RESEARCH SUMMARY

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This document is a summary of my published work and some submitted/nearly submitted work. This is not meant to be a full research statement since it does not include a vision for the future. This is a high-level discussion without many of the theorem statements or references (as the pertinent information are in the listed papers).

Continuous Amortization.

- M. Burr, S. Gao, and E. Tsigaridas. The Complexity of the Plantinga-Vegter Curve and Surface Approximation Algorithms. In *Proceedings of the 42nd International Symposium on Symbolic and Algebraic Computation.* 61-68, Association for Computing Machinery, 2017.
- M. Burr. Applications of Continuous Amortization to Bisection-based Root Isolation. *Journal of Symbolic Computation.* 77, 78-126, 2016.
- M. Burr and F. Krahmer. `SqFreeEVAL`: an almost optimal real-root isolation algorithm. *Journal of Symbolic Computaiton.* 47(2), 131-152, 2012.

This project deals with the complexity of subdivision-based algorithms. A subdivision-based algorithm is one that iteratively and adaptively subdivides an initial domain until some test succeeds on every subdomain. These types of algorithms are common in symbolic algebra and frequently used in other sciences, but their complexity is less well understood. In particular, these algorithms are adaptive, performing more subdivisions near difficult features while quickly terminating near easier features. Therefore, a tight complexity analysis must respond to this adaptivity. The work in these papers develops the theory of continuous amortization, which is a unifying technique for studying the complexity of subdivision-based algorithm.

These papers introduce and develop the technique of continuous amortization. Consider the case where we start an interval I and iteratively bisect subintervals J of I whenever they fail a test. We define a local size bound as a function $F : I \to \mathbb{R}_+$ with the property that for each $x \in I$, $F(x)$ is smaller than the width of the smallest subinterval J containing x which must be subdivided, i.e.,

$$F(x) \leq \min_{\substack{J \ni x \\ J \text{ not terminal}}} \operatorname{width}(J).$$

In the case of a bisection algorithm, we prove that the number of regions produced by the bisection algorithm is bounded above by the maximum of 1 and the integral

$$\int_I \frac{2dx}{F(x)}.$$

In this way, we turn a complicated problem into an integral, which may be easier to evaluate.

In the third paper in the list, we introduce this technique and apply it to an evaluation-based algorithm for isolating the real roots of a polynomial. This results in a straight-forward integral to evaluate to compute the number of intervals formed by the subdivision and the result is a nearly optimal state-of-the-art bound on the complexity. In the second paper, I extend the technique of continuous amortization to a general setting, allowing for different types of subdivisions, general measure spaces, and generalizing the integral to compute other quantities, such as the bit complexity. In this paper, I provide complexity bounds for 6 algorithms, either matching or improving upon the known bounds. This shows that the method of continuous amortization is applicable in general. In the first paper on the list, we use continuous amortization to provide a complexity bound for a curve/surface approximation algorithm in two/three dimensions by Plantinga and Vegter. This is the first complexity bound for this algorithm and we show, by example, that our bound is tight and the complexity of the algorithm is exponential.

This series of papers develops the technique of continuous amortization. We show that the technique is a unifying technique as it can recover the work in several other papers. Also, since the integral often involves geometric properties, the complexity analyses explicitly isolate and describe the challenging cases for each algorithm.

Currently, I am working to generalize the ISSAC paper into a journal version. I am also studying other applications of continuous amortization, such as its application to continued fractions.

Solving Polynomial Systems.

- M. Burr, J. Xu, and C. Yap. An Approach for Certifying Homotopy Continuation Paths: Univariate Case. In *Proceedings of the 43rd International Symposium on Symbolic and Algebraic Computation.* 399-406, 2018.
- M. Burr, S. Choi, B. Galehouse, and C. Yap. Complete subdivision algorithms II: isotopic meshing of general algebraic curves. *Journal of Symbolic Computation.* 47(2), 153-166, 2012.

