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Sampled-data $\mathcal{H}_\infty$ filtering for linear parameter varying systems

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In this paper, we address the sampled-data filter design problem for continuous-time linear parameter-varying (LPV) systems. The filtering error system obtained from augmenting a continuous-time LPV system and the sampled-data filter is a hybrid system. The sampled-data filter design objective is to ensure the error system stability and a prescribed level of the induced energy-to-energy gain (or $\mathcal{H}_\infty$ norm) from the disturbance input to the estimation error. To this purpose, we employ a *lifting method* to derive an equivalent discrete-time LPV representation for the continuous-time LPV system. In the present study, the sampled-data filter synthesis conditions are formulated in terms of linear matrix inequality optimisation problems. The viability of the proposed design method to cope with variable sampling rates is illustrated through numerical examples, where reliable estimation of the LPV system outputs is achieved.

Keywords: sampled-data filter; lifting method; linear parameter-varying systems

1. Introduction

Filters utilise the output measurements of a dynamic system to estimate the states or a linear combination of the states of the system. The performance of a filter is often assessed in terms of a measure of the state estimation error that is the difference between the actual and the estimated state. The literature on various versions of Kalman filtering technique is rich (see, e.g. Grewal and Andrews 2001; Eubank 2006). Using the statistical information of the exogenous disturbance input of the system, the Kalman filter minimises the variance of the state estimation error. In contrast, when the statistical information is unknown, the $\mathcal{H}_\infty$ filtering method can be proposed to minimise the energy of the estimation error signal for the worst bounded energy disturbance input (Nagpal and Khargonekar 1991; Geromel, Bernussou, Garcia, and Oliveria 2000). Other performance measures such as energy-to-peak gain, peak-to-peak gain or a combination of these objectives from the disturbance input to the estimation error signal can be also utilised for the filtering design problem (Skelton, Iwasaki, and Grigoriadis 1998).

Among several factors that affect the search for improved filter design strategies, one can mention the challenges posed by signal recovery and estimation under time-varying dynamics. Recently, linear parameter-varying (LPV) systems theory has led to significant steps forward in the study of time-varying systems. LPV systems constitute a class of linear time-varying systems whose dynamics depends on time-varying parameters, also known as scheduling parameters. When such parameters are available in real time, they can be employed for control and filter synthesis purposes resulting in less conservative conditions compared to fixed robust controllers and filters (Rugh and Shamma 2000). In addition, within the framework of quasi-LPV, we can model a large class of nonlinear systems as LPV systems. In a quasi-LPV system, the scheduling parameters are not only a function of exogenous signals but also of the system states. Some of the recent studies in this area have addressed the filter design problem in LPV systems, especially in the continuous-time domain. In Mahmoud and Boujarwah (2001), the $\mathcal{H}_\infty$ filtering problem for a class of polytopic LPV systems is considered. The design of fault detection and isolation filters for LPV systems has been another area of interest for researchers (see, e.g. Abdalla, Nobrega, and Grigoriadis 2001; Grenaille, Henry, and Zolghadri 2008). In addition, the authors in Mohammadpour and Grigoriadis (2008) addressed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter design for LPV systems, where the system contains delay in the states. While the aforementioned references examine the LPV filter design problem in continuous-time domain, our main concern in this paper is to develop an LPV filter design method that is implemented in discrete time.

In the past few decades, advances in computing devices has led to efficient ways to digitally implement controllers and filters for continuous-time physical systems (Chang, Tsai, and Shieh 2002). Digital implementation of the filters results in a mixture of continuous-time and discrete-time signals and systems forming a hybrid dynamical system. In a typical hybrid process, the measurable output signals

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are periodically sampled with an analog-to-digital (A/D) converter. Then, the digitised outputs are processed using a digital device (controller or filter) and fed to the plant after being converted to analog signals through a digital-to-analog (D/A) converter. Due to the aforementioned hybrid nature of the system, there has been a need to adapt the continuous-time filtering theory to capture this level of complexity. The issue of digital implementation has been studied primarily within the area of digital control theory. Therein, the existing methods only approximately cope with the behaviour of the continuous-time signals in the control system since the behavior of such systems can be captured and studied only at the sampling instants (Yamamoto 1990; Franklin, Powell, and Workman 1997). In contrast to the traditional approaches, sampled-data control theory provides an exact solution method for the analysis and synthesis of sampled-data control systems with the inter-sample behavior taken into account (Chen and Francis 1995). Within this context, Bamieh and Pearson (1992) presented a framework to design an $\mathcal{H}_\infty$ controller for sampled-data systems. Using a lifting technique, they solved the sampled-data control problem in terms of an equivalent discrete-time system, where the plant is augmented with the sampler and hold devices and is lifted to a system with a finite-dimensional state-space representation and with infinite-dimensional input and output spaces (see Yamamoto 1990; Chen and Francis 1995). The lifting technique was shown to preserve the input–output energy-to-energy gain of the closed-loop hybrid system. Tan and Grigoriadis (2000) and Tan, Grigoriadis, and Wu (2002) used the idea of lifting technique and applied it to the LPV sampled-data systems, where they solved energy-to-energy and energy-to-peak gain problems to design state feedback and output feedback controllers. A benefit of this formulation is that the sampling interval can be varying as a function of the scheduling parameters. This is the case in event-sampling systems, such as engines where the sampling interval is a function of the engine speed. There have also been some additional recent efforts on sampled-data control design for LPV systems (see, e.g. Lawrence 2001; Farret, Duc, and Harcaut 2002).

The lifting method essentially maps the hybrid system to the discrete-time domain in an equivalent representation. As an alternative method, Fridman, Seuret, and Richard (2004) introduced a sampled-data $\mathcal{H}_\infty$ control and filtering methodology that maps the hybrid system to the continuous-time domain (see also Fridman, Shaked, and Suplin 2005; Suplin, Fridman, and Shaked 2007, 2009). In this approach, the digital control law is represented as a delayed control and thus the augmentation of the plant and the controller (or the filter) leads to a state-delay system. Comparing the two approaches described above, the lifting method is more cumbersome but results in an improved performance, while the input delay approach is more conservative due to the introduction of delay to the system. In addition, the input delay method can be extended for systems with intrinsic delay, as well as uncertain sampling times or uncertain system matrices.

The contribution of the present paper is as follows. We employ the lifting method to synthesise a discrete-time filter for a continuous-time LPV system. In that aspect the obtained discrete-time filter captures the inter-sample behavior of the system. The corresponding synthesis conditions to guarantee energy-to-energy (or H_∞) performance objective on the filtering error are formulated in terms of linear matrix inequalities (LMIs). To this purpose, we assume that the scheduling parameters of the LPV system are piecewise constant. A simplified version of this work has appeared in Ramezanifar, Mohammadpour, and Grigoriadis (2012), wherein the lifting of the original LPV plant was performed approximately.

The notation used in this paper is standard. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the set of real vectors of dimension n and the set of real $n \times n$ matrices, respectively. Given a symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$, $X > 0$ ($X \geq 0$) denotes matrix positive definiteness (semi-definiteness). The notation $(\cdot)^T$ denotes the transpose of a real matrix. Given a real $n \times m$ matrix Y with rank r, the orthogonal complement $Y^\perp$ is defined as the $(n - r) \times n$ matrix that satisfies $Y^\perp Y = 0$ and $Y^\perp Y^{\perp\top} > 0$. The $\mathcal{L}_2[a, b]$ norm of a continuous-time signal is defined as $\|f\|_{\mathcal{L}_2[a,b]} = (\int_b^a |f(t)|^2)^{\frac{1}{2}}$. The space of the time series with a finite $\mathcal{L}_2[a, b]$ norm is called the signal space $\mathcal{L}_2[a, b]$. The l_2 norm of a discrete signal is defined as $\|f\|_{l_2} = (\sum_{k=0}^{\infty} |f(k)|^2)^{\frac{1}{2}}$. Finally, $(\cdot)^*$ denotes the adjoint of an operator on the Hilbert space.

The paper is organised as follows. Section 2 presents the problem statement. In Section 3, we present the lifting method employed to find an equivalent discrete-time LPV state-space representation of a continuous-time LPV system. Next, we propose a solution to the LPV sampled-data filtering problem by designing a discrete-time LPV filter for the discrete-time LPV system obtained using the lifting method. As an alternative solution, in Section 4, we describe the conventional procedure to first design a continuous-time filter and then discretise the designed filter using a discretisation method. Section 5 illustrates the proposed LPV sampled-data filtering design using a numerical example. We also present the results of comparative studies between the LPV sampled-data design and the approximate discretisation. Section 6 concludes the paper.

