{1.8}$$

where $E = 0.577...$ is the Euler constant.

Differential Eq. (1.6) is effectively the equation of the minimum surface [18], since it has no term with the hydrostatic pressure: it may be neglected compared to the capillary forces. However, the hydrostatics plays a significant role in the problem through the condition at infinity (the first condition of (1.7)). This condition itself is the result of matching of the principal terms in the inner and outer expansions. It is necessary to take into account condition at infinity (1.7) of the term on the order of unity in the right-hand side, because height $h(x, y)$ in problem (1.1)–(1.4) is measured from the liquid level at infinity $r \to \infty$, but infinity is not covered by inner expansion (1.4) and (1.6)–(1.8). Thus, for the inner expansion there is no characteristic point in plane (x_*, y_*) with known absolute value $h(x_*, y_*)$ which may be used as an origin of function $h^{(i)}(x, y)$. Consequently, when we solve boundary value problem (1.4) and (1.6)–(1.8) or, in other words, when we integrate the equation in partial derivatives, the problem of determining the integration constant arises. Only considering the term on the order of unity in the boundary condition at infinity allows us to hope for an effective solution to this problem.

Note that earlier [13] it was not suggested in the reasoning of the asymptotic formulation of type (1.4), (1.6)–(1.8) that the contact line Γ_c has such vertical segments passing along the sharp edges. The assumption about the presence of such jumps introduces no contradiction; we need just to specify the character of the boundary condition in segment C^AC^B. It is thought that segment C^AC^B of the fiber sharp edge is simultaneously the edge of meniscus Σ (see Fig. 1c). However, in terms of function $h^{(i)}(x, y)$ it is impossible to write this condition, and, formally, in the setting of problem (1.4), (1.6)–(1.8) we specify no boundary condition. Note that angle α between the meniscus and fiber surface at its face end is assumed undetermined, and, in general, not constant in the entire segment (see Fig. 1c). With this specification, the asymptotic setting of problem (1.4), (1.6)–(1.8) admits the existence of the liquid level jumps at the sharp edges of the fiber.

If in the problem results the principal term of inner expansion $h^{(i)}(x, y)$ is found, then we may determine principal term $h^{(u)}(x, y)$ of the asymptotic expansion in powers of ε approximating solution $h(x, y)$ to problem (1.1)–(1.4) uniformly in the entire plane x, y by the Van Dyke's formula [19]

$$h^{(u)}(x, y) = h^{(i)}(x, y) + 2\pi^{-1}\cos\gamma\left[K_0(r\varepsilon^{1/2}) + \ln(2^{-1}re^E\varepsilon^{1/2})\right]. \tag{1.9}$$

We only need to determine the solution to problem (1.4), (1.6)–(1.8).

2. ANALOGY TO FILTRATION OF ABNORMALLY VISCOUS LIQUIDS: CHAPLYGIN TRANSFORMATION

As shown in [13], to solve the problem of type (1.4), (1.6)–(1.8) we may employ the analogy to the problems of flow of abnormally viscous liquids through a porous continuum [20]. We determine the fictitious hydrodynamic flux of liquid:

$$\mathbf{J} = -\nabla h^{(i)}\left|\nabla h^{(i)}\right|^{-1}J, \quad J = \left(1 + \left|\nabla h^{(i)}\right|^2\right)^{-1/2}\left|\nabla h^{(i)}\right|. \tag{2.1}$$

The value of flux $J = |\mathbf{J}|$ with θ, which is the inclination angle of the flux vector $\mathbf{J}$ to the horizon, are the variables of the velocity hodograph.

Equation (1.6) in terms of function $\mathbf{J}(x, y)$ takes form $\nabla \cdot \mathbf{J} = 0$. Consequently, $\mathbf{J}$ is the flux of the incompressible fluid, and, therefore, we may introduce stream function $\psi(x, y)$ [21]. Then the projections of flux $\mathbf{J}$ onto the coordinate axes are determined by the formulas:

$$J\cos\theta = \psi_y, \quad J\sin\theta = -\psi_x. \tag{2.2}$$

The relations (2.1) between values J and $\left|\nabla h^{(i)}\right|$ may be written in another form:

$$\left|\nabla h^{(i)}\right| = \Phi(J), \quad \Phi(J) = J(1 - J^2)^{-1/2}, \tag{2.3}$$

where conditions $\Phi(J) \geq 0$ and $\Phi'(J) \geq 0$ are satisfied.

Hence, Eq. (1.6) may be represented in the form of a system of equations describing the fictitious flow of an abnormal fluid through the porous continuum [20]:

$$\Omega: \quad \nabla h^{(i)} = -\Phi(J)\,\mathbf{J}/J, \quad \nabla \cdot \mathbf{J} = 0. \tag{2.4}$$

Here, $h^{(i)}(x, y)$ is the analog of the pressure head, and $\mathbf{J}(x, y)$ is the analog of the flow velocity. Figure 1b depicts the character of the flow in the case with $\gamma < \pi/2$ by arrows. Note that some flows following law (2.3) and (2.4) were first investigated by Sokolovskii [22].

The last two boundary conditions in (1.7) may be rewritten in terms of the velocity hodograph. Indeed, directly from Fig. 1b it follows that

$$AB: \quad \theta = 0; \quad AC^A: \quad \theta = -\pi/2. \tag{2.5}$$

In terms of the hydrodynamic flow, boundary condition (1.8) takes the simple form

$$BC^B: \quad \mathbf{J} \cdot \mathbf{n} = \cos\gamma.$$

Taking expression (1.4) into consideration, we may also reorganize it in terms of the velocity hodograph

$$BC^B: \quad J\cos\theta = \cos\gamma. \tag{2.6}$$

From the boundary condition at infinity (1.7), and taking into account formulas (2.1), the estimations follow

$$A: \quad \nabla h^{(i)} = -\frac{2\cos\gamma}{\pi r^2}\mathbf{r} + O(r^{-2})\Big|_{r\to\infty}, \quad J = \frac{2\cos\gamma}{\pi r} + O(r^{-2})\Big|_{r\to\infty}. \tag{2.7}$$

As a result, problems (1.4) and (1.6)–(1.8) reduce to the problem of the filtration flow of liquid following the Sokolovskii law (2.3) and (2.4) in the domain Ω with boundary conditions (2.5) and (2.6) and the sink at infinity (2.7).