This project deals with the development of algorithms using interval arithmetic for approximating the solutions to polynomial systems. Finding accurate approximations to the solutions to systems of polynomials is a major problem in the fields of symbolic algebra and numerical algebraic geometry. In this work, our main tools involve interval arithmetic and interval methods for functions. Interval arithmetic extends the standard arithmetic operations to intervals, for example, the sum of two intervals $[a, b]+[c, d] = [a+c, b+d]$ is another interval consisting of all possible sums of points in the two input intervals. Interval methods for functions extend these ideas to functions so that if f is a function (usually a polynomial in our case) and J is an interval, interval methods use interval arithmetic to define $\square f(J)$ which is an easily computed over-approximation to $f(J)$, i.e., $f(J) \subseteq \square f(J)$. Such methods allow for one-sided tests and work well with subdivision-based methods. The work in these papers develops algorithms for producing certified approximations to solutions of polynomial systems.

The algorithms in the papers listed above also use separation bounds or homotopy continuation. Separation bounds provide *a priori* lower bounds on the distances between features of a variety. For instance, one can derive separation bounds on the distance between isolated solutions of a system of polynomial equations. These bounds are usually given in terms of the size of the input, e.g., the degree of the polynomials and the number of bits in the coefficients. Homotopy continuation is a technique for finding the roots of a system of polynomial equations by starting with a simple system whose roots can be found easily and deforming the system into the desired system. As the systems are deformed, path tracking techniques are used to follow the roots from the initial system to the target system. With the development of Bertini, homotopy continuation is rapidly becoming a useful tool for applications. Homotopy continuation systems, like Bertini, are often not certified. In other words, even though there are some safeguards in place, there is a chance that the tracked paths jumped between paths. Most certified homotopy continuation algorithms, on the other hand, use Newton steps instead of gradient-based steps (which Bertini uses), thereby requiring a large number of homotopy steps.

In the second paper in the list, we use interval arithmetic and separation bounds to develop a completely numerical algorithm for approximating (possibly) singular curves in the plane. This is the first algorithm to use only evaluation to be able to accurately approximate the solutions to real singular curves. Previously, approximating singular curves requires the theory of resultants, but this work illustrates how to compute such an approximation without this technique (iterative resultants in 3 and higher dimensions frequently become bottlenecks for computation due to coefficient swell). The approximation that we provide is proved to be ambient isotopic to the underlying curve and within specified Hausdorff distance. In the first paper in the list, we use interval arithmetic to certify homotopy continuation paths for a univariate polynomial. In other words, we use the over-approximation from interval arithmetic to construct a tube that contains the tracked path and no other paths. Since the entire computation is certified, the final result is also certified. This method is the first certified algorithm to use gradient-based steps, so our algorithm combines certification with the more efficient steps of methods like Bertini.

This work illustrates how to effectively include interval arithmetic-based methods in computations in symbolic algebra and numerical algebraic geometry. This allows for easy and efficient certification tests in many cases.

Currently, I am working to prepare the homotopy continuation paper for publication. I am also working to extend these techniques to systems in higher dimensions. I am also consulting with a graduate student at Georgia Tech concerning the use of interval arithmetic in his project.

Asymptotic Purity.

- M. Burr. Asymptotic purity for very general hypersurfaces of $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree (k, k). *Central European Journal of Mathematics.* 10(2), 530-542, 2012.
- M. Burr. Asymptotic cohomological vanishing theorems and applications of real algebraic geometry to computer science. Ph.D. Thesis, *New York University*, 2010

This project deals with positivity and vanishing theorems in algebraic geometry. Let X be a smooth projective variety of dimension n over $\mathbb{C}$. The asymptotic Riemann-Roch formula states that for a divisor D,

$$\chi(X, \mathcal{O}_X(mD)) = D^n \cdot \frac{m^n}{n!} + o(m^n)$$

is a polynomial in m where D^n is the self-intersection number of D. Since the Euler characteristic can also be written as an alternating sum of the dimensions of groups in the sheaf cohomology, we also have

$$\chi(X, \mathcal{O}_X(mD)) = \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{O}_X(mD)).$$

A natural question is then, which cohomology groups contribute to the growth of the Euler characteristic, i.e., for which i is

$$\widehat{h}^i(X, D) := \limsup_{m \to \infty} \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!}$$

nonzero. D is called asymptotically pure if there is at most one $\widehat{h}^i$ which is nonzero; in this case, $\widehat{h}^i = (-1)^i D^n$. We say that X is asymptotically pure if every divisor D (not just the effective ones) on X is asymptotically pure. Some examples of asymptotically pure varieties are curves, abelian varieties, flag varieties, varieties with Picard number 1. Moreover, surfaces are asymptotically pure if and only if they do not contain any negative curves, i.e., curves with a negative self-intersection number.