2. Problem statement

Consider a stable nth-order LPV system with the following state-space representation

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t) \\ z(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) \\ y(t) &= C_2(\rho(t))x(t), \end{aligned} \tag{1}$$

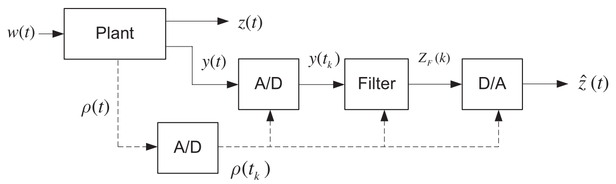

Figure 1. The block diagram of the hybrid system.

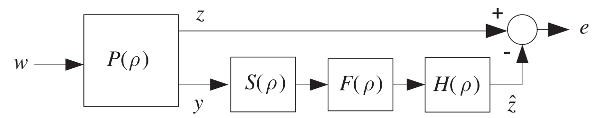

Figure 2. Estimation error system.

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated, $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector and $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector containing process noise. The system matrices $A(\cdot)$, $B_1(\cdot)$, $C_1(\cdot)$, $D_{11}(\cdot)$ and $C_2(\cdot)$ are real continuous functions of a time-varying parameter vector $\rho(t)$ and of appropriate dimensions. It is assumed that the parameter vector is bounded piecewise constant. We first describe the sampling scenario we consider in this study. We assume time intervals $[0, t_1)$, $[t_1, t_2)$, $\ldots, [t_k, t_{k+1}), \ldots$ that are not necessarily equi-spaced with t_k's being the sampling instants. For the sake of brevity, throughout this paper, k will be used to represent t_k, and the length of the kth interval will be represented by τ_k, i.e. $\tau_k = t_{k+1} - t_k$.

Next, we consider an nth-order discrete-time parameter-varying filter F described by the following state-space representation

$$
\begin{aligned}
x_F(k+1) &= A_F(\rho(k))x_F(k) + B_F(\rho(k))y(k)\\
z_F(k) &= C_F(\rho(k))x_F(k) + D_F(\rho(k))y(k),
\end{aligned} \tag{2}
$$

where $x_F(k)$, $y(k)$ and $z_F(k)$ represent the discrete-time filter state vector, the discrete samples of measurement data, i.e. $y(k) = y(t_k)$ and the filter output, respectively. All the system matrices are defined to be of appropriate dimensions. In the aforementioned filter structure, not only the measured output signal $y(t)$ is sampled, but also the parameter vector $\rho(t)$ is sampled synchronously at t_k (for $k = 0, 1, 2, \ldots$). Using $z_F(k)$, we build a continuous-time stepwise signal $\hat{z}(t)$ as $\hat{z}(t) = z_F(k)$ for $t_k \le t < t_{k+1}$ in order to estimate the signal $z(t)$ in Equation (1). The filter design problem described above is a hybrid filtering problem, where the physical system has a continuous dynamics, while the filter to estimate the plant output is implemented in a digital computer. Figure 1 shows the configuration of the hybrid system under study, the interconnection of the open-loop continuous-time system and the discrete-time filter, along with the signal conversion devices. We assume in this paper that the A/D converter is an ideal sampler, the D/A converter is a zero-order hold and that the quantisation errors are neglected. In Figure 1, the dependency of the converters on the parameter $\rho(t_k)$ remarks that sampling and holding frequency is not necessarily constant and may vary arbitrarily according to the exogenous parameter(s). It is noted that, in a typical LPV system, the parameter vector $\rho(t)$ varies continuously and is assumed to be measurable in real time, i.e. the parameter space is

$$
\begin{aligned}
\mathcal{F}_{\mathcal{P}}^{v} \equiv \{\rho : \rho(t) \in C(\mathbb{R}, \mathbb{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \le v_i\\
i = 1, 2, \ldots, s \quad \forall t \in \mathbb{R}_+\},
\end{aligned} \tag{3}
$$

where $C(\mathbb{R}, \mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathcal{P}$ is a compact set of $\mathbb{R}^s$ and $\{v_i\}_{i=1}^s$ are nonnegative numbers. However, according to the configuration in Figure 1, in the current study we can measure it only at sampling instants. Therefore, we assume that in the continuous-time system, the parameter vector does not change in between two consecutive samples. Hence, the set of all admissible trajectories for the parameter vector $\rho(t)$ in Equation (1) is defined as

$$
\begin{aligned}
\mathcal{E}_{\mathcal{P}}^{v} \triangleq \{\rho : \rho(t) \in \mathcal{P}, \rho(t_k + t) = \rho(t_k), |\rho_i(t_{k+1}) - \rho_i(t_k)|\\
\le v_i, k \in \mathbb{Z}^+, i = 1, 2, \ldots, s\ \forall t \in [0, \tau_k).\}.
\end{aligned} \tag{4}
$$

Although this assumption seems restrictive, but it is valid in many practical systems, where during the sampling instants, the parameter changes are insignificant and without the loss of generality, it could be neglected.

Figure 2 shows the estimation error defined as $e(t) = z(t) - \hat{z}(t)$ along with the filtering problem configuration where P and F are the plant and filter, and S and H are sampling and holding devices, respectively. For the error system that relates the disturbance signal $w(t)$ to the estimation error signal $e(t)$, the induced $\mathcal{L}_2$-gain (or the $\mathcal{H}_\infty$-norm) is defined as

$$
\|T_{we}\|_{i,2} = \sup_{\rho \in \mathcal{E}_{\mathcal{P}}^{v}} \sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}, \tag{5}
$$

where T_{we} is the operator mapping the disturbance $w(t)$ to the estimation error $e(t)$. This quantity, also known as the energy-to-energy gain of the augmented system, indicates the worst case output energy $\|e\|_{\mathcal{L}_2}$ over all bounded energy disturbances $\|w\|_{\mathcal{L}_2}$ for all admissible parameter vectors $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$. In this paper, we aim to design the filter F so that the following conditions are satisfied:

- The filtering error system is asymptotically stable, and
- The energy-to-energy gain of the filtering error system is minimised, i.e.

$$
\min_F \|T_{we}\|_{i,2}. \tag{6}
$$

Instead of the optimal design problem (6), one can solve the γ-suboptimal energy-to-energy gain problem, in which a filter F is sought such that

$$\|T_{we}\|_{i,2} < \gamma, \tag{7}$$

where γ is a given positive scalar. If the inequality (7) holds true, then the estimation error energy will be bounded by $\gamma\|w\|_{\mathcal{L}_2}$ for any nonzero disturbance $w(t)$ with bounded energy. That is, as long as $w(t) \in \mathcal{L}_2 - \{0\}$, regardless of its nature, the energy of the error signal does not exceed a specific bound.

Remark 1: In this paper, we only consider the full-order filter design problem, where the filter has the same order as the plant. It is, however, noted that the results presented in this paper can be extended to design reduced-order filters as well, using the approach in Grigoriadis and Watson (1997).

Remark 2: It is noted that in Equation (1), we assume there is no feed through matrix D_{21} influencing the measurement signal $y(t)$ in order for the sampling operator to be well defined (Bamieh and Pearson 1992). This is not a restrictive assumption and when it is not the case, we can cascade $y(t)$ with a strictly proper filter to relax this requirement.

Preceding to the discussion and for further justification, we slightly change the aforementioned filtering problem, where we convert the configuration of the filtering problem to the well-known control design problem. This is done so that we can benefit from the existing techniques developed for sampled-data control design. Figure 3(a) illustrates the new configuration. In this arrangement, we construct a new plant in which $e(t) = z(t) - \hat{z}(t)$ is the signal to be controlled and $\hat{z}(t)$ is the control input. The state-space representation of the new augmented plant Q in Figure 3(a) is

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t)\\ e(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))\hat{z}(t)\\ y(t) &= C_2(\rho(t))x(t),\end{aligned} \tag{8}$$

where $D_{12} = -I$. The main reason for this rearrangement will be described in the following section. Next, we have to augment the sample and hold devices with the plant Q so that we can employ the so-called lifting technique. To this purpose, we form a new plant G by augmenting the system Q and the sample and hold devices as shown in Figure 3(b) described by

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t)\\ e(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))z_F(k)\\ y(k) &= C_2(\rho(t_k))x(t_k).\end{aligned} \tag{9}$$

It is important to note that using this configuration the effect of both converters is directly taken into account within the design process. A simpler approach to solve the sampled-data filtering problem without casting it into the control design problem has been addressed by Ramezanifar et al. (2012); however, that is not as accurate as the one proposed in this paper since only the sampling device is augmented with the plant before the lifting technique is applied.