For filtration flows of abnormally viscous liquids, it is necessary to apply the Chaplygin transformation [23] to the variables of the velocity hodograph [20, 22, 24, 25]. Using the flow function introduced by Eqs. (2.2), we proceed to functions $h^{(i)}(x,y)$ and $\psi(x,y)$ in the system of equations (2.4) given by

$$h_x^{(i)} = -\Phi(J)\cos\theta, \quad h_y^{(i)} = -\Phi(J)\sin\theta; \quad \psi_x = -J\sin\theta, \quad \psi_y = J\cos\theta, \tag{2.8}$$

where J and $\Phi(J)$ are expressed through $\left|\nabla h^{(i)}\right|$ by formulas (2.1) and (2.3). Afterwards, we transform Eqs. (2.8) to the variables of the velocity hodograph J and θ and obtain

$$\frac{\Phi^2(J)}{J\Phi'(J)}\psi_J = -h_\theta^{(i)}, \quad \frac{\Phi(J)}{J^2}\psi_\theta = h_J^{(i)}.$$

We substitute into it the concrete form, Eq. (2.3), of the function $\Phi(J)$ and find

$$J(1-J^2)^{1/2}\psi_J = -h_\theta^{(i)}, \quad \psi_\theta = J(1-J^2)^{1/2}h_J^{(i)}. \tag{2.9}$$

Analysis of this system shows that instead of variable J it is necessary to introduce a new variable t such that

$$J^{-1}(1-J^2)^{-1/2}dJ = -dt.$$

Integrating this differential relation, we find the explicit relation between J and t

$$J = \cosh^{-1}t, \quad t = \operatorname{arccosh}(J^{-1}), \quad \Phi(J) = \sinh^{-1}t. \tag{2.10}$$

Note that by reason of its meaning quantity $\left|\nabla h^{(i)}\right|$ in the physical plane can vary from zero to infinity. We take relation (2.3) between $\left|\nabla h^{(i)}\right|$ and J as well as relation (2.10) between J and t into consideration and obtain

$$\left|\nabla h^{(i)}\right| \to 0, \quad J \to 0, \quad t \to \infty; \quad \left|\nabla h^{(i)}\right| \to \infty, \quad J \to 1, \quad t \to 0. \tag{2.11}$$

Consequently, everywhere in domain closure $\bar{\Omega}$ including its boundary Γ and infinity, value J can vary in range $0 < J \le 1$.

As a result of the transition from variable J to variable t, system of equations (2.9) takes the form of Cauchy–Riemann relations [26]:

$$\psi_t = h_\theta^{(i)}, \quad \psi_\theta = -h_t^{(i)}.$$

Hence, we may introduce complex variables W and χ:

$$W = -h^{(i)} + i\psi, \quad \chi = t + i\theta.$$

Note that W means the complex potential of the flow and χ is the logarithm of the complex conjugate of the flow velocity; only the flow is potential not in the physical, but in any auxiliary plane associated with plane W by the conformal map. If necessary, we may associate this auxiliary plane as well as planes W and χ with physical plane x, y and the associating map cannot be conformal.

As a result of the Chaplygin transformation, filtration flow problem (2.3)–(2.7) reduces to the determination of one of the functions of complex variable $W(\chi)$ or $\chi(W)$. If, let us say, function $\chi(W)$ is determined, then we may return to physical plane x, y by writing the Khristianovich formulas (following from the system of equations (2.8)) of the total differentials of functions $x(h^{(i)}, \psi)$ and $y(h^{(i)}, \psi)$ [20, 24] given by

$$dx = -\cos\theta\sinh t\, dh^{(i)} - \sin\theta\cosh t\, d\psi, \quad dy = -\sin\theta\sinh t\, dh^{(i)} + \cos\theta\cosh t\, d\psi. \tag{2.12}$$

As a consequence of integration of these formulas we can find functions $x(h^{(i)}, \psi)$ and $y(h^{(i)}, \psi)$, and, hence, recover function $h^{(i)}(x, y)$ which is the solution to the problem of construction of the inner asymptotic expansion, Eqs. (1.4), (1.6)–(1.8).

Thus, the problem reduces to the determination of function $\chi(W)$. In the theory of filtration of abnormal liquids, many nontrivial solutions were constructed which are exact in the sense of determination of the closed form of function $\chi(W)$ [20]. In this context, the method of the velocity hodograph reduces, in fact, to the recovering of the form of planes W and χ and the construction of conformal map $W \to \chi$ if these planes have canonical form. Let us determine the boundary conditions for complex variables W and χ for the possible reconstruction of the form of the corresponding planes.

3. ANALYSING THE FORM OF PLANES OF HODOGRAPH OF χ AND COMPLEX POTENTIAL W

Conditions (2.5) are actually the conditions for variable χ in boundary segments BA and AC^A. Let us obtain the condition for variable W in these segments. Because of the symmetry, rays BA and AC^A are the stream lines, and on each of them the value of ψ is constant (see Fig. 1b). We may find the values of these constants by evaluating the flow character at infinity. Substituting estimation (2.7) into expression (2.1), we obtain that in the neighborhood of infinity $\mathbf{J} \approx -\nabla h^{(i)}$; therefore, there will be a potential flow from a sink of power $4\cos\gamma$ located at infinity [21]. We take function ψ equal to the total power of the source on stream line AB and obtain the boundary conditions for variable W:

$$AB: \quad \psi = 4\cos\gamma; \quad AC^A: \quad \psi = 3\cos\gamma. \tag{3.1}$$

Afterwards, taking expression (2.10) of quantity J in terms of t into consideration, from boundary condition (2.6) we obtain the expression relating the real and imaginary parts of variable χ in boundary segment BC^B

$$BC^B: \quad \cos\theta = \cos\gamma\cosh t, \tag{3.2}$$

and this relation is actually the equation of curve BC^B in plane χ. Further, we may deduce from it the values of complex variable χ at points B and C^B. Indeed, for point B, taking the first condition in (2.5) into consideration, we obtain from Eq. (3.2)

$$B: \quad \theta = 0, \quad t_B = \operatorname{arccosh}(|\cos\gamma|^{-1}). \tag{3.3}$$

For point C^B we have $\left|\nabla h^{(i)}\right| \to \infty$, and considering formula (2.11) from Eq. (3.2) we obtain

$$C^B: \quad \theta = -\gamma, \quad t = 0. \tag{3.4}$$

Concerning variable W, there is no explicit condition following directly from the Young–Laplace condition (1.3) in segment BC^B or from its analog (3.2).

Let us assess the behavior of variables W and χ at point A. It follows directly from the boundary condition at infinity (1.7) that at this point the value of $h^{(i)} \to -\infty$. Then, from estimation (2.7), accounting for the limiting relations (2.11), it follows that at point A the quantity $J \to 0$ and $t \to \infty$. As a result, we have

$$A: \quad h^{(i)} \to -\infty, \quad t \to \infty. \tag{3.5}$$

Now we determine the conditions for variables W and χ in boundary segment $C^A C^B$. This segment is simultaneously the fiber boundary and the boundary of meniscus Σ. Since in this segment $\left|\nabla h^{(i)}(x, y)\right| \to \infty$, from the second relation in Eq. (2.11), the boundary condition for χ in segment $C^A C^B$ follows,

$$C^A C^B: \quad t = 0. \tag{3.6}$$

Further, we find the boundary condition for variable W in segment $C^A C^B$. The entire segment in plane x, y is projected onto single point C. Function $h^{(i)}(x, y)$ in it is not completely determined, as was discussed from the beginning. Concerning function $\psi(x, y)$, the arguments were reported in [14] for the fact that function $\psi(x, y)$ is sufficiently determined at point C.

We bring another argument for this inference. It is well known that, in the filtration flows following the Darcy law [27], fully admissible are the situations in which, at the single points of the physical plane, function $\psi(x, y)$ is not completely determined: those are the points at which the flow peculiarities are located, including point sources (sinks), dipoles, etc. However, in the filtration flows following Sokolovskii law (2.3) and (2.4), the value of flow J is bounded from above, $J \le 1$, and, therefore, no point peculiarities of the flow in the final plane are admitted (though, the sources or sinks at infinity are fully admissible; they do not need $J \to \infty$). Consequently, in the flows following the Sokolovskii law, function $\psi(x, y)$ is completely determined everywhere in the closure of domain Ω including the domain boundary, but except for infinity.