In the second item in the list, my dissertation, I prove that complete simplicial toric varieties are asymptotically pure if and only if they are projective spaces or their quotients by finite subgroups of the torus action. In the first paper, I prove that very general hypersurfaces of $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree (k, k) are asymptotically pure. The proof technique for the second result requires looking at the flat family of hypersurfaces of bidegree (k, k) and computing the cohomology for a special (nonreduced) subscheme in this family. It turns out that the long exact sequence in cohomology has an $SL(n+1, \mathbb{C})$-equivariant map, and representation theory can be used to understand the kernel and cokernel of this map, thereby computing the dimensions of the cohomology groups of interest.

It is expected that asymptotic purity is related to a (strong) notion of positivity since asymptotically pure varieties must have equality between big and ample cones.

I am a co-PI on a submitted SC EPSCoR/IDeA Stimulus grant. If funded, our team would work on this project as well as related questions.

Data Depth.

- M. Burr and R. Fabrizio. Error Probabilities for Halfspace Depth. *Statistics & Probability Letters.* 124, 33-40, 2017.
- M. Burr, E. Rafalin, and D. Souvaine. Dynamic maintenance of half-space depth for points and contours. *arXiv.* Technical report. arXiv:1109.1517 [cs.CG], 2011.
- T. Abbott, M. Burr, M. Chan, E. Demaine, M. Demaine, J. Hugg, D. Kane, S. Langerman, J. Nelson, E. Rafalin, K. Seyboth, V. Yeung. Dynamic ham-sandwich cuts in the plane. *Computational Geometry: Theory and Applications.* 42(5), 419-428, 2009.
- M. Burr, E. Rafalin, and D. Souvaine. simplicial depth: an improved definition, analysis, and efficiency for the finite sample case. In R. Liu, R. Serfling, D. Souvaine, editors, *Data Depth: Robust Multivariate Analysis, Computational Geometry, and Applications,* volume 72 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science,* 195-209. American Mathematical Society, 2006.

The first paper in this list is joint work with an undergraduate student, Robert Fabrizio.

This project deals with a connection between geometry and statistics. Data depth functions measure the centrality of a query point q with respect to a sample $\mathcal{F}_n$ or a distribution. These functions are nonparametric, do not depend on any assumptions about the underlying distribution, and are based purely on the geometry of the sample or distribution. Three of the most commonly studied data depth functions are convex hull peeling depth, half-space depth, and simplicial depth: Let $\mathcal{F}_n$ be a sample of size n in $\mathbb{R}^d$ and q a query point. Let $\mathcal{H}$ be the set of all half-spaces in $\mathbb{R}^d$ and $\mathcal{S}$ be the set of all simples in $\mathbb{R}^d$ whose vertices come from $\mathcal{F}_n$. Then, the convex-hull peeling depth of q is the minimum number of times one needs to iteratively remove the convex hull of $\mathcal{F}_n$ so that q is outside the remaining points. The half-space depth of q can be computed in several ways: it is the fraction of data points that need to be removed from q so that q is outside the convex hull of the remaining points, and, alternatively, the half-space depth of q is the minimum fraction of data points in a half-space containing q, i.e.,

$$HD(q;\mathcal{F}_n) = \frac{1}{n}\left(\min_{\substack{q\in H\\ H\in\mathcal{H}}} \#(H\cap\mathcal{F}_n)\right).$$

The simplicial depth of q is the fraction of simplices formed by the data which contain the query point, i.e.,

$$SD(q;\mathcal{F}_n) = \binom{n}{d}^{-1} \#(q\in S)_{S\in\mathcal{S}}.$$

These functions have been proved to enjoy many pleasant properties; moreover, they are of interest to both the statistics community because they are nonparametric and the discrete geometry community because they are based on geometry. This series of papers studies all three of these depths. The work in these papers develops the theory for and algorithms for computing data depth functions, which make the functions more interesting to practitioners.