3. Solution to the hybrid filtering problem

In this section, we first describe the lifting method that we will use in this paper to discretise a continuous-time system.

3.1. Discretising using lifting technique

We consider a signal $f(t) \in \mathcal{L}_2[0, \infty)$. By signal lifting, we mean breaking $f(t)$ into the intervals $[0, t_1)$, $[t_1, t_2)$, $\ldots$, $[t_k, t_{k+1})$, $\ldots$ and constructing a sequence of signals denoted by $f_k(t)$, whose elements are defined as $f_k(t) = f(t_k + t)$ for $0 \le t < t_{k+1} - t_k$ or equivalently $0 \le t < \tau_k$. Collecting all the elements in a vector, we define $\underline{f} = [\ldots, f_{-1}(t), f_0(t), f_1(t), \ldots]^T$. It is evident that each element $f_k(t)$ belongs to $\mathcal{L}_2[0, \tau_k]$. Next, we consider the continuous-time LPV system G as illustrated in Figure 3(b). One can think of G as an operator acting on the input pair $w(t)$ and $z_F(k)$ to provide the output pair $e(t)$ and $y(k)$. The lifting of the system G is the process of finding an operator

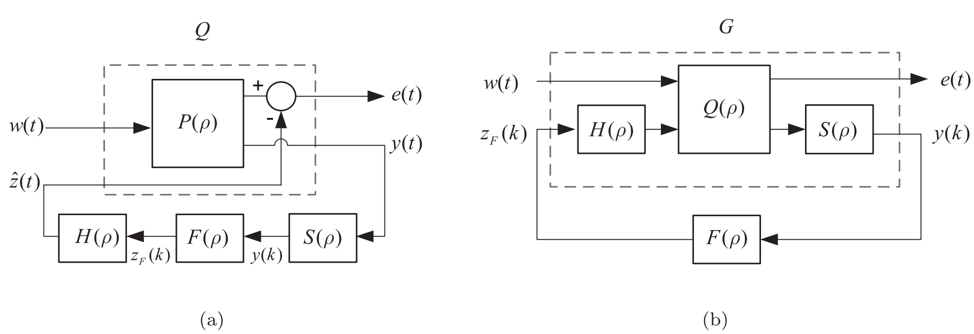

Figure 3. (a) Describing the filter in control configuration. (b) Augmenting the sample and hold devices.

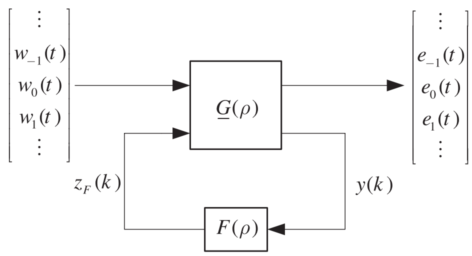

Figure 4. The lifted sampled-data system.

denoted by $\underline{G}$ that maps the lifted signal $[\underline{w}^T, z_F(k)^T]^T$ to the lifted signal $[\underline{e}^T, y(k)^T]^T$ as depicted in Figure 4, in the sense that both systems have equivalent closed-loop $\mathcal{H}_\infty$ norm.

Next, we derive the state-space realisation describing the lifted system $\underline{G}$. The integral solution to the state-space representation (9) is

$$\begin{aligned} x(t_k + t) &= \Phi(t_k + t, t_k)x(t_k) \\ &\quad + \int_{t_k}^{t_k+t} \Phi(t_k + t, s)B_1(\rho(s))w(s)ds \end{aligned} \tag{10}$$

for $t \in [0, \tau_k)$, where $\Phi(t_2, t_1) = \exp\left(\int_{t_1}^{t_2} A(\rho(\xi))d\xi\right)$ (for $0 \leq t_1 \leq t_2 < \tau_k$) is the corresponding state transition matrix. Since we have assumed the parameter space is piecewise constant, the state transition matrix becomes $\Phi(t_k + t, t_k) = \exp(A(\rho(t_k))\tau_k)$. Thus, if we change the integral variable and use the lifted signal definition, Equation (10) can be simplified as

$$x_k(t) = e^{A(\rho(t_k))t}x_k(0) + \int_0^t e^{A(\rho(t_k))(t-s)}B_1(\rho(t_k))w_k(s)ds$$

for $t \in [0, \tau_k)$. Note that $x_k(t) = x_{k+1}(0)$ for $t = \tau_k$. Similarly, the lifted output signal is

$$\begin{aligned} e_k(t) &= C_1(\rho(t_k))\{e^{A(\rho(t_k))t}x_k(0) \\ &\quad + \int_0^t e^{A(\rho(t_k))(t-s)}B_1(\rho(t_k))w_k(s)ds\} \\ &\quad + D_{11}(\rho(t_k))w_k(t) + D_{12}(\rho(t_k))z_F(k), \end{aligned}$$

and

$$y_k(0) = C_2(\rho(t_k))x_k(0).$$

Finally, we can represent the state-space realisation of $\underline{G}$, i.e. the lifted version of G, as

$$\begin{aligned} x_{k+1}(0) &= A_d(\rho(k))x_k(0) + \underline{B}_1(\rho(k))w_k(s) \\ e_k(t) &= \underline{C}_1(\rho(k))x_k(0) + \underline{D}_{11}(\rho(k))w_k(s) \\ &\quad + D_{12}(\rho(k))z_F(k) \\ y_k(0) &= C_2(\rho(k))x_k(0), \end{aligned} \tag{11}$$

where $A_d = e^{A(\rho(t_k))\tau_k}$ and

$$\begin{aligned} \underline{B}_1 &: \mathcal{L}_2[0, \tau_k] \to \mathbb{R}^n, \underline{B}_1 w_k \\ &= \int_0^{\tau_k} e^{A(\rho(t_k))(\tau_k - s)}B_1(\rho(t_k))w_k(s)ds, \\ \underline{C}_1 &: \mathbb{R}^n \to \mathcal{L}_2[0, \tau_k], (\underline{C}_1 x_k)(t) = C_1(\rho(t_k))e^{A(\rho(t_k))t}x_k, \\ \underline{D}_{11} &: \mathcal{L}_2[0, \tau_k] \to \mathcal{L}_2[0, \tau_k], (\underline{D}_{11} w_k)(t) \\ &= C_1(\rho(t_k)) \int_0^t e^{A(\rho(t_k))(t-s)}B_1(\rho(t_k))w_k(s)ds \\ &\quad + D_{11}(\rho(t_k))w_k(t). \end{aligned} \tag{12}$$

The lifted system (11) has infinite-dimensional input and output spaces but its state-space realisation is finite-dimensional with the dimension equal to that of the original system. The question is now how to describe this system using a discrete-time LPV model such that the stability and an upper bound on the $\mathcal{H}_\infty$ norm of the closed-loop system is preserved. Indeed, at this stage we seek for the lifted system's state-space matrices that would be implemented in discrete time by sampling the input vector and parameter signals at discrete-time instants. This equivalent discrete-time system is determined to be

$$\begin{aligned} x_d(k+1) &= A_{dd}(\rho(k))x_d(k) + B_{1d}(\rho(k))w_d(k) \\ &\quad + B_{2d}(\rho(k))z_F(k) \\ e_d(k) &= C_{1d}(\rho(k))x_d(k) + D_{12d}(\rho(k))z_F(k) \\ y_d(k) &= C_2(\rho(k))x_d(k), \end{aligned} \tag{13}$$

with the matrices A_{dd} and B_{1d} given by

$$\begin{aligned} A_{dd} &= A_d + \underline{B}_1\underline{D}_{11}^*(\gamma^2 I - \underline{D}_{11}\underline{D}_{11}^*)^{-1}\underline{C}_1 \\ B_{2d} &= \underline{B}_1\underline{D}_{11}^*(\gamma^2 I - \underline{D}_{11}\underline{D}_{11}^*)^{-1}\underline{D}_{12}. \end{aligned} \tag{14}$$

In addition, the matrices B_{1d}, C_{1d} and D_{12d} are given by

$$\begin{aligned} B_{1d}B_{1d}^T &= \gamma^2\underline{B}_1(\gamma^2 I - \underline{D}_{11}^*\underline{D}_{11})^{-1}\underline{B}_1^* \\ \begin{bmatrix} C_{1d}^T \\ D_{12d}^T \end{bmatrix}\begin{bmatrix} C_{1d} & D_{12d} \end{bmatrix} &= \gamma^2\begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix}(\gamma^2 I - \underline{D}_{11}\underline{D}_{11}^*)^{-1} \\ &\quad \times \begin{bmatrix} \underline{C}_1 & \underline{D}_{12} \end{bmatrix}. \end{aligned} \tag{15}$$

The following theorem states the equivalence of the initial hybrid LPV system and the lifted discrete-time LPV system with respect to stability and energy-to-energy gain.

