2.17

2.17 Equation 2.12 provides the relationship between pressure and elevation in the atmosphere for those regions in which the temperature varies linearly with elevation. Derive this equation and verify the value of the pressure given in Table C.2 in Appendix C for an elevation of 5 km.

$$\int_{p_1}^{p_2} \frac{dp}{p} = -\frac{g}{R}\int_{z_1}^{z_2} \frac{dz}{T} \qquad \text{(Eq. 2.9)}$$

Let $p_1 \sim p_a$ for $z_1 = 0$, $p_2 \sim p$ for $z_2 = z$, and $T = T_a - \beta z$.

Thus,

$$\int_{p_a}^{p} \frac{dp}{p} = -\frac{g}{R}\int_{0}^{z} \frac{dz}{T_a - \beta z}$$

or

$$\ln\frac{p}{p_a} = -\frac{g}{R}\left[-\frac{1}{\beta}\ln\left(T_a - \beta z\right)\right]_0^z = \frac{g}{R\beta}\left[\ln\left(T_a-\beta z\right) - \ln T_a\right]$$

$$= \frac{g}{R\beta}\ln\left(1 - \frac{\beta z}{T_a}\right)$$

and taking logarithm of both sides of equation yields

$$p = p_a\left(1 - \frac{\beta z}{T_a}\right)^{\frac{g}{R\beta}} \qquad \text{(Eq. 2.12)}$$

For $z = 5\,km$ with $p_a = 101.33\,kPa$, $T_a = 288.15\,K$, $g = 9.807\,\frac{m}{s^2}$, $\beta = 0.00650\,\frac{K}{m}$, $R = 287\,\frac{J}{kg\cdot K}$,

$$p = (101.33\,kPa)\left[1 - \frac{(0.0065\,\frac{K}{m})(5\times10^3\,m)}{288.15\,K}\right]^{\frac{9.807\,\frac{m}{s^2}}{(287\,\frac{J}{kg\cdot K})(0.0065\,\frac{K}{m})}}$$

$$= 5.40\times10^4\,\frac{N}{m^2}$$

(From Table C.2 in Appendix C, $p = 5.405\times10^4\,\frac{N}{m^2}$.)