Lecture Notes: Applying the KF

When formulating a problem for the Kalman filter, one takes the following steps:

1. Determine the state variables (things of interest).
2. Write the state transition equations (description of nominal expected behavior of the state variables).
3. Define the dynamic noise(s). This describes the possible deviations during a state transition.
4. Determine the observation variables (sensor readings).
5. Write the observation equations (relating the sensor readings to the state variables).
6. Define the measurement noise(s). This describes the possible corruptions during a sensor reading.
7. Characterize the covariance matrices for the state variables, dynamic noises, and measurement noises.
8. Characterize the state transition matrix and observation matrix.
9. Check all matrices in the KF equations to make sure the sizes are appropriate.

These steps will be demonstrated via an example. In this example, we will formulate the KF for a 2D constant velocity model.

The first step is to describe the things of interest. For this example, the state X_t can be defined as:

$$X_t = \begin{bmatrix} x_t \\ y_t \\ \dot{x}_t \\ \dot{y}_t \end{bmatrix} \tag{1}$$

where x_t and y_t are the position and $\dot{x}_t$ and $\dot{y}_t$ are the velocity.

The second step is to write a set of state transition equations that describe the typical expected behavior of the state variables. For this example, we are assuming a constant 2D velocity model. We can write state transition equations for this model as:

$$\begin{aligned} x_{t+1} &= x_t + T\dot{x}_t \\ y_{t+1} &= y_t + T\dot{y}_t \\ \dot{x}_{t+1} &= \dot{x}_t \\ \dot{y}_{t+1} &= \dot{y}_t \end{aligned} \tag{2}$$

The third step is to define the dynamic noise(s). For this example, we will assume that random accelerations can happen between sensor samples, and define the dynamic noises as:

$$\text{dyn noise} = \begin{bmatrix} 0 \\ 0 \\ N(0, \sigma_{a_1}^2) \\ N(0, \sigma_{a_2}^2) \end{bmatrix} \tag{3}$$

The fourth step is to list the observation variables. For this example, we will assume that we can sense the x and y positions, denoting a sensor reading as:

$$Y_t = \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} \tag{4}$$

The fifth step is to write out the observation equations. For this example, these can be written as:

$$\begin{aligned} \tilde{x}_t &= x_t \\ \tilde{y}_t &= y_t \end{aligned} \tag{5}$$

The sixth step is to define the measurement noise(s). For this example, we will assume that both sensor readings are corrupted by noise that can be modeled as:

$$\text{meas noise} = \begin{bmatrix} N(0, \sigma_{n_1}^2) \\ N(0, \sigma_{n_2}^2) \end{bmatrix} \tag{6}$$

The seventh step is to characterize the covariance matrices. There are three of them. The covariance of the state variables can be written as:

$$\text{cov(state)} = \text{cov(X)} = S_t = \begin{bmatrix} \sigma_{x_t}^2 & \sigma_{x_t,y_t} & \sigma_{x_t,\dot{x}_t} & \sigma_{x_t,\dot{y}_t} \\ \sigma_{x_t,y_t} & \sigma_{y_t}^2 & \sigma_{y_t,\dot{x}_t} & \sigma_{y_t,\dot{y}_t} \\ \sigma_{x_t,\dot{x}_t} & \sigma_{y_t,\dot{x}_t} & \sigma_{\dot{x}_t}^2 & \sigma_{\dot{x}_t,\dot{y}_t} \\ \sigma_{x_t,\dot{y}_t} & \sigma_{y_t,\dot{y}_t} & \sigma_{\dot{x}_t,\dot{y}_t} & \sigma_{\dot{y}_t}^2 \end{bmatrix} \tag{7}$$

The covariance of the dynamic noises can be written as:

$$\text{cov(dyn noise)} = Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{a_1}^2 & \sigma_{a_1,a_2} \\ 0 & 0 & \sigma_{a_1,a_2} & \sigma_{a_2}^2 \end{bmatrix} \tag{8}$$

The covariance of the measurement noises can be written as:

$$\text{cov(meas noise)} = R = \begin{bmatrix} \sigma_{n_1}^2 & \sigma_{n_1,n_2} \\ \sigma_{n_1,n_2} & \sigma_{n_2}^2 \end{bmatrix} \tag{9}$$

The eighth step is to define the state transition and observation matrices. The state transition matrix Φ can be obtained by looking at equation 2. For the example, it is defined as:

$$\Phi = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The observation matrix M can be obtained by looking at equation 5. For the example, it is defined as:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{11}$$

It is important to note that all the work up until now has been to formulate the model for the filtering problem. None of these equations are implemented. The equations provide a theoretical model of how we expect the system being tracked to behave. Once we have built the model, we can then implement it in the Kalman filter. Before doing so it is wise to check that all the KF equations are plausible by checking matrix sizes for the formulated model:

$$\overbrace{X_{t,t-1}}^{4\times1} = \overbrace{\Phi}^{4\times4} \overbrace{X_{t-1,t-1}}^{4\times1} \tag{12}$$

$$\overbrace{S_{t,t-1}}^{4\times4} = \overbrace{\Phi}^{4\times4} \overbrace{S_{t-1,t-1}}^{4\times4} \overbrace{\Phi^T}^{4\times4} + \overbrace{Q}^{4\times4} \tag{13}$$

$$\overbrace{K_t}^{4\times2} = \overbrace{S_{t,t-1}}^{4\times4} \overbrace{M^T}^{4\times2} [\overbrace{M}^{2\times4} \overbrace{S_{t,t-1}}^{4\times4} \overbrace{M^T}^{4\times2} + \overbrace{R}^{2\times2}]^{-1} \tag{14}$$

$$\overbrace{X_{t,t}}^{4\times1} = \overbrace{X_{t,t-1}}^{4\times1} + \overbrace{K_t}^{4\times2} (\overbrace{Y_t}^{2\times1} - \overbrace{M}^{2\times4} \overbrace{X_{t,t-1}}^{4\times1}) \tag{15}$$

$$\overbrace{S_{t,t}}^{4\times4} = [\overbrace{I}^{4\times4} - \overbrace{K_t}^{4\times2} \overbrace{M}^{2\times4}] \overbrace{S_{t,t-1}}^{4\times4} \tag{16}$$