Theorem 3.1: *Consider two dynamical systems, one of which is formed by the interconnection of the continuous-time system (9) with the sampled-data system (2), and second one is formed by interconnecting the discrete-time system (13) with Equation (2). The following statements are equivalent provided that* $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)} < \gamma$.

- *The former system is stable and has the energy-to-energy gain less than γ.*
- *The latter system is stable and has the energy-to-energy gain less than γ.*

Proof: Please refer to Chen and Francis (1995).

In order to apply Theorem 3.1, the $\mathcal{L}_2[0, \tau_k)$ induced gain of $\underline{D}_{11}$, as well as several other operator compositions must be evaluated. For a complete discussion on the evaluation of these operators, the reader is referred to Chen and Francis (1995), where the design for the LTI case has been addressed. As a quick reference, the procedure for evaluating the aforementioned operators is presented in Appendix.

Using the lifted LPV discrete-time system (13), the next step is to design a discrete-time parameter-dependent filter represented by Equation (2). We discuss the design procedure in the following section.

3.2. Discrete-time filter design for discrete-time LPV systems

In this section, we consider an LPV system represented by Equation (13), where the objective is to design a discrete-time system F described by the state-space representation (2) such that the energy-to-energy gain from the disturbance w_d to the estimation error e_d is less than γ with γ being a given positive scalar. First, we present preliminaries in the form of two lemmas that are required for the discussions in this section. For a proof of the two lemmas, the interested reader is referred to Skelton et al. (1998).

Lemma 3.2: *Consider a stable discrete-time LPV system represented by*

$$\begin{aligned} x_d(k+1) &= \mathfrak{A}_d(\rho(k))x_d(k) + \mathfrak{B}_d(\rho(k))w_d(k) \\ y_d(k) &= \mathfrak{C}_d(\rho(k))x_d(k), \end{aligned}$$

and let γ be a given positive scalar. Then, the energy-to-energy gain of the system from w_d to y_d is less than γ if and only if there exists a parameter-dependent symmetric positive definite matrix $P(\rho(k))$ such that

$$\begin{bmatrix} \mathfrak{A}_d(\rho(k)) & \mathfrak{B}_d(\rho(k)) \\ \mathfrak{C}_d(\rho(k)) & 0 \end{bmatrix}^T \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \mathfrak{A}_d(\rho(k)) & \mathfrak{B}_d(\rho(k)) \\ \mathfrak{C}_d(\rho(k)) & 0 \end{bmatrix} < \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Lemma 3.3: *Consider the matrices Γ, Λ, Θ and a symmetric matrix R. There exists a matrix F such that the quadratic matrix inequality*

$$(\Theta + \Gamma F \Lambda)^T R(\Theta + \Gamma F \Lambda) < Q \tag{16}$$

has a solution if and only if the following conditions hold true

$$\Lambda^{T\perp}(Q - \Theta^T R\Theta)\Lambda^{T\perp T} > 0 \tag{17}$$

$$\Gamma^{\perp}(R^{-1} - \Theta Q^{-1}\Theta^T)\Gamma^{\perp T} > 0. \tag{18}$$

In this case, all the possible solutions for the matrix F are parameterised by

$$F = -\Omega\Gamma^T R\Theta\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + \Psi^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}, \tag{19}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and

$$\begin{aligned} \Phi &= (Q - \Theta^T R\Theta + \Theta^T R\Gamma\Omega\Gamma^T R\Theta)^{-1} \\ \Psi &= \Omega - \Omega\Gamma^T R\Theta(\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi)\Theta^T R\Gamma\Omega \\ \Omega &= (\Gamma^T R\Gamma)^{-1}. \end{aligned}$$

The following theorem gives the solution to the problem mentioned at the beginning of this section.

Theorem 3.4: *For a given positive scalar γ, there exists an nth-order system F represented in state-space form by Equation (2) to make the energy-to-energy gain of the system (13) from w_d to e_d less than γ, if and only if there exist parameter-dependent matrices $X > 0$ and $Y > 0$ such that for all admissible parameters, there is a feasible solution to the set of LMIs*

$$\begin{aligned} C_2^{T\perp}\big(&Y(\rho(k-1)) - A_{dd}^T Y(\rho(k))A_{dd} \\ &- C_{1d}^T C_{1d}\big)C_2^{T\perp T} > 0, \end{aligned} \tag{20}$$

$$\begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp} \left(\begin{bmatrix} X(\rho(k)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A_{dd} & B_{1d} \\ C_{1d} & 0 \end{bmatrix}^T \times \begin{bmatrix} X(\rho(k-1)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{dd} & B_{1d} \\ C_{1d} & 0 \end{bmatrix} \right) \begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp T} > 0, \tag{21}$$

and

$$\begin{bmatrix} Y(\rho(k)) & \gamma I \\ \gamma I & X(\rho(k)) \end{bmatrix} \geq 0. \tag{22}$$

Proof: To solve the γ-suboptimal $\mathcal{H}_\infty$ design problem, we first examine the closed-loop representation of systems (13) and (2) (where $y(k)$ and $y_d(k)$ are the same). Defining $\bar{x}_d(k) = [x_d^T(k), x_F^T(k)]^T$, we have

$$\begin{aligned} \bar{x}_d(k+1) &= (\bar{A} + \bar{B}F\bar{M})\bar{x}_d(k) + \bar{D}w_d(k) \\ e_d(k) &= (\bar{C} + \bar{H}F\bar{M})\bar{x}_d(k), \end{aligned}$$

where

$$\bar{A} = \begin{bmatrix} A_{dd}(\rho(k)) & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B_{2d}(\rho(k)) & 0 \\ 0 & I \end{bmatrix}$$
$$\bar{M} = \begin{bmatrix} C_2(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, \bar{D} = \begin{bmatrix} B_{1d}(\rho(k)) \\ 0 \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} C_{1d}(\rho(k)) & 0 \end{bmatrix}, \bar{H} = \begin{bmatrix} D_{12d}(\rho(k)) & 0 \end{bmatrix}.$$

In addition, the matrix F defined by

$$F = \begin{bmatrix} D_F(\rho(k)) & C_F(\rho(k)) \\ B_F(\rho(k)) & A_F(\rho(k)) \end{bmatrix} \tag{23}$$

includes the unknown matrices corresponding to the filter state-space representation. Next, we use Lemma 3.2 as the LMI-based condition to ensure that there is a solution to the γ-suboptimal filtering problem. The lemma states that there exists a γ-suboptimal LPV filter if and only if there exists a parameter-dependent matrix $P(\rho(k)) > 0$ such that

$$\begin{bmatrix} \bar{A} + \bar{B}F\bar{M} & \bar{D} \\ \bar{C} + \bar{H}F\bar{M} & 0 \end{bmatrix}^T \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix}$$
$$\times \begin{bmatrix} \bar{A} + \bar{B}F\bar{M} & \bar{D} \\ \bar{C} + \bar{H}F\bar{M} & 0 \end{bmatrix} < \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Next, we use Lemma 3.3 to determine a set of LMI conditions to ensure the existence of the filter F. The associated matrices are determined to be

$$\Theta = \begin{bmatrix} \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} \bar{B} \\ \bar{H} \end{bmatrix} = \begin{bmatrix} B_{2d} & 0 \\ 0 & I \\ D_{12d} & 0 \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} \bar{M} & 0 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \tag{24}$$

$$R = \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, Q = \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \tag{25}$$

It can be verified that

$$\Gamma^{\perp} = \begin{bmatrix} \begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix},$$
$$\Lambda^{T\perp} = \begin{bmatrix} C_2^{T\perp} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}. \tag{26}$$

Next, we partition P as

$$P(\rho(k)) = \begin{bmatrix} Y(\rho(k)) & Y_{12}(\rho(k)) \\ Y_{12}^T(\rho(k)) & Y_{22}(\rho(k)) \end{bmatrix}. \tag{27}$$

Then, the solvability condition (17) leads to the LMI (20). In addition, the solvability condition (18) leads to a matrix inequality problem, in which the matrix P^{-1} appears. Then, applying the congruence transformation $\mathcal{T} = \text{diag}(\gamma I, \gamma I)$ and introducing

$$\gamma^2 P^{-1}(\rho(k)) = \begin{bmatrix} X(\rho(k)) & X_{12}(\rho(k)) \\ X_{12}^T(\rho(k)) & X_{22}(\rho(k)) \end{bmatrix}$$

leads to Equation (21). The (1,1) entries of the two matrices P and P^{-1} are related through

$$Y - \gamma^2 X^{-1} = Y_{12} Y_{22}^{-1} Y_{12}^T \geq 0, \tag{28}$$

which implies that

$$Y(\rho(k)) - \gamma^2 X^{-1}(\rho(k)) \geq 0,$$

or equivalently the LMI (22).