Further, since the function $\psi(x, y)$ is completely determined at point C, that is, $\psi = \psi_{C^B} = \psi_{C^A}$, and since the quantity ψ_{C^A} is known, see formula (3.1), we may write the desired boundary condition for variable W in the segment $C^A C^B$

$$C^A C^B: \quad \psi = 3\cos\gamma. \tag{3.7}$$

Therefore, using determined conditions (2.5), (3.2)–(3.4) and (3.6) for variable $\chi = t + i\theta$, we may reconstruct the form of entire plane χ (see Fig. 2a). Note that the form of segment BC^B in plane χ is completely determined by expression (3.2); though, it is not canonical. We may determine the angles at vertices B and C^B of domain Ω^χ. Indeed, using Eqs. (3.3) and (3.4) as well as expression (3.2) as the implicitly given equation of curve BC^B in plane χ: $\theta = \theta_\Gamma(t)$, we can estimate derivatives $\dfrac{d\theta_\Gamma}{dt}$ at points B and C^B [14]:

$$B \in BC^B: \quad \frac{d\theta_\Gamma}{dt} \to \infty, \quad C^B \in BC^B: \quad \frac{d\theta_\Gamma}{dt} = 0.$$

Hence, both angles at vertices B and C^B of domain Ω^χ are right.

Similarly, using the found conditions (3.1), (3.5), and (3.7) for $W = -h^{(i)} + i\psi$, we can reconstruct the form of plane W itself (Fig. 2b). Since there is yet no boundary condition for W in segment BC^B, the configuration of the segment remains undetermined, which is emphasized by the dashed line used. Also, the exact positions of points B, C^A, and C^B remain undetermined. At the same time, the angles at vertices B and C^B of domain Ω^W are completely determined [14].

Indeed, the parametric equation of boundary BC^B in plane W may be formally prescribed in form BC^B: $\psi = \psi_\Gamma(h^{(i)})$. For boundary function $\psi_\Gamma(h^{(i)})$ we have the differential relation

$$BC^B: \quad \frac{d\psi_\Gamma}{dh^{(i)}} = \frac{\psi_y}{h_y^{(i)}}.$$

Taking expressions (2.8) and (2.10) into consideration, we transform this relation into

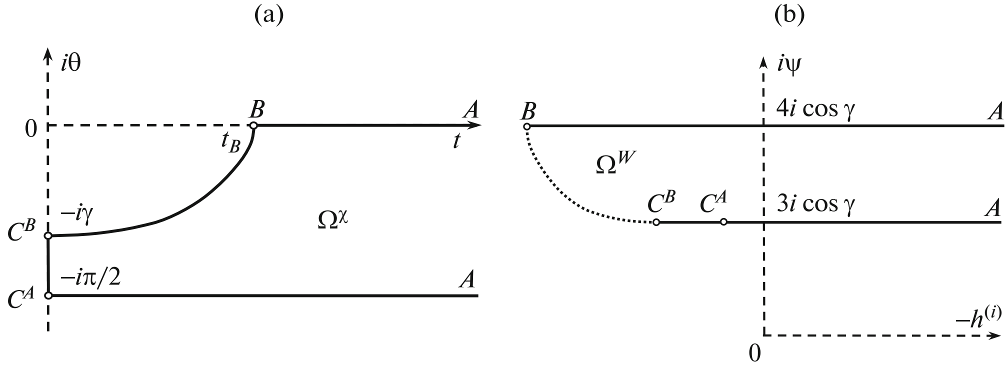

Fig. 2.

$$BC^B: \quad \frac{d\psi_\Gamma}{dh^{(i)}} = -\frac{\tanh t}{\tan\theta}.$$

At points B and C^B the quantities of derivative $\dfrac{d\psi_\Gamma}{dh^{(i)}}$ are determined by known values $\chi = t + i\theta$ as

$$B \in BC^B: \quad \frac{d\psi_\Gamma}{dh^{(i)}} = -\frac{\tanh t_B}{\tan\theta}\bigg|_{\theta\to 0} \to -\infty, \qquad C^B \in BC^B: \quad \frac{d\psi_\Gamma}{dh^{(i)}} = \frac{\tanh t}{\tan\gamma}\bigg|_{t\to 0} \to 0.$$

Consequently, in domain Ω^W the angle at vertex B is right, and the angle at vertex C^B is straight (Fig. 2b).

Let us explain where the jump of function $h^{(i)}(x, y)$ at point C comes from [14]. On the one hand, at this point the value of $\left|\nabla h^{(i)}\right| \to \infty$, and, therefore, quantities $J = 1$ and $t = 0$ are determined straight away. Then, from the Young–Laplace condition (3.2), we determine angle θ at point C if we approach this point along the domain boundary from the side of point B, $\theta = -\gamma$. On the other hand, straight line AC is the line of the flow symmetry which allows determining angle θ at point C if we approach it along the domain boundary from the side of point A, $\theta = -\pi/2$. Thus, if we approach point C along the domain boundary from different sides, then θ takes different values. Therefore, we need to introduce two points C^A and C^B and state that the entire boundary segment C^AC^B in plane $\chi = t + i\theta$ corresponds to boundary point C. Furthermore, since planes $\chi = t + i\theta$ and $W = -h^{(i)} + i\psi$ are related by the conformal map, for which the principle of boundary matching is valid, domain boundary Ω^W contains segment C^AC^B also. Hence, variable W at point C of plane x, y has a jump. However, stream function $\psi = \operatorname{Im}(W)$ for the flows following the Sokolovskii law is completely determined at all points of final plane x, y and admits no jumps. Therefore, it is function $h^{(i)}(x, y) = -\operatorname{Re} W$ that has a jump at point C.

4. PARAMETERIZATION OF SOLUTION AND DETERMINATION OF VELOCITY HODOGRAPH $\chi(\zeta)$

The form of domain Ω^W is not completely determined, which makes the problem of the direct determination of function $\chi(W)$ difficult. Therefore, similarly to the problems of the theory of ideal fluid jets [28, 29], it is necessary to use the technique of solution parameterization. The technique involves the introduction of the auxiliary plane of complex variable ζ of canonical form and reduction of the determination problem of function $\chi(W)$ to the two simpler determination problems of functions $\chi(\zeta)$ and $W(\zeta)$.

In place of $\zeta = \xi + i\eta$, let us take the complex plane where rectangle $\Omega^\zeta = ABC^BC^A$ reflects the domain of the flow with correspondence of the points depicted in Fig. 3a. According to the conformal map theory [26], only three boundary points of domain Ω^ζ can be prescribed arbitrarily. Therefore, we can choose only one of two values $K > 0$ and $K' > 0$ arbitrarily. Let it be K; however, we will choose the value of K

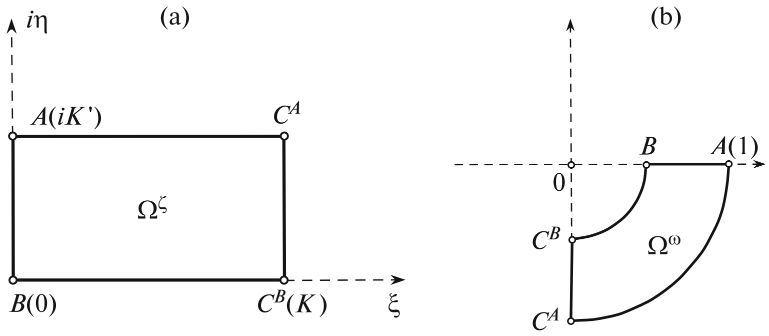

Fig. 3.

later. Consequently, the ratio of the rectangle sides K'/K is the undetermined parameter to be determined by the construction of conformal map $\chi(\zeta)$.

