Lecture Notes: EKF sinusoid example

Sometimes, it is difficult to formulate an appropriate filter model for a tracking problem. Suppose we want to track something moving sinusoidally. For example, consider a spool of thread unwinding in a machine. Imagine we want to track the position on the spool from which the thread is coming. This position moves up and down the spool as it unwinds.

It is difficult to formulate state transition equations for this problem, because we want the dependent variable to be time. For example, we might want to try something like this:

$$f(x_t, a_t) = \left[x_t = \sin\tfrac{t}{10} + a_t\right] \tag{1}$$

However, using this equation, the following state does not depend upon the previous state; it only depends upon the time. This is not a viable formulation for state-space filtering, because there is no uncertainty in the state of the system with respect to time.

We can however write a state model and state transition equations to get something similar that we can use for filtering:

$$f(x_t, a_t) = \begin{bmatrix} x_{t+1} = x_t + \dot{x}_t T \\ \dot{x}_{t+1} = \dot{x}_t + a_t \\ h_{t+1} = \sin\frac{x_t}{10} \end{bmatrix} \tag{2}$$

In this case, we use the variable x to represent something like time, and $\dot{x}$ along with dynamic noise a_t to represent uncertainty in the propagation of the sinusoid over time. The variable h provides the actual value of the sinusoid.

Using this model, we have three state variables:

$$X_t = \begin{bmatrix} x_t \\ \dot{x}_t \\ h_t \end{bmatrix} \tag{3}$$

The state transition equations were given above, where a_t is a random sample drawn from $N(0, \sigma_a^2)$ representing an uncertainty in the propagation of the sinusoid over time.

For observations, we will consider a sensor that detects the current height of the sinusoid d_t:

$$Y_t = \left[d_t\right] \tag{4}$$

For example, this could be accomplished by using a light sensor to detect the thread position as it spins off a spool. The observation equations for this model are:

$$g(x_t, n_t) = \left[d_t = h_t + n_t\right] \tag{5}$$

where n_t is a random sample drawn from $N(0, \sigma_n^2)$ representing measurement noise.

In order to use this model in the EKF, we must calculate the four Jacobians. The derivative of the state transition equations with respect to the state variables is:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial(x, \dot{x}, h)} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ \frac{1}{10}\cos\frac{x}{10} & 0 & 0 \end{bmatrix} \tag{6}$$

The derivative of the state transition equations with respect to the dynamic noises is:

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial(0, a_t, 0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

The values in the matrix $\frac{\partial f}{\partial x}$ must be calculated every iteration. They are calculated using values from the current filtered estimate of the state variables.

The derivative of the observation equations with respect to the state variables is:

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial(x, \dot{x}, h)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The derivative of the observation equations with respect to the measurement noises is:

$$\frac{\partial g}{\partial n} = \frac{\partial g}{\partial(n_t)} = \begin{bmatrix} 1 \end{bmatrix} \tag{9}$$

These Jacobians are both fairly simple because that portion of this model is linear. Therefore all the derivatives are constant.

To finish this example we must look at the covariances. The covariance of the dynamic noises is:

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

The covariance of the measurement noises is

$$R = \begin{bmatrix} \sigma_n^2 \end{bmatrix} \tag{11}$$

The covariance of the state, S, is a 3×3 matrix with the variances of $x, \dot{x}, h$ along the diagonal and the (hopefully very small) covariances in the other elements.

Finally, in order to verify that the problem has been configured properly, one should go through all the EKF equations and make sure that the matrix sizes match. Doing this will reveal that the Kalman gain matrix K is 3×1, and everything fits.

$$\overbrace{X}^{3\times1} = \overbrace{f}^{3\times1} \tag{12}$$

$$\overbrace{S}^{3\times3} = \overbrace{\left(\frac{\partial f}{\partial x}\right)}^{3\times3} \overbrace{S}^{3\times3} \overbrace{\left(\frac{\partial f}{\partial x}\right)^T}^{3\times3} + \overbrace{\left(\frac{\partial f}{\partial a}\right)}^{3\times3} \overbrace{Q}^{3\times3} \overbrace{\left(\frac{\partial f}{\partial a}\right)^T}^{3\times3} \tag{13}$$

$$\overbrace{K}^{3\times1} = \overbrace{S}^{3\times3} \overbrace{\left(\frac{\partial g}{\partial x}\right)^T}^{3\times1} \left[\overbrace{\left(\frac{\partial g}{\partial x}\right)}^{1\times3} \overbrace{S}^{3\times3} \overbrace{\left(\frac{\partial g}{\partial x}\right)^T}^{3\times1} + \overbrace{\left(\frac{\partial g}{\partial n}\right)}^{1\times1} \overbrace{R}^{1\times1} \overbrace{\left(\frac{\partial g}{\partial n}\right)^T}^{1\times1} \right]^{-1} \tag{14}$$

$$\overbrace{X}^{3\times1} = \overbrace{X}^{3\times1} + \overbrace{K}^{3\times1} (\overbrace{Y}^{1\times1} - \overbrace{g}^{1\times1}) \tag{15}$$

$$\overbrace{S}^{3\times3} = \left[\overbrace{I}^{3\times3} - \overbrace{K}^{3\times1} \overbrace{\left(\frac{\partial g}{\partial x}\right)}^{1\times3} \right] \overbrace{S_{t,t-1}}^{3\times3} \tag{16}$$