Remark 1: It is noted that the inequality conditions in Theorem 3.4 are parameterised LMIs. To solve this infinite-dimensional LMI problem, we initially pick some basis functions to represent the dependency of the matrix variables on the LPV parameters, e.g. by selecting first-order polynomials as

$$X = X_0 + \rho X_1, Y = Y_0 + \rho Y_1. \tag{29}$$

Next, the set of LMI problem is solved to determine X_0, X_1, Y_0 and X_1 at the selected grid points. The results are finally checked on a finer grid.

Remark 2: It is important to note that in Theorem 3.4, the matrix inequalities (20), (21) and (22) are not linear in terms of γ, since the associated matrices A_{dd}, B_{1d}, B_{2d}, C_{1d} and D_{12d} obtained from (14) and (15) are themselves dependent on γ. So, in order to find the optimum γ, we decrease γ successively in a loop and solve the feasibility problem of the LMIs (with A_{dd}, B_{1d}, B_{2d}, C_{1d} and D_{12d} updated accordingly). The search is terminated as soon as the set of LMIs become infeasible.

Remark 3: At each sampling instant, the set of LMIs depends on the LPV parameter vector at both kth and $(k-1)$th samples. Hence, there is a need to store parameters during the design process. One can also replace $\rho(k-1)$ by $r(k)$ in the corresponding LMIs and treat it as a new parameter vector $r \in \mathcal{E}_{\mathcal{P}}^{v}$. In this case, the feasibility or optimisation problem corresponding to the sampled-data filter design should be solved over the new parameter space $\mathcal{E}_{\mathcal{P}}^{v} \times \mathcal{E}_{\mathcal{P}}^{v}$.

After solving the LMIs associated with Theorem 3.4 offline, the filter matrices (23) are determined at each sampling instant as following:

Step 1: The scheduling parameter ρ is measured.

Step 2: For a predetermined value of γ and the current value of ρ, the discrete-time system matrices in Equation (13) are updated, using the process given in Appendix.

Step 3: The matrices Θ, Γ and Λ in (24) are determined. Using X_0, X_1, Y_0 and Y_1, from Equation (29) X and Y are calculated. Once X and Y are calculated, Y_{12} and Y_{22} can be determined from the factorisation problem (28) using singular value decomposition.

Step 4: Next, P is found from Equation (27) and subsequently R and Q in Equation (25) are determined.

Step 5: Finally, F is obtained from Equation (19). By partitioning matrix F, the filter matrices A_F, B_F, C_F and D_F are then obtained from Equation (23).

4. Continuous-time LPV filter discretisation

A conventional solution to the sampled-data filter problem is to design a continuous-time LPV filter for the continuous-time LPV plant and then apply a standard discretisation method to find a discrete-time representation of the filter. In this section, we present the LMI-based conditions to design a continuous-time filter for a given continuous-time LPV system. Then, we discuss the use of trapezoidal approximation method to discretise the designed filter. We first present two lemmas that are important in the proof of the main results of this section. For more details, please refer to Skelton et al. (1998).

Lemma 4.1: *Consider a stable continuous-time LPV system represented by*

$$\begin{aligned}\dot{x}(t) &= \mathfrak{A}(\rho(t))x(t) + \mathfrak{B}(\rho(t))w(t)\\ y(t) &= \mathfrak{C}(\rho(t))x(t) + \mathfrak{D}(\rho(t))w(t)\end{aligned}$$

and let γ be a given positive scalar. Then energy-to-energy gain of the system from w to y is less than γ if and only if there exists a parameter-dependent symmetric positive definite matrix $P(\rho(t))$ that satisfies the following matrix inequality

$$\begin{bmatrix} \dot{P}(\rho) + P(\rho)\mathfrak{A}(\rho) + \mathfrak{A}^T(\rho)P(\rho) & P(\rho)\mathfrak{B}(\rho) & \mathfrak{C}^T(\rho) \\ \mathfrak{B}^T(\rho)P(\rho) & -\gamma^2 I & \mathfrak{D}^T(\rho) \\ \mathfrak{C}(\rho) & \mathfrak{D}(\rho) & -I \end{bmatrix} < 0$$

Lemma 4.2: *Consider the matrices Γ, Λ, Θ and a symmetric matrix R. There exists a matrix F such that the quadratic matrix inequality*

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0 \tag{30}$$

has a solution if and only if the following conditions hold true

$$\Gamma^{\perp}\Theta\Gamma^{\perp T} < 0 \tag{31}$$

$$\Lambda^{T\perp}\Theta\Lambda^{T\perp T} < 0. \tag{32}$$

In this case, all the possible solutions for the matrix F are parameterised by

$$F = -R^{-1}\Gamma^T\Phi\Lambda^T\Psi + \Omega^{\frac{1}{2}}L\Psi^{\frac{1}{2}}, \tag{33}$$

where Φ, R and L are free parameters satisfying

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} > 0, \quad R > 0, \quad \|L\| < 1$$

and Φ and Ψ are defined by

$$\begin{aligned}\Omega &= R^{-1} - R^{-1}\Gamma^T(\Phi - \Phi\Lambda^T\Psi\Lambda\Phi)\Gamma R^{-1}\\ \Psi &= (\Lambda\Phi\Lambda^T)^{-1}.\end{aligned}$$

4.1. Procedure for the design of continuous-time filters for LPV systems

Considering the LPV system represented by Equation (1), we define a continuous-time filter F described by the following state-space representation

$$\begin{aligned}\dot{x}_F(t) &= A_F(\rho(t))x_F(t) + B_F(\rho(t))y(t)\\ \hat{z}(t) &= C_F(\rho(t))x_F(t) + D_F(\rho(t))y(t).\end{aligned} \tag{34}$$

The design objective is to ensure that the energy-to-energy gain from the disturbance w to the estimation error e becomes less than γ, where γ is a given positive scalar and the estimation error is defined as $e(t) = z(t) - \hat{z}(t)$. The following theorem gives the solution to this problem.

Theorem 4.3: *For a given positive scalar γ, there exists an nth-order filter (34) to solve the γ-suboptimal continuous-time filtering problem if and only if there exist parameter-dependent matrices $X > 0$ and $Y > 0$ such that for all admissible parameters, there is a feasible solution to the set of LMIs*

$$\begin{bmatrix} \dot{X}(\rho) + A(\rho)X(\rho) + X(\rho)A^T(\rho) & B_1(\rho) \\ B_1^T(\rho) & -\gamma^2 I \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} C_2^{T\perp}(\rho) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \dot{Y}(\rho) + Y(\rho)A(\rho) + A^T(\rho)Y(\rho) & C_1(\rho) \\ C_1^T(\rho) & -I \end{bmatrix} \times \begin{bmatrix} C_2^{T\perp}(\rho) & 0 \\ 0 & I \end{bmatrix}^T < 0, \tag{36}$$

and

$$Y(\rho(t)) \geq X(\rho(t)). \tag{37}$$

Proof: Defining $\bar{x}(t) = [x^T(t), x_F^T(t)]^T$, the estimation error dynamics is given by

$$\begin{aligned}\dot{\bar{x}}(t) &= (\bar{A} + \bar{B}F\bar{M})\bar{x}(t) + \bar{D}w(t)\\ e(t) &= (\bar{C} + \bar{H}F\bar{M})\bar{x}(t) + \bar{E}w(t),\end{aligned} \tag{38}$$

where

$$\begin{aligned}\bar{A} &= \begin{bmatrix} A(\rho(t)) & 0\\ 0 & 0\end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & 0\\ 0 & I\end{bmatrix},\\ \bar{M} &= \begin{bmatrix} C_2(\rho(t)) & 0\\ 0 & I\end{bmatrix},\\ \bar{D} &= \begin{bmatrix} B_1(\rho(t))\\ 0\end{bmatrix}, \bar{C} = \begin{bmatrix} C_1(\rho(t)) & 0\end{bmatrix},\\ \bar{H} &= \begin{bmatrix} -I & 0\end{bmatrix}, \bar{E} = D_{11}\end{aligned}$$

and

$$F = \begin{bmatrix} D_F(\rho(t)) & C_F(\rho(t))\\ B_F(\rho(t)) & A_F(\rho(t))\end{bmatrix}. \tag{39}$$