Conformal map $\chi(\zeta)$ can be constructed despite the noncanonical form of domain Ω^χ. To verify, we turn to condition (3.2) on BC^B. Taking the relation between the hyperbolic and trigonometric functions into account, we write the condition as follows

$$BC^B: \quad \cos\theta - \cos\gamma\cos(it) = 0$$

and reformulate it after identity transformation into the form [30]:

$$BC^B: \quad \tan\frac{\theta - it}{2}\tan\frac{\theta + it}{2} = q, \quad q = \frac{1-\cos\gamma}{1+\cos\gamma} < 1. \tag{4.1}$$

Here, we express θ and t in terms of χ and $\overline{\chi}$ and obtain

$$BC^B: \quad \left|\tanh\frac{\chi}{2}\right|^2 = q. \tag{4.2}$$

We introduce new complex variable ω related to variable χ by the conformal map

$$\omega(\chi) = \tanh\frac{\chi}{2}; \quad \chi(\omega) = \ln\frac{1+\omega}{1-\omega}, \tag{4.3}$$

where the logarithm denotes its single-valued branch which is real on AB.

Domain Ω^ω of plane ω has the canonical form. Indeed, because of condition (4.2) the circle arc corresponds to boundary segment BC^B in plane ω

$$BC^B: \quad |\omega| = q^{1/2}.$$

The segment of the real axis corresponds to boundary segment AB in plane ω whereas the segment of the imaginary axis corresponds to boundary segment C^AC^B, according to the form of domain Ω^χ (Fig. 2a) and formula (4.3). Finally, on boundary segment AC^A we have

$$AC^A: \quad \omega = \tanh\frac{2t - i\pi}{4} = i\frac{\tanh(t/2) - i}{\tanh(t/2) + i},$$

and, therefore, the arc of the unit circle corresponds to this boundary segment in plane ω, AC^A: $|\omega| = 1$. Thus, domain Ω^ω has the form of a quarter of the circle ring (Fig. 3b). This allows determining conformal maps $\zeta \to \omega$ and $\omega \to \zeta$ [26]

$$\zeta(\omega) = 2\pi^{-1} iK \ln(q^{-1/2}\omega), \quad \omega(\zeta) = q^{1/2} e^{-\frac{i\pi}{2K}\zeta}. \tag{4.4}$$

The logarithm denotes its single-valued branch which is real on boundary segment AB. We substitute values $\omega = 1$ and $\zeta = iK'$ corresponding to point A into formulas (4.4) and determine the relation of ratio K'/K of the sides of rectangle Ω^ζ with already determined parameter q (see formula (4.1))

$$\frac{K'}{K} = -\frac{1}{\pi}\ln q; \quad q = e^{-\pi K'/K}. \tag{4.5}$$

As a result, the determination of function $\chi(\zeta)$ reduces to the construction of conformal map $\zeta \to \chi$ as a combination of conformal maps (4.3) and (4.4)

$$\chi(\zeta) = \ln\left[\left(1 + q^{1/2} e^{-\frac{i\pi}{2K}\zeta}\right)\left(1 - q^{1/2} e^{-\frac{i\pi}{2K}\zeta}\right)^{-1}\right]. \tag{4.6}$$

The logarithm again denotes its single-valued branch which is purely real on boundary segment AB.

We proceed to the problem of determining function $W(\zeta)$. We suggest solving it in two stages: firstly, formulate and solve the boundary value problem for derivative $W'(\zeta)$ and, afterwards, reconstruct function $W(\zeta)$ itself by integration.

5. FORMULATION OF BOUNDARY VALUE PROBLEM FOR FUNCTION $W'(\zeta)$

The parametric equation of boundary BC^B in plane W can be given by two functions of argument ξ of the point of boundary BC^B in plane ζ,

$$BC^B: \quad h^{(i)} = h_\Gamma(\xi), \quad \psi = \psi_\Gamma(\xi); \quad \xi \in [0, K].$$

For these boundary functions the following differential relations are valid:

$$BC^B: \quad \begin{cases} \dfrac{dh_\Gamma}{d\xi} = \dfrac{dh_\Gamma}{dy}\dfrac{dy_\Gamma}{d\xi} = \left(h_y^{(i)} + h_x^{(i)}\dfrac{dx_\Gamma}{dy_\Gamma}\right)\Bigg|_\Gamma \dfrac{dy_\Gamma}{d\xi}, \\[2ex] \dfrac{d\psi_\Gamma}{d\xi} = \dfrac{d\psi_\Gamma}{dy}\dfrac{dy_\Gamma}{d\xi} = \left(\psi_y + \psi_x\dfrac{dx_\Gamma}{dy_\Gamma}\right)\Bigg|_\Gamma \dfrac{dy_\Gamma}{d\xi}. \end{cases}$$

Taking formulas (2.6), (2.8), and (2.10) and conditions $x_\Gamma \equiv 0$ into consideration, we rewrite these relations in the form:

$$BC^B: \quad \frac{dh_\Gamma}{d\xi} = -\frac{\sin\theta}{\sinh t}\frac{dy_\Gamma}{d\xi}, \quad \frac{d\psi_\Gamma}{d\xi} = \cos\gamma\frac{dy_\Gamma}{d\xi}. \tag{5.1}$$

We construct the linear combination of Eqs. (5.1) to eliminate derivative $\dfrac{dy_\Gamma}{d\xi}$. As a result, we obtain the boundary differential relation:

$$BC^B: \quad \sin\theta\frac{d\psi_\Gamma}{d\xi} + \cos\gamma\sinh t\frac{dh_\Gamma}{d\xi} = 0. \tag{5.2}$$

Simultaneously, everywhere in plane ζ including boundary BC^B, another differential relation is satisfied:

$$BC^B: \quad W'(\zeta) = \frac{d}{d\xi}(-h^{(i)} + i\psi) = -\frac{dh^{(i)}}{d\xi} + i\frac{d\psi}{d\xi}.$$

According to this, derivatives $\dfrac{dh_\Gamma}{d\xi}$ and $\dfrac{d\psi_\Gamma}{d\xi}$ contained in relation (5.2) are the real and imaginary parts of analytical function $W'(\zeta)$ on boundary BC^B.