In the first paper in the list, an undergraduate student and I improved the convergence for the half-space depth of a point to the half-space depth for the underlying distribution. In this paper, the approach is of note because we took a high dimensional problem and turned it into many lower dimensional problems via projection. This type of reduction has the potential to be useful in other projects. The second paper presents an algorithm for maintaining the half-space depth of points in the plane when the data set is dynamic, i.e., points are added to and removed from the data set. We came up with a very efficient algorithm both in theory and in practice. An interesting side result of this work was that we proved a structural result about the level sets for half-space depth, i.e., that they don't change much during a single insertion or deletion. The third paper deals with maintaining a ham sandwich cut dynamically , i.e., where points are added to or deleted from a collection of points. My contribution to this project focuses on the portion concerning ham sandwich cuts for data sets with bounded convex hull peeling depth. In the last paper, we provide a new definition of simplicial depth that avoids certain computational instabilities and proves bounds between the simplicial depth and the half-space depth of a query point with respect to the same data set. We also provided a first example of a data set where simplicial depth does not have maximality at the center, showing that simplicial depth does not have all of the desirable properties that half-space depth enjoys.

These papers have improved the theoretical underpinnings and algorithmic efficiency for data depth functions. Data depth functions are of interest to statisticians, and these papers show how to compute with them more effectively, thereby making the data depth functions more useful in practice.

I am continuing to work with Robert Frabrizio now that he is a Masters student at Clemson as well as an applied statistician on using data depth functions in practice.

Computability.

- M. Burr and C. Wolf. Computability at zero temperature. *arXiv.* arXiv:1809.00147 [math.DS], 2018.
- M. Burr, M. Schmoll, and C. Wolf. On the Computability of Rotation Sets and their Entropies. *Ergodic Theory and Dynamical Systems.* 2018

This project deals with the connections between dynamical systems and computation. If one were to naïvely attempt to study dynamical systems on a computer, it is very likely that one will observe atypical behavior. For example, the "$2x \mod 1$" dynamical system given by the map $f : [0, 1) \to [0, 1)$ where

$$f(x) = 2x \pmod 1$$

doubles x and then retains the fractional part. Since the built-in floating point numbers on a computer only represent dyadic numbers, i.e., $\mathbb{Z}\left[\frac{1}{2}\right]$, and dyadic numbers map to zero after under iterates of the map f, by performing naïve experiments, one might guess that 0 is an attracting point of the system. This is not true, in fact 0 is a repelling point and the behavior of dyadic numbers is somewhat unusual. Therefore, naïve implementations are likely to give meaningless results; this paper, on the other hand, illustrates how to perform calculations whose output has an accuracy guarantee and is important for those interested in experimental work.

This paper studies the computability properties of the generalized rotation set and the localized entropy of the points in the rotation set. The rotation set is a generalization of Poincaré's rotation number for circle homeomorphisms. We provide general conditions on a dynamical system which guarantee that the rotation set of the system is computable, i.e., can be approximated on a computer to arbitrary precision. In addition, we provide general conditions for the computation of the localized entropy function in the interior of the rotation set. We show that these general conditions are satisfied by shifts maps (hence their rotation sets and localized entropy are computable), and we also provide an example that indicates that the localized entropy may not be computable on the boundary of the rotation set.

Previous papers only studied specific computations, such as the computability of the Julia set for specific maps. This is the first paper which provides a general framework and conditions for computability based on the properties of the dynamical system without focusing on a particular system.

Currently, I am continuing to work on side projects that were discovered during the production of this paper. In particular, I am correcting an error in the previously published literature and studying the computability properties of the entropy function on the boundary of the rotation set.

Edge Ideals.

- M. Burr and D. Lipman. Quadratic-Monomial Generated Domains from Mixed Signed, Directed Graphs. *International Journal of Algebra and Computation.* 2018

This paper is joint work with my Ph.D. student, Drew Lipman, who successfully defended in May 2017 and is now working at an R&D company in Austin.