Next, we apply Lemma 4.1 to the augmented system (38) to obtain

$$\begin{bmatrix} \dot{P} + P(\bar{A} + \bar{B}F\bar{M}) + (\bar{A} + \bar{B}F\bar{M})^T P & P\bar{D} & (\bar{C} + \bar{H}F\bar{M})^T\\ \bar{D}^T P & -\gamma^2 I & \bar{E}^T\\ \bar{C} + \bar{H}F\bar{M} & \bar{E} & -I\end{bmatrix} < 0.$$

This matrix inequality can be cast in the form of (30) with

$$\begin{aligned}\Gamma &= \begin{bmatrix} P\bar{B}\\ 0\\ \bar{H}\end{bmatrix}, \Lambda = \begin{bmatrix} \bar{M} & 0 & 0\end{bmatrix},\\ \Theta &= \begin{bmatrix} \dot{P} + P\bar{A} + \bar{A}^T P & P\bar{D} & \bar{C}^T\\ \bar{D}^T P & -\gamma^2 I & \bar{E}^T\\ \bar{C} & \bar{E} & -I\end{bmatrix}.\end{aligned} \tag{40}$$

One can readily obtain that

$$\begin{aligned}\Gamma^{\perp} &= \begin{bmatrix} \begin{bmatrix} \bar{B}\\ \bar{H}\end{bmatrix}^{\perp} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} P^{-1} & 0 & 0\\ 0 & 0 & I\\ 0 & I & 0\end{bmatrix},\\ \Lambda^{T\perp} &= \begin{bmatrix} \bar{M}^{T\perp} & 0 & 0\\ 0 & 0 & I\end{bmatrix},\end{aligned}$$

where

$$\begin{bmatrix} \bar{B}\\ \bar{H}\end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 & 0\end{bmatrix}, \bar{M}^{T\perp} = \begin{bmatrix} C_2^{T\perp} & 0\end{bmatrix}.$$

Before we proceed to apply the solvability conditions in Lemma 4.2, we partition $P(\rho(t))$ and $P^{-1}(\rho(t))$ as

$$\begin{aligned}P(\rho(t)) &= \begin{bmatrix} Y(\rho(t)) & Y_{12}(\rho(t))\\ Y_{12}^T(\rho(t)) & Y_{22}(\rho(t))\end{bmatrix},\\ P^{-1}(\rho(t)) &= \begin{bmatrix} X(\rho(t)) & X_{12}(\rho(t))\\ X_{12}^T(\rho(t)) & X_{22}(\rho(t))\end{bmatrix}.\end{aligned} \tag{41}$$

The solvability condition (31) becomes

$$\dot{X}(\rho) + A(\rho)X(\rho) + X(\rho)A^T(\rho) + \frac{1}{\gamma^2}B_1(\rho)B_1^T(\rho) < 0.$$

Applying the Schur complement on the above inequality yields (35). In addition, the $(1, 1)$ entries of the two matrices P and P^{-1} are related through

$$Y - X = Y_{12}Y_{22}^{-1}Y_{12}^T \geq 0 \tag{42}$$

that yields (37). Once the matrices X and Y are found, the matrices Y_{12} and Y_{22} can be determined from the factorisation problem (42). Subsequently, the matrix P in Equation (41) is calculated. Substituting the obtained matrices in Equation (40), the filter state-space matrices in Equation (39) are computed by Equation (33).

Remark 1: In the matrix inequalities (35) and (36), the (1,1) entries include a derivative term that can be replaced by $\dot{X} = \frac{\partial X}{\partial \rho}\dot{\rho}$ and $\dot{Y} = \frac{\partial Y}{\partial \rho}\dot{\rho}$, respectively. Due to the affine dependency of matrix inequalities on $\dot{\rho}$, it is only required to solve the feasibility problem at vertices of $\dot{\rho}$. Therefore, one can replace the term $\dot{X}$ with $\sum_{i=1}^{s} \pm(v_i \frac{\partial X}{\partial \rho})$ and $\dot{Y}$ with $\sum_{i=1}^{s} \pm(v_i \frac{\partial Y}{\partial \rho})$ (Wu and Grigoriadis 2001), where v_i is defined in Equation (3). The summation means that every combination of + and − should be included in the inequality. That is, the corresponding inequalities actually represent 2^s different combinations in the summation.

4.2. *Trapezoidal discretisation of the continuous-time LPV filter*

Among various options for discretisation of a continuous-time dynamic system, we employ the trapezoidal approximation that is a counterpart of the bilinear transformation for LPV systems (Tóth, Lovera, Heuberger, and Van den Hof 2009). The proposed formulation in this section is adopted from the work of Apkarian (1997). It is, however, slightly tailored for nonuniform sampling periods. This approach is moderately accurate and advantageously reduces the computational cost.

Considering the sampling interval $t_k \leq t < t_{k+1}$, we assume that for the continuous-time filter (34), the state vector $x_F(t_k)$ is known. Then, at the end of sampling interval, we have

$$\begin{aligned} x_F(t_{k+1}) &= x_F(t_k) + \int_{t_k}^{t_{k+1}} \big(A_F(\rho(\tau))x(\tau) \\ &\quad + B_F(\rho(\tau))y(\tau)\big)d\tau \\ \hat{z}(t_k) &= C_F(\rho(t_k))x_F(t_k) + D_F(\rho(t_k))y(t_k). \end{aligned} \tag{43}$$

Using the trapezoidal approximation for the integral part in Equation (43) and with a simplified notation, we obtain

$$\begin{aligned} x_F(k+1) &\approx x_F(k) + \frac{\tau_k}{2}\big(A_F(\rho(k))x_F(k) \\ &\quad + B_F(\rho(k))y(k) + A_F(\rho(k+1))x_F(k+1) \\ &\quad + B_F(\rho(k+1))y(k+1)\big). \end{aligned}$$

Next, we gather all the terms corresponding to the sampling time t_{k+1} and rename them $x_d(k+1)$, i.e.

$$\begin{aligned} x_d(k+1) &= \left(I - \frac{\tau_k}{2}A_F(\rho(k+1))\right)x_F(k+1) \\ &\quad - \frac{\tau_k}{2}B_F(\rho(k+1))y(k+1), \end{aligned} \tag{44}$$

which implies that

$$\begin{aligned} x_F(k) &= \left(I - \frac{\tau_{k-1}}{2}A_F(\rho(k))\right)^{-1}(x_d(k) \\ &\quad + \frac{\tau_{k-1}}{2}B_F(\rho(k))y(k)). \end{aligned} \tag{45}$$

Finally, we substitute Equations (44) and (45) into Equation (43) to obtain a discrete-time representation of the filter designed in continuous time. The following theorem characterises the filter state-space matrices for the implementation purposes.