Therefore, relation (5.2) is, in fact, the boundary equation that links the real and imaginary parts of function $W'(\zeta)$ on this boundary,

$$BC^B: \quad \mathrm{Re}\left\{\overline{G_\Gamma(\xi)} i W'(\zeta)\right\} = 0, \tag{5.3}$$

where $G_\Gamma(\xi)$ is the auxiliary function of point $\zeta = \xi \in [0, K]$ of boundary BC^B,

$$BC^B: \quad G_\Gamma(\xi) = \sin\theta(\xi) + i\cos\gamma \sinh t(\xi). \tag{5.4}$$

Using conditions (3.1) and (3.7) for function $W(\zeta)$, we find the conditions for function $W'(\zeta)$ in the remaining segments of domain boundary $\partial\Omega^\zeta$

$$AC^A: \quad \mathrm{Im}\, W'(\zeta) = 0; \quad AB \cup C^A C^B: \quad \mathrm{Re}\, W'(\zeta) = 0. \tag{5.5}$$

We also estimate the behavior of function $W'(\zeta)$ in the vicinity of all key points of domain boundary $\partial\Omega^\zeta$: A, B, C^A, and C^B. According to boundary conditions (3.7), harmonic function $\psi(\xi,\eta)$ has a jump at point A. Consequently, conjugate harmonic function $-h^{(i)}(\xi,\eta)$ has a logarithmic character at point A [26]. This allows estimating the behavior of function $W(\zeta)$ in the neighborhood of point A,

$$\zeta \sim iK': \quad W(\zeta) = -\frac{2\cos\gamma}{\pi}\ln\left(\zeta - iK'\right) + O(1) + O\left(\zeta - iK'\right). \tag{5.6}$$

According to this, function $W'(\zeta)$ has the simple pole at point A. At points B, C^B, and C^A we may estimate the behavior of function $W'(\zeta)$ comparing the angles at the corresponding vertices of domains Ω^W and Ω^ζ [26]. As a result, we have

$$\begin{gathered} W'(\zeta): \quad \text{point } A \text{ is the simple pole,} \quad \text{point } B \text{ is regular,} \\ \text{point } C^A \text{ and } C^B \text{ are the simple zeros.} \end{gathered} \tag{5.7}$$

As a consequence, we obtain mixed boundary value problem (5.3)–(5.5) and (5.7) for function $W'(\zeta)$ which is regular everywhere in the closure of domain Ω^ζ except for point A where it has the simple pole.

6. SOLVING PROBLEM FOR FUNCTION $W'(\zeta)$

In the theory of boundary value problems for the analytical function, the problems with the boundary condition of type (5.3) are called the Hilbert homogeneous problems, and their solution can be constructed analytically [31]. To this end, it is necessary to transform the problem statement such that in condition (5.3) we place the boundary condition of some analytical function $g(\zeta)$ of complex variable ζ instead of boundary function $G_\Gamma(\xi)$.

We multiply boundary function $G_\Gamma(\xi)$ by $\cos\theta(\xi)$. Then, taking condition (3.2) into account, from definition (5.4), we derive the relation:

$$BC^B: \quad \arg\{G_\Gamma(\xi)\} = \arg\{i\sinh\overline{\chi}\} = \arg\{i/\sinh\chi\}. \tag{6.1}$$

Here, we take into consideration that

$$\sinh\overline{\chi} = \sinh t(\xi)\cos\theta(\xi) - i\sin\theta(\xi)\cosh t(\xi).$$

The zero right-hand side of condition (5.3) allows multiplying it by any real factor. We choose $\cos\theta(\xi)\left|\sinh\chi\right|^{-2}$ as this factor. Therefore, according to relation (6.1), we may replace boundary function $G_\Gamma(\xi)$ in condition (5.3) by the following function (we take formulas (4.5) and (4.6) into account):

$$g(\zeta) = \frac{i}{\sinh\chi(\zeta)} = -\sin\frac{\pi\left(\zeta - iK'\right)}{2K}. \tag{6.2}$$

As a result, boundary Eq. (5.3) for function $W'(\zeta)$ takes the form

$$BC^B: \quad \mathrm{Re}\left\{\overline{g(\zeta)} i W'(\zeta)\right\} = 0, \tag{6.3}$$

where $g(\zeta)$ is the function of complex variable ζ analytical everywhere in the closure of domain Ω^{ζ}. Considering its form (6.2), we may state that, in the boundary segments of domain Ω^{ζ} distinct from BC^B, this function satisfies the homogeneous relations

$$AB: \quad \operatorname{Re} g(\zeta) = 0; \quad C^A C^B \cup C^A A: \quad \operatorname{Im} g(\zeta) = 0$$

as the desired function $W'(\zeta)$ does, see formula (5.5). Then we can seek the solution to boundary value problem (6.3) in the multiplicative form

$$W'(\zeta) = C_0 g(\zeta) f(\zeta) \tag{6.4}$$

up to an arbitrary real factor C_0. Here, $f(\zeta)$ is the function satisfying the homogeneous boundary conditions on the boundaries of domain Ω^{ζ}

$$AB \cup BC^B \cup C^A A: \quad \operatorname{Im} f(\zeta) = 0; \quad C^A C^B: \quad \operatorname{Re} f(\zeta) = 0. \tag{6.5}$$

The qualitative behavior of sought function $f(\zeta)$ in the neighborhood of all characteristic points of domain boundary $\partial\Omega^{\zeta}$ can be estimated (i), by using estimates (5.7) obtained before for the behavior of function $W'(\zeta)$ in the closure of domain Ω^{ζ} and (ii), from the fact that function $g(\zeta)$ has the simple zero at point A ($\zeta = iK'$) and differs from both zero and infinity at all remaining points of the closure of domain Ω^{ζ}. As a result, we obtain

$$\begin{aligned} f(\zeta): \quad & \text{point } A \text{ is the second-order pole,} \quad \text{point } B \text{ is regular,} \\ & \text{point } C^A \text{ and } C^B \text{ are the simple zeros.} \end{aligned} \tag{6.6}$$

Due to the multiplicative representation of function $W'(\zeta)$ in the form of Eq. (6.4), the problem of its determination reduces to the problem of determination of function $f(\zeta)$, analytical in rectangle Ω^{ζ}, which satisfies homogeneous boundary conditions (6.5) on the rectangle boundary and behaves at the rectangle vertices according to data (6.6). Because of homogeneous boundary conditions (6.5), function $f(\zeta)$ may be symmetrically extended through the boundary of rectangle Ω^{ζ} to the entire plane ζ, and, therefore, this function is the elliptic (doubly periodic) function. According to the theory of elliptic functions [32], function $f(\zeta)$ can be reconstructed from its poles and zeros.

We use the apparatus of the Jacobi elliptic functions [30]. It is common to write them as functions depending on the variable and some additional parameter $m \in [0,1]$. Recall that up to now parameter $K > 0$ remains arbitrary (see Section 4). To use the apparatus of the Jacobi elliptic functions directly in plane ζ, we need to choose parameters K and m in a special manner [26, 30]:

$$K = \frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right], \quad m = 16q\left[1 + \sum_{n=1}^{\infty} q^{n^2+n}\right]^4 \left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^{-4}. \tag{6.7}$$

It is well known [30, 32] that Jacobi elliptic function $\operatorname{cn}(\zeta|m)$ has the simple zero at point C^B and the simple pole at point A in rectangle Ω^{ζ} presented in Fig. 3a while Jacobi elliptic function $\operatorname{dn}(\zeta|m)$ has the simple zero at point C^A and the simple pole at point A. Hence, elliptic function $f(\zeta)$ is the multiplication of the following two functions:

$$f(\zeta) = \operatorname{cn}(\zeta|m)\operatorname{dn}(\zeta|m). \tag{6.8}$$

Direct verification can prove that found function $f(\zeta)$ satisfies boundary condition (6.5).