This project deals with a connection between graphs and toric coordinate rings. Starting with a graph G with vertices $V = \{v_1, \ldots, v_n\}$ and edges E, it is possible to construct a map $\rho : E \to \mathbb{R}^n$ where edge $v_iv_j \in E$ maps to $e_i + e_j$ in $\mathbb{R}^n$. Moreover, $\mathbb{Z}_+\rho(E)$ is an affine subsemigroup of $\mathbb{Z}^n$ and the corresponding semigroup ring $k[G]$ is called the edge ring. On the other hand, given a quadratic-monomial generated subring R of $k[x_1, \ldots, x_n]$ there is a corresponding graph G, where G has edge v_iv_j if and only if $x_ix_j \in R$. In this case, the edge ring of G is R. In previous work, it had been shown, for example, that the normality and Serre's R_1 condition for $k[G]$ can be detected in terms of combinatorial properties of the graph. For example, $k[G]$ is normal if and only if G satisfies the odd cycle condition: any two odd cycles in the same component of G either share a vertex or have a single edge connecting them. This work provides stronger connections between ring theoretic properties and combinatorics, and combinatorial structures often provide pleasant methods to find interesting examples in the ring theory.

These paper extend the definition and constructions to all quadratic-monomial generated subrings of the Laurent polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Combinatorially, we represent $x_i^{-1}x_j^{-1}$ as a negative-signed edge $-v_iv_j$ and $x_i^{-1}x_j$ as a directed edge (v_i, v_j). In these papers, we extend the conditions for normality and Serre's R_1 to this case. This case is significantly more complicated than the previous work because the negative powers allow for cancellation. We developed more subtle definitions and new proofs to deal with this generalization. Somewhat surprisingly, we we able to show that a mixed signed, directed graph has an associated signed graph (a graph with no directed edges) which has the same normality and R_1 properties as the original graph, this allows one to reduce the directed case to the signed case. Moreover, we provide simple examples of graphs whose edge rings are normal as well as a simple example of an edge ring which is not Cohen-Macaulay.

These papers provide a complete classification of quadratic-monomial generated subrings of the Laurent polynomial ring. Moreover, they provide simple examples and straightforward ways to check whether the edge ring has various interesting commutative algebraic properties.

Other Projects.

- M. Burr, S. Gao, and F. Knoll. Optimal Bounds of Johnson-Lindenstrauss Transformations. *Journal of Machine Learning Research.* 2018.

This project deals with the dimension reduction problem, in particular whether it is possible to map a collection of vectors in $\mathbb{R}^d$ to $\mathbb{R}^k$ while nearly preserving their pairwise distances. For a fixed set of vectors $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$ and an error $0 < \varepsilon < \frac{1}{2}$, if k is sufficiently large, then there exists a linear map $A : \mathbb{R}^d \to \mathbb{R}^k$ such that for all pairs i and j, the ℓ_2 norms between the images Ax_i and Ax_j is close to the ℓ_2 norm between x_i and x_j. In other words,

$$(1-\varepsilon)\|x_i - x_j\|_2 \leq \|Ax_i - Ax_j\|_2 \leq (1+\varepsilon)\|x_i - x_j\|_2.$$

Such a transformation is called a Johnson-Lindenstrauss transformation. The existence of such a transformation is equivalent to the question of whether, for fixed $x \in \mathbb{R}^d$, there is a probability distribution $\mathcal{D}$ on matrices such that

$$\mathrm{Prob}_{A\sim\mathcal{D}}[(1-\varepsilon)\|x\|_2 \leq \|Ax\|_2 \leq (1+\varepsilon)\|x\|_2] > 1-\delta.$$

It is known that if $k > C_1\varepsilon^{-2}\log\frac{1}{\delta}$, then such a distribution exists, and if $k < C_2\varepsilon^{-2}\log\frac{1}{\delta}$, then no such distribution exists (for constants C_1 and C_2). In this paper, we show that as $\varepsilon, \delta \to 0$, the constants C_1 and C_2 both approach 4. Therefore, showing that, in the limit, the threshold of a Johnson-Lindenstrauss transformation is $4\varepsilon^{-2}\log\frac{1}{\delta}$.

- M. Burr, A. Cheng, R. Coleman, and D. Souvaine. An intuitive approach to measuring protein surface curvature. *PROTEINS: Struture, Function, and Bioinformatics.* 61, 1068-1074, 2005.

This project deals with the problem of curvature approximation or least-squares sphere-fitting with applications to protein modeling. This problem is known to be challenging in the case of proteins because many of the known methods include assumptions which are not appropriate for protein modeling. In this work, we use inversive geometry as a way to turn the sphere fitting problem into a hyperplane-fitting problem. Our method is computationally efficient and produced very good results. Moreover, this method is appropriate for protein fitting as it works even when we are trying to approximate a surface patch.