Theorem 4.4: *Consider the LPV filter (34) designed in continuous time. The sampled dynamics of this filter is represented by the following discrete-time state-space model*

$$\begin{aligned} x_d(k+1) &= A_{Fd}(\rho_k)x_d(k) + B_{Fd}(\rho_k)y(k) \\ \hat{z}_d(k) &= C_{Fd}(\rho_k)x_d(k) + D_{Fd}(\rho_k)y(k), \end{aligned}$$

where

$$\begin{aligned} A_{Fd} &= \left(I + \frac{\tau_k}{2}A_F\right)\left(I - \frac{\tau_{k-1}}{2}A_F\right)^{-1} \\ B_{Fd} &= \frac{\tau_k + \tau_{k-1}}{2}\left(I - \frac{\tau_{k-1}}{2}A_F\right)^{-1}B_F \\ C_{Fd} &= C_F\left(I - \frac{\tau_{k-1}}{2}A_F\right)^{-1} \end{aligned}$$

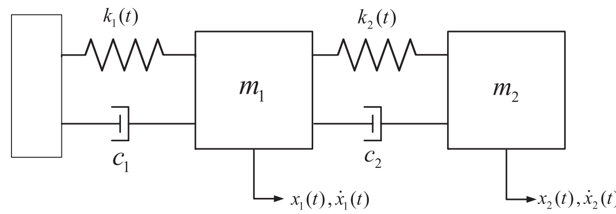

Figure 5. The double mass-spring-damper system.

$$D_{Fd} = \frac{\tau_{k-1}}{2}C_F\left(I - \frac{\tau_{k-1}}{2}A_F\right)^{-1}B_F + D_F.$$

This discrete-time system is then placed in the Filter block in Figure 1. It is noted that the filter state-space matrices are functions of ρ(k) and updated at each sampling instant.

5. Simulation results

In this section, we present some numerical results obtained from applying the proposed sampled-data LPV filter design methods. We consider a forth-order resonant system corresponding to a double mass-spring-damper system with nonlinear springs as shown in Figure 5. The dynamic model of the system is described by

$$\begin{aligned} m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - k_2x_2 - c_2\dot{x}_2 &= w_1(t) \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 &= w_2(t), \end{aligned}$$

where m_1 and m_2 are masses, k_1 and k_2 are spring stiffnesses, c_1 and c_2 are the damping coefficients and $w_1(t)$ and $w_2(t)$ are external force disturbances acting on the masses. The objective is to design a sampled-data filter to estimate the mass velocities using the measurement of the positions. The associated state-space model of the system is as follows:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1(t)+k_2(t)}{m_1} & -\frac{c_1+c_2}{m_1} & -\frac{k_2(t)}{m_1} & -\frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2(t)}{m_2} & \frac{c_2}{m_2} & -\frac{k_2(t)}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\ z(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}, \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}. \end{aligned} \tag{46}$$

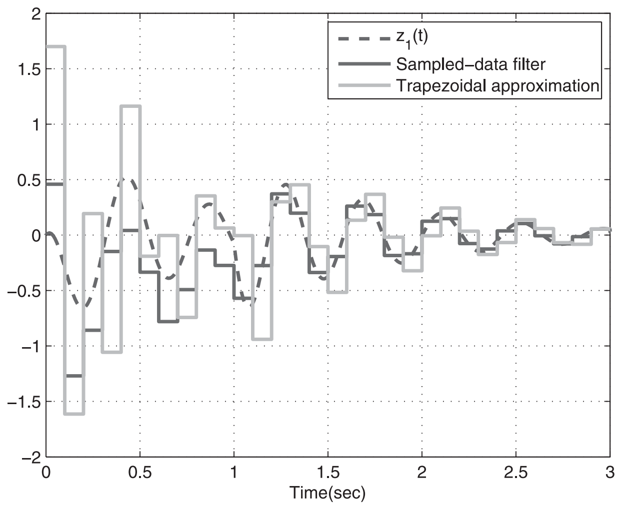

Figure 6. The estimation results of $z_1(t)$ for the sampling rate of 0.1 sec.

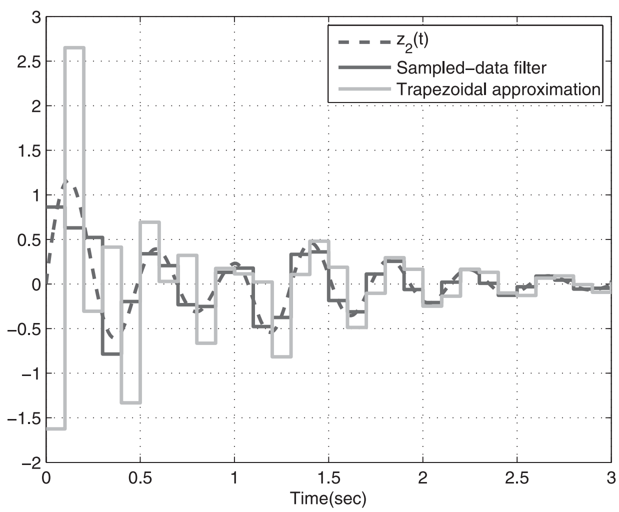

Figure 7. The estimation results of $z_2(t)$ for the sampling rate of 0.1 sec.

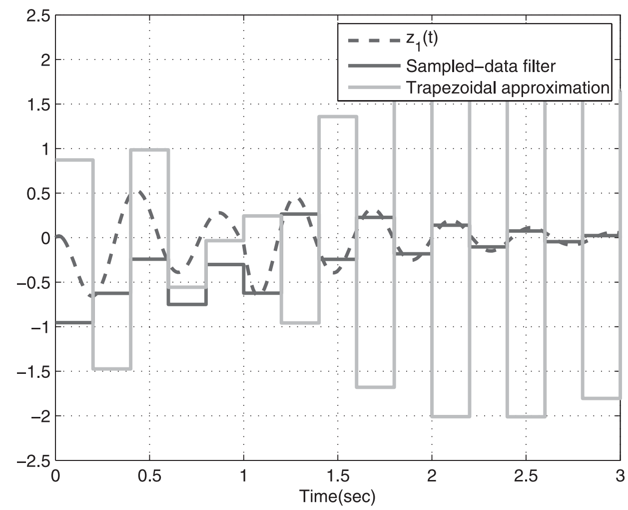

Figure 8. The estimation results of $z_1(t)$ for the sampling rate of 0.2 sec.

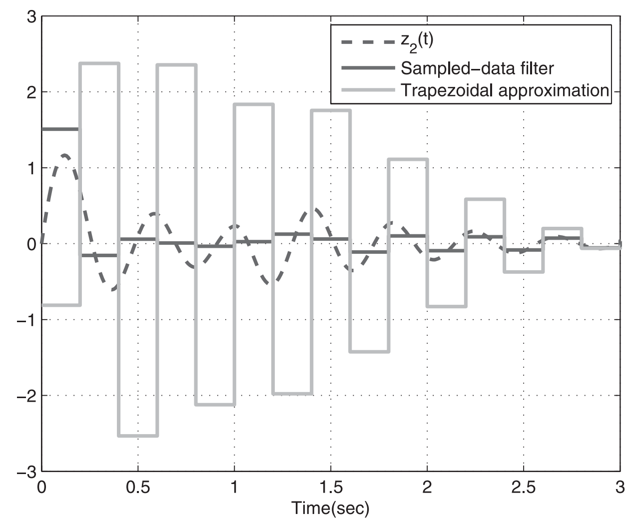

Figure 9. The estimation results of $z_2(t)$ for the sampling rate of 0.2 sec.

The parameters are assumed to be

$$\begin{aligned} m_1 &= m_2 = \frac{1}{11}[\mathrm{kg}] \\ k_1 &= k_2 = 8 + 2\sin(t)[\mathrm{N/m}] \\ c_1 &= c_2 = 0.5[\mathrm{N.s/m}]. \end{aligned} \tag{47}$$

We assume that the *sine* term in Equation (47) corresponds to the LPV parameter, i.e. $\rho(t) = \sin(t)$, whose functional representation is not known a priori but since it is a time-dependent variable, it can be produced in a digital device. We note that the parameter space is $[-1, 1]$. It is also assumed that the system is affected by an input disturbance signal $w_1(t) = 1$ for $t \in [0, 1]$ and $w_1(t) = 0$ otherwise. We consider three designs corresponding to different sampling rates. First, we design a discrete-time filter for the case of a constant sampling rate $\tau_k = 0.1$. Figures 6 and 7 illustrate the estimates of $z_1(t)$ and $z_2(t)$, respectively, using the proposed sampled-data method along with the actual outputs of the continuous-time system. For comparison purposes, we have also shown in these figures the trapezoidal approximation of an LPV filter designed in continuous time. As observed, the estimation performance using the discretisation of the continuous-time filter design is inferior to that of the proposed sampled-data design. It is emphasised that, if we use the rectangular approximation for discretisation of the continuous-time filter, the estimation performance

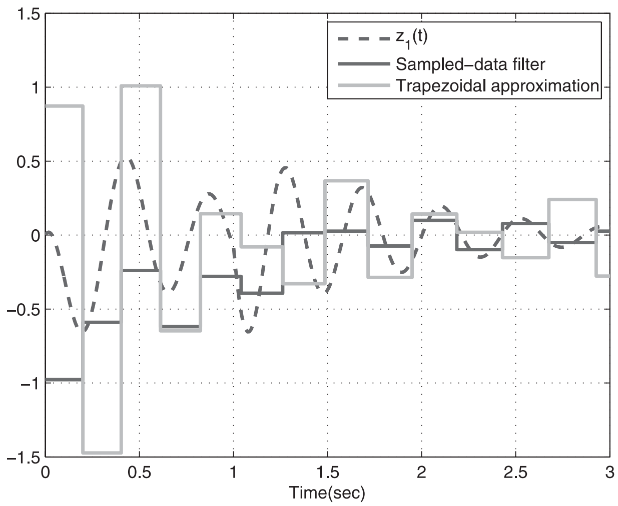

Figure 10. The estimation results of $z_1(t)$ with variable sampling rate.