We substitute expressions (6.2) and (6.8) into relation (6.4) and conclude that

$$W'(\zeta) = C_0 \operatorname{cn}(\zeta|m)\operatorname{dn}(\zeta|m)\sin\frac{\pi(\zeta - iK')}{2K}. \tag{6.9}$$

We may determine factor C_0 by evaluating the simple pole of function $W'(\zeta)$ at point A: $\zeta = iK'$. Indeed, the behavior of Jacobi functions $\operatorname{cn}(\zeta|m)$ and $\operatorname{dn}(\zeta|m)$ in the neighborhood of this point is well known [32],

$$\zeta \sim iK': \quad \mathrm{cn}(\zeta|m) = -\frac{i}{m^{1/2}(\zeta - iK')} + O(\zeta - iK'), \quad \mathrm{dn}(\zeta|m) = -\frac{i}{\zeta - iK'} + O(\zeta - iK'). \tag{6.10}$$

Taking the known behavior of sine at small arguments into consideration, from Eq. (6.9) we determine the estimation of the behavior of function $W'(\zeta)$ in the neighborhood of point A:

$$\zeta \sim iK': \quad W'(\zeta) = -\frac{\pi C_0}{2Km^{1/2}\left(\zeta - iK'\right)} + O(\zeta - iK'). \tag{6.11}$$

Then we compare it with estimation (5.6) obtained earlier to the behavior of function $W(\zeta)$ in the same neighborhood and find

$$C_0 = 4Km^{1/2}\pi^{-2}\cos\gamma. \tag{6.12}$$

Thus, the form of function $W'(\zeta)$ is completely established.

7. RECONSTRUCTION OF FUNCTION $h^{(i)}(x, y)$

Since the problem was parameterized by the introduction of auxiliary plane $\zeta = \xi + i\eta$, to obtain function $h^{(i)}(x, y)$ it is necessary to determine functions $h^{(i)}(\xi, \eta)$, $x(\xi, \eta)$, and $y(\xi, \eta)$. Firstly, we integrate function $W'(\zeta)$ in the form of Eq. (6.9), for instance, from point C^A. Taking boundary condition (3.7) into consideration, we have

$$W(\zeta) = -h_{C^A} + 3i\cos\gamma + I(\zeta), \quad I(\zeta) = C_0 \int_{K+iK'}^{\zeta} \mathrm{cn}\,\zeta\,\mathrm{dn}\,\zeta \sin\frac{\pi\left(\zeta - iK'\right)}{2K} d\zeta, \tag{7.1}$$

where we introduce one more undetermined parameter of the problem h_{C^A} which is the meniscus level at point C^A. Note that due to the presence of a simple pole (6.11) in derivative $W'(\zeta)$ at point A: $\zeta = iK'$, integral $I(\zeta)$ has a logarithmic singularity at this point

$$\zeta \sim iK': \quad I(\zeta) = -2\pi^{-1}\cos\gamma\ln\left(\zeta - iK'\right) + O(1), \tag{7.2}$$

where we have taken formula (6.12) into account. As a consequence, we obtain

$$h^{(i)}(\xi, \eta) = h_{C^A} - \mathrm{Re}\{I(\zeta)\}. \tag{7.3}$$

To determine the functions $x(\xi, \eta)$ and $y(\xi, \eta)$, we have the differential relations resulting from the Khristianovich formulas (2.12) (indices ξ and η denote the corresponding partial derivatives)

$$\begin{aligned} x_\xi &= -\cos\theta\sinh t\, h_\xi^{(i)} - \sin\theta\cosh t\,\psi_\xi, \quad y_\xi = -\sin\theta\sinh t\, h_\xi^{(i)} + \cos\theta\cosh t\,\psi_\xi, \\ x_\eta &= -\cos\theta\sinh t\, h_\eta^{(i)} - \sin\theta\cosh t\,\psi_\eta, \quad y_\eta = -\sin\theta\sinh t\, h_\eta^{(i)} + \cos\theta\cosh t\,\psi_\eta, \end{aligned} \tag{7.4}$$

where the partial derivatives of functions $h^{(i)}(\xi, \eta)$ and $\psi(\xi, \eta)$ can be associated with the function $W'(\zeta)$ using the differential relations [26]

$$W'(\zeta) = -h_\xi^{(i)} + i\psi_\xi = \psi_\eta + ih_\eta^{(i)}.$$

Since the form of functions $\chi(\zeta)$ and $W'(\zeta)$ is known, formulas (7.4) allow finding the partial derivatives of functions $x(\xi, \eta)$ and $y(\xi, \eta)$. Then, with integrals

$$\begin{aligned} &BC^B: \quad x(\xi, 0) = 0, \quad y(\xi, 0) = \int_0^\xi (y_\xi)_{\eta=0} d\xi, \\ &x(\xi_0, \eta_0) = \int_0^{\eta_0} (x_\eta)_{\xi=\xi_0} d\eta, \quad y(\xi_0, \eta_0) = y(\xi_0, 0) + \int_0^{\eta_0} (y_\eta)_{\xi=\xi_0} d\eta, \end{aligned} \tag{7.5}$$

we determine functions $x(\xi, \eta)$ and $y(\xi, \eta)$ firstly in segment BC^B and then at an arbitrary point ξ_0, η_0 of domain Ω^ζ.

We compute integrals (7.3) and (7.5) by the standard means of integration using MATLAB everywhere except for the neighborhood of point A, i.e., in the neighborhood of infinity in the physical plane. Thus, we determine function $h^{(i)}(x,y)$ indirectly. It remains to link all auxiliary parameters of the problem with the constitutive parameters or, in other words, to solve the problem of parameters.

8. THE PROBLEM OF PARAMETERS

The problem of construction of inner asymptotic expansion (1.4) and (1.6)–(1.8) as well as initial boundary value problem (1.1)–(1.4) contains of only two parameters: contact angle γ and Bond number ε. They must determine all auxiliary parameters of the problem. With the second formula of Eq. (4.1) and with formulas (4.5), (6.7), and (6.12), contact angle γ determines auxiliary parameters q, K, K', m, and C_0 uniquely. Thus, there remains just one undetermined parameter h_{C^A} to be found, which is the integration constant appearing in reconstruction of function $W(\zeta)$.

To this end, we can use the condition at infinity (1.7), including the term of order unity. However, in the obtained analytical solution there is no explicit dependence $h^{(i)}(x,y)$; it is indirect because of the introduction of auxiliary plane ζ. Therefore, to estimate the type of condition at infinity (1.7), we need to perform several intermediate estimations.

Firstly, we estimate the behavior of function $h^{(i)}(\xi,\eta)$ in the neighborhood of point $A \in AC^A$

$$\zeta = \xi + iK', \quad \xi \sim 0: \quad \begin{cases} h^{(i)}(\xi,\eta) = 2\pi^{-1}\cos\gamma \ln\xi - B_0 + O(\xi), \\ h^{(i)}(\xi,\eta) = h_{C^A} - \operatorname{Re}\{I(\xi + iK')\}. \end{cases} \tag{8.1}$$

The first estimation was obtained with formula (5.6) where the term of order unity is denoted by B_0. The second estimation was deduced with formula (7.1). Comparing these estimations, we obtain

$$h_{C^A} = \lim_{\xi\to 0}\left[2\pi^{-1}\cos\gamma\ln\xi + \operatorname{Re}\{I(\xi + iK')\}\right] - B_0. \tag{8.2}$$

The logarithmic term in the square brackets is exactly compensated by logarithmic singularity (7.2) in integral $I(\xi + iK')$; therefore, the limit of the expression in the square brackets is finite and can be determined analytically [32]

$$\lim_{\xi\to 0}\left[\frac{2\cos\gamma}{\pi}\ln\xi + \operatorname{Re} I(\xi + iK')\right] = \frac{2\cos\gamma}{\pi}\left[1 - \frac{2K}{\pi} - \ln\frac{\sqrt{1-m}}{2} - \frac{1+q}{1-q} + \frac{A_0}{2}\right], \tag{8.3}$$

where

$$A_0 = \sum_{n=1}^{\infty}\frac{1+q^{2n+1}}{1-q^{2n+1}}\frac{(2n+1)^{-1} - (-1)^n}{n(n+1)}. \tag{8.4}$$

Thus, according to formula (8.2), the determination of parameter h_{C^A} reduces to the determination of auxiliary parameter B_0 arising in estimation (8.1). To find it we employ the boundary condition at infinity (1.7) directly. We restrict ourselves to the neighborhood of point $A \in AC^A$ and rewrite this condition in the form:

$$x = 0, \quad y \sim -\infty: \quad h^{(i)}(x,y) = -2\pi^{-1}\cos\gamma\ln(2^{-1}|y|e^{E}\varepsilon^{1/2}) + O(|y|^{-1}). \tag{8.5}$$

To compare condition (8.5) with first estimation (8.1), we need to find the behavior of $y(\xi)$ in the neighborhood of point $A \in AC^A$. We use the second differential relation of Eq. (7.4) and the fact that on AC^A: $\theta = -\pi/2$. It follows that

$$AC^A: \quad y_\xi = \sinh t\, h_\xi^{(i)} = -\sinh t \operatorname{Re}\{W'(\zeta)\}\big|_{\zeta=\xi+iK'}. \tag{8.6}$$

In the neighborhood of point A, the behavior of function $W'(\zeta)$ is known, see formula (6.11). We estimate the behavior of function $\sinh t$ there. In the neighborhood of point A: $t \sim +\infty$ and

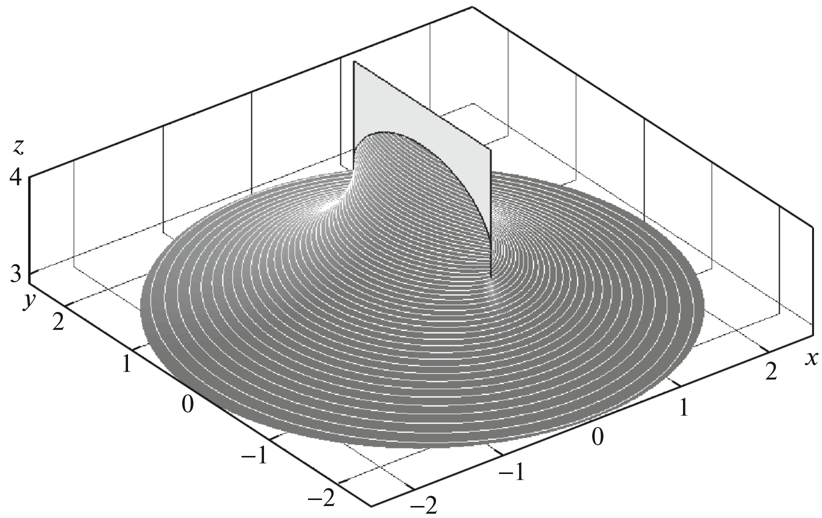

Fig. 4.

$\sinh t = e^t/2 + O(e^{-t})$. Taking $t = \mathrm{Re}\,\chi(\zeta)$ into account and employing the obtained form (4.6) of the function $\chi(\zeta)$, we find the estimation of $\sinh t$ in the neighborhood of point $A \in AC^A$

$$\zeta = \xi + iK', \quad \xi \sim 0: \quad \sinh t = 2K\pi^{-1}\xi^{-1} + O(\xi). \tag{8.7}$$

We substitute expressions (6.11) and (8.7) into relation (8.6) and may estimate the behavior of derivative y_ξ in the neighborhood of point $A \in AC^A$, and after integration we may estimate the behavior of function $y(\xi)$ as well

$$\zeta = \xi + iK', \quad \xi \sim 0: \quad y(\xi) = -C_0 m^{-1/2}\xi^{-1} + O(1).$$

We substitute this expression into condition (8.5) with allowance for formula (6.12) and acquire one more estimation of the behavior of function $h^{(i)}(\xi,\eta)$ in the neighborhood of point $A \in AC^A$:

$$\zeta = \xi + iK';, \quad \xi \sim 0: \quad h^{(i)}(\xi,\eta) = 2\pi^{-1}\cos\gamma\left[\ln\xi - \ln(2\pi^{-2}K\cos\gamma e^E\varepsilon^{1/2})\right] + O(\xi). \tag{8.8}$$

Comparing this estimate (8.8) with the first estimate of (8.1), we obtain

$$B_0 = 2\pi^{-1}\cos\gamma\ln(2\pi^{-2}K\cos\gamma e^E\varepsilon^{1/2}). \tag{8.9}$$

As a result, we substitute the acquired expressions of Eqs. (8.3) and (8.9) into formula (8.2) and find the last undetermined parameter of the problem:

$$h_{C^A} = \frac{2\cos\gamma}{\pi}\left[1 - \frac{2K}{\pi} - \ln\left(\frac{K}{\pi^2}e^E\sqrt{(1-m)\varepsilon}\cos\gamma\right) - \frac{1+q}{1-q} + \frac{A_0}{2}\right]. \tag{8.10}$$

The notation of (8.4) is employed. The problem of parameters has been solved.

9. ANALYSIS OF RESULTS

Applying formulas (7.3) and (7.5) and considering expression (8.10), we can find the principal term of inner asymptotic expansion $h^{(i)}(x,y)$, and, afterwards, we can find the principal term of uniform asymptotic expansion $h^{(u)}(x,y)$ using the Van Dyke's formula (1.9). As a result, we may reconstruct the form of meniscus $\Sigma^{(u)}$ approaching sought meniscus Σ uniformly in the entire physical plane.

In Fig. 4 we presented an example of meniscus $\Sigma^{(u)}$ on the ribbon-like fiber for Bond number $\varepsilon = 10^{-5}$ and contact angle $\gamma = \pi/8$ where the vertically located light gray plate is the fiber (its edges and the contact

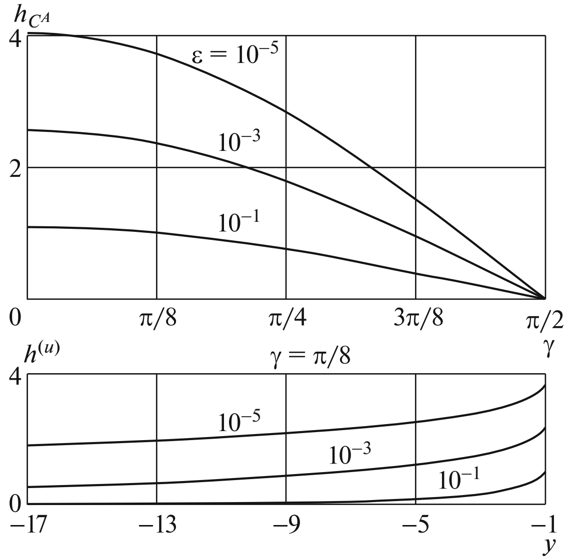

Fig. 5.

line are depicted by the black color); the dark gray surface is the meniscus; and the light lines are the contour lines on its surface. The lower contour lines located far from the fiber differ slightly from the circle. As the level increases, the lines approach the fiber and become oval, but retain the smoothness. The first contour line touching the fiber is line $h^{(u)} = h_{C^A}$ which remains smooth everywhere also. All contour lines in interval $(h_{C^A}, h_{C^B}]$ touch the fiber in its sharp edge and have a kink. The contour lines in interval (h_{C^B}, h_B) cross the fiber at the points of contact line Γ_c and also have a kink.