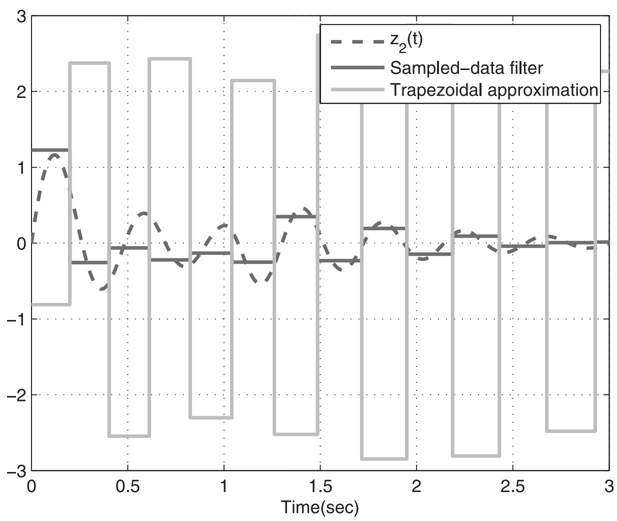

Figure 11. The estimation results of $z_2(t)$ with variable sampling rate.

even worsens compared to the trapezoidal approximation. In this example the optimal value of γ is obtained to be 2.5. It is noted that the output tracking is even improved for lower sampling rates than $\tau_k = 0.1$. In the second scenario, we examine the case of a constant sampling rate $\tau_k = 0.2$, which is quite large with respect to the frequency of the output signals. Figures 8 and 9 show the estimation results using the sampled-data and trapezoidal approximation methods. While the latter method fails to provide a good estimate, the former provides reliable estimates of the output of the continuous-time LPV system. In this case, the optimal value of γ is determined to be 4.4. Finally, we consider a case with a variable sampling rate, in which the sampling rate changes according to the pattern

$$t_{k+1} = t_k + 0.2(1 + 0.5\sin(0.2t_k)).$$

Starting from $t_0 = 0$, the pattern above is associated with a time-varying sampling. Figures 10 and 11 show acceptable estimation results that the sampled-data LPV filter can provide. The optimal value of γ is obtained to be 5.3. Also shown in these figures are the results of trapezoidal approximation of an LPV filter designed in continuous time. It is obvious that the sampled-data filter exhibits better performance.

6. Concluding remarks

In this paper, we presented a sampled-data filter design method for stable continuous-time LPV systems using the lifting technique. The design method consisted of obtaining an equivalent discrete-time LPV system by employing the lifting method and subsequently design a discrete-time LPV filter for the lifted system. It was shown that the sampled-data approach effectively handles large and even variable sampling rates. Numerical results demonstrated the viability of the proposed sampled-data filtering method. In addition, to compare the results with conventional filtering methods, we designed a continuous-time LPV filter and discretised it by means of trapezoidal approximation. This approach is a fast solution alternative for the sampled-data control and filtering problems that reduces the computational effort at the cost of accuracy. This method can give favourable results specially when the LMI optimisation leads to a sufficiently small $\mathcal{H}_\infty$ norm for the estimation error system. The simulation results demonstrated the improved estimation performance achieved using the developed LPV sampled-data filter design method compared to the discretisation of a filter designed in continuous time. The improvement was observed for three cases with small, large and variable sampling rates.

The authors are currently developing sampled-data control and filtering methodologies for LPV systems that include time-varying state delay and uncertainty.

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Appendix

In this section, we summarise the procedure to compute $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$, as well as the matrix-valued representation for the operator compositions (14) and (15). The interested reader is referred to Chen and Francis (1995), where a complete discussion is presented for LTI systems. In order to compute $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$, we define

$$\begin{aligned} S &= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\ &= \exp\left\{\tau_k \begin{bmatrix} -A^T(\rho(t_k)) & -C_1^T(\rho(t_k))C_1(\rho(t_k)) \\ \gamma^{-2}B_1(\rho(t_k))B_1^T(\rho(t_k)) & A(\rho(t_k)) \end{bmatrix}\right\}, \end{aligned}$$

for $\gamma > 0$. It is shown by Chen and Francis (1995) that $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$ is equal to the largest value of γ, for which the matrix S_{11} has a zero eigenvalue. The procedure for evaluating the operator compositions at each sampling instant is performed by taking the following steps:

Step 1: Define the square matrix U as

$$U = \begin{bmatrix} A(\rho(t_k)) & 0_{n\times n_z} \\ 0_{n_z\times n} & 0_{n_z\times n_z} \end{bmatrix},$$

and

$$\begin{aligned} E &= \begin{bmatrix} -A^T(\rho(t_k)) & -C_1^T(\rho(t_k))C_1(\rho(t_k)) \\ \frac{1}{\gamma^2}B_1(\rho(t_k))B_1^T(\rho(t_k)) & A(\rho(t_k)) \end{bmatrix} \\ X &= [\,C_1(\rho(t_k)) \quad D_{12}(\rho(t_k))\,]^T [\,0 \quad C_1(\rho(t_k))\,] \\ Y &= [\,C_1(\rho(t_k)) \quad 0\,]^T [\,C_1(\rho(t_k)) \quad D_{12}(\rho(t_k))\,]. \end{aligned}$$

Using the aforementioned definitions, we introduce

$$\begin{bmatrix} P & M & L \\ 0 & Q & N \\ 0 & 0 & R \end{bmatrix} = \exp\left\{\tau_k \begin{bmatrix} -U^T & X & 0 \\ 0 & E & Y \\ 0 & 0 & U \end{bmatrix}\right\}.$$

Next, we partition Q and R in the above equation as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \; R = \begin{bmatrix} R_{11} & 0 \\ 0 & I \end{bmatrix}.$$

Consequently, the matrix A_d, that appears in the first subequation in (14), is determined to be

$$A_d(\rho(k)) = R_{11}. \tag{A1}$$

Step 2: In this step, the system matrices A_{dd} and B_{2d} in Equation (13) are obtained. Having

$$F = [\,F_1 \quad F_2\,] = [\,(Q_{11}^{-1})^T \quad 0\,]M^T R$$

and utilising (A1), we determine

$$\begin{aligned} A_{dd}(\rho(k)) &= A_d(\rho(k)) + F_1 \\ B_{2d}(\rho(k)) &= F_2. \end{aligned}$$

Step 3: By means of a matrix factorisation (e.g. using Cholesky factorisation), one can find B_{1d} in Equation (13) satisfying

$$B_{1d}(\rho(k))B_{1d}^T(\rho(k)) = \gamma^2 Q_{12}Q_{11}^{-1}.$$

Step 4: Finally, C_{1d} and D_{12d} in Equation (13) are found. To this end, we first define the matrix V as

$$V = [C_1(\rho(t_k)) \quad D_{12}(\rho(t_k))]. \tag{A2}$$

Defining

$$\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} = \exp\left\{\tau_k \begin{bmatrix} -U & V^T V \\ 0 & U \end{bmatrix}\right\},$$

and

$$J = R^T M \begin{bmatrix} Q_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} N - R^T L + P_{22}^T P_{12},$$

the two matrices C_{1d} and D_{12d} are found satisfying

$$[\,C_{1d}(\rho(k)) \quad D_{12d}(\rho(k))\,]^T [\,C_{1d}(\rho(k)) \quad D_{12d}(\rho(k))\,] = J$$

through a matrix factorisation. It is emphasised that since in this study our focus is on the matrices depending on a piecewise-constant parameter, we need to repeat the aforementioned steps at each sampling instant. This is done to update the system matrices in Equations (14) and (15).