For the contour line from interval $(h_{C^A}, h_{C^B}]$, we can assess the value of angle 2α at the kink point (Fig. 1c). The vicinity of the fiber is the action zone of the inner expansion where $h^{(u)} \approx h^{(i)}$. For the tangent to contour line $h^{(i)} = \text{const}$, the following relations obtain, according to the Khristianovich formulas (2.12):

$$h^{(i)} = \text{const}: \quad dx = -\sin\theta\cosh t\, d\psi, \quad dy = \cos\theta\cosh t\, d\psi \Rightarrow \frac{dy}{dx} = -\cot\theta.$$

This means that the tangent to the contour line forms angle $\alpha = -\theta$ with the y axis [14]. In segment C^AC^B, according to Fig. 2a, θ varies from $-\pi/2$ to $-\gamma$. As a result, we find that in this segment angle α changes from value γ at point C^B to $\pi/2$ at point C^A. Thus, the meniscus is locally a helicoid in the neighborhood of every point of segment C^AC^B [33].

The configurations of contact line Γ_c and meniscus $\Sigma^{(u)}$ are determined in general by two nondimensional parameters: contact angle γ and Bond number $\varepsilon \ll 1$. In particular, formula (8.10) defines the dependence of height h_{C^A} of the liquid level at point C^A on γ and ε. In the upper part of Fig. 5 we provide curves $h_{C^A}(\gamma)$ for different ε. In general, the decrease of any parameter, γ or ε, results in the increasing value h_{C^A}. Note that the finite values $h_{C^A}(\varepsilon)$ correspond to the limit $\gamma \to 0$. The character of the influence of Bond number ε on the meniscus form is demonstrated in the lower part of Fig. 5 where curves AC^A are presented which are the profiles of the meniscus in its section by the plane coinciding with the fiber plane. All three profiles correspond to one value of contact angle $\gamma = \pi/8$ and different values ε. For $\varepsilon = 10^{-1}$ at distance $r = \sqrt{x^2 + y^2}$ equal to 10, the meniscus achieves nearly the zero level. With the decrease in value of ε, the achievement of the zero level by the meniscus moves consequently towards the large values of r.

At the same time, quantity ε has no effect on the form of meniscus Σ^u in the vicinity of the fiber completely determined by angle γ, and quantity ε has no effect on the form of contact line Γ_c if we ignore the absolute values of $h^{(u)}$. Figure 6 shows the configurations of contact line BC^B referred to point B for dif-

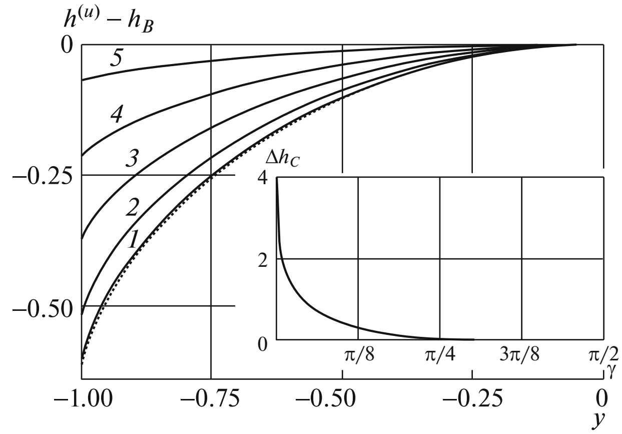

Fig. 6.

ferent values of contact angle γ with uniform steps of 0.1π from $\gamma = 0.05\pi$ for curve *1* up to $\gamma = 0.45\pi$ for curve *5*. For $\gamma < 0.05\pi$ the relative configuration of contact line BC^B almost ceases to vary: the dashed line shown for the case with $\gamma = 10^{-5}$ differs slightly from curve *1*.

The jump of the liquid level at the fiber edges $\Delta h_C = h_{C^B} - h_{C^A}$, according to formula (7.1) is determined by the integral

$$\Delta h_C = C_0 \operatorname{Re}\left\{ \int_K^{K+iK'} \operatorname{cn}\zeta \operatorname{dn}\zeta \sin\frac{\pi(\zeta - iK')}{2K} d\zeta \right\}, \tag{9.1}$$

which does not include singularities. Utilizing the properties of the Jacobi elliptic function [32], this integral can be computed similarly to integral $I(\zeta)$ in formula (7.1) in the case of the certain upper limit, see formula (8.3). As a consequence, we acquire

$$\Delta h_C = \frac{\cos\gamma}{\pi}\left\{ \frac{2K}{\pi}\left[2 - (1+q)m^{1/2}q^{-1/2}\right] + 1 - \ln(4q) - \sum_{n=1}^{\infty}(-1)^n \frac{(1+q)q^n - 2q^{2n+1}}{n(n+1)(1-q^{2n+1})} \right\}, \tag{9.2}$$

from which it clearly follows that the value of liquid level jump at the fiber edges Δh_C is also independent of Bond number ε and is determined only by the value of contact angle γ. In the right lower corner of Fig. 6, we draw dependence $\Delta h_C(\gamma)$. Conspicuous is the fact that jump Δh_C increases unboundedly for $\gamma \to 0$ in contrast to value h_{C^A}. This is established from formula (9.2) rigorously.

Indeed, taking into account both dependence (4.1) between γ and q and the second relation in (6.7), for $\gamma \to 0$ we have $mq^{-1} \approx 16$ and $q \approx \gamma^2/4$. Therefore, from formula (9.2) it follows that

$$\gamma \to 0: \quad \Delta h_C \approx -2\pi^{-1}\ln\gamma.$$

Such unrealistic behavior of jump Δh_C at small values of contact angle γ can be explained by the fact that on contact line Γ_c, $|\nabla h(x,y)| \to \infty$ for $\gamma \to 0$ and, in the vicinity of the ribbon-like fiber, probably, another boundary layer arises that is not taken into consideration in the scheme of the asymptotic analysis [13].

In this regard, we note that the case of small γ and, especially $\gamma = 0$, requires almost always a separate consideration even for the fibers with the smooth profile. For instance, in work [11] mentioned above, where numerical analysis was performed for the meniscus configuration on the fiber with an elliptic profile, the case of small γ was simply excluded, and the ultimately small calculation value of contact angle γ was approximately $\pi/3$.

10. CONCLUSIONS

Within the asymptotic approach [13] we constructed the analytical solution to the problem of configuration of the external meniscus formed under the immersion of the ribbon-like fiber into a liquid in the case when the fiber width was small compared to the capillary length. We analyzed the effect of the Bond number and the contact angle on the meniscus shape. In full correspondence with the conclusions obtained earlier [14], we confirmed the presence of a finite jump of the liquid level at the sharp edges for any values of the contact angle different from $\pi/2$. The jump tends to infinity as the contact angle decreases to zero. Therefore, an adjustment of the asymptotic approach [13] is needed in the case when both the fiber profile is nonsmooth and the contact angle value is small.

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