Degradation Modeling for Real-Time Estimation of Residual Lifetimes in Dynamic Environments

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Abstract

This paper presents a methodology for modeling degradation signals from components functioning under dynamically-evolving environment conditions. We utilize in-situ sensor signals related to the degradation process, as well as the environment conditions, to predict and update, in real-time, the distribution of a component's residual lifetime. The model assumes that the time-dependent rate at which a component's degradation signal increases (or decreases) is affected by the severity of the current environmental or operational conditions. These conditions are assumed to evolve as a continuous-time Markov chain. Unique to our model is the union of historical data with real-time, sensor-based data to update the signal parameters, environment parameters, and the residual lifetime distribution of the component within a Bayesian framework.

Index Terms: Degradation model, residual lifetime distribution, reliability, Bayesian estimation.

1 Introduction

The use of real-time sensor data for continuously monitoring critical engineering components in complex systems (wind turbine systems, aircraft navigation systems, smart structures, nuclear reactor cooling systems, etc.) holds significant promise for not only assessing the current health of components, but for dynamically predicting their future residual lifetimes. Recent advances in sensor technologies, especially those related to sensor miniaturization and improved energy consumption, have enabled the health and performance monitoring of complex engineering systems and the environments in which they operate. Sensor signals acquired from functioning components are often correlated with the severity of the underlying physical degradation process. Naturally, many degradation models developed to date focus on using these degradation-based signals (degradation signals for short) to predict the residual lifetime of partially-degraded components (cf. Doksum and Hóyland (1992)). However, in most engineering applications, the evolution of these signals is also affected by the environmental (or operational) conditions in which the components function. For example, when power generators, motors, or even machine tools rotate at higher operational speeds, they generate vibration amplitudes that are higher than those generated at lower speeds. By additionally incorporating signals that capture the time-varying nature of environment or operating conditions, the effects of these conditions on degradation can be analyzed by examining changes in the degradation signal stemming from significant environmental changes.

To date, the vast majority of failure models assume that prevailing environmental conditions are temporally-invariant, or have no effect on degradation and failure processes, as noted by Cha and Mi (2007). Some models have focused on characterizing the environment as a random shock process and estimating the stationary lifetime distribution (cf. Teng and Pham (2006)). The concept of intrinsic age has also been used to model the effects of the environment on component lifetime distributions (cf. Özekici (1995)). This school of thought assumes that a component experiences distinct rates of aging when operating in different environments. Therefore, rather than focusing on the real age of the component, these models examine the intrinsic age in different environments. This literature focuses only on environmental effects on the lifetime distributions and/or failure rates of a population of similar units; it does not, however, consider the effects of the environment on the underlying degradation process of a specific unit. Numerous other proportional hazard models

(PHMs) have been used to capture the effects of the environment by incorporating covariates (see Lee et al. (2004)). Section 2 discusses differences between our approach and the PHM approach in greater detail.

In this paper, we develop a methodology for modeling degradation signals from components functioning in dynamically-evolving environments. Specifically, we generalize the models presented in Bian and Gebraeel (2011, 2013), which assume that the environment or operating condition evolves according to a deterministic profile, by investigating a model in which these conditions transition stochastically between different states. It is important to note that the term "environment" is used generically here and does not (necessarily) refer to the ambient environment of the component, which is presumed to be uncontrollable. The environment process may also refer to operating conditions, which can be viewed as controllable (by an operator). Even if the operating condition is controllable, the impact on the component is the same, i.e., degradation is affected by dynamically-evolving conditions whether those changing conditions are induced by an operator or by the ambient environment. For this reason, we refer to the process that influences or modulates degradation as the "environment process" with the understanding that it may imply operating conditions. Our techniques employ in-situ sensor signals related to the degradation process, as well as the environment conditions, to estimate and update the probability distribution of a component's residual lifetime. The model assumes that the time-dependent rate at which a component's degradation signal increases (or decreases) is affected by the severity of the current environmental or operational conditions. Additionally, we account for the reality that transitions in the environmental and operational conditions may induce upward or downward jumps in the amplitude of the degradation signal, depending on the nature of the changes.

This modeling framework is most appropriate when environment transitions are not instantaneous but their effects can significantly influence the rate of degradation. For example, consider the load induced on an aircraft engine in different operating regimes. The engine load varies as the aircraft is exposed to several distinct (and known) flight conditions: takeoff, maximum climb, maximum cruise, loiter, flight idle, taxi, ground idle, and cutoff. Moreover, if inclement weather is encountered during flight, the aircraft may change altitude, accelerate, or decelerate to avoid (or minimize the effects of) turbulence. These changes induce distinct loading patterns on the aircraft's engines, and these changes are usually significant. (Similar behaviors are encountered with

generators or gas/steam turbines in response to seasonal power demand.) Residual life distribution prediction with some uncertainty in environmental conditions has been explored by Liao and Tseng (2006) who estimated the residual life distribution when the future environment is constant but has some variability. By contrast, we model significant mean shifts in the environmental conditions.

To estimate residual life distributions (RLDs), we take into consideration the future evolution of environmental conditions that evolve as a temporally-homogeneous continuous-time Markov chain (CTMC) whose transition and state dwell times are random. CTMC models have been used extensively for modeling the ambient environment of degrading systems (cf. Kiessler et al. (2002)). More recently, CTMCs have been widely used to model the dynamics of power supply and demand in so-called smart grid environments. The state of the CTMC represents renewable resources such as wind speed intervals and/or directions and solar radiation (Zhu et al. (2011)). Dynamic power demand (load) has also been modeled as a CTMC (Bu et al. (2011)). These environment conditions affect the degradation processes of units in the system. For instance, the degradation of hydrodynamic/journal bearings and gears in wind turbines is influenced by time-varying wind speeds, and generators are subjected to fluctuating power loads that vary the rate of degradation.

The remainder of the paper is organized as follows. Section 2 reviews the literature related to lifetime estimation, including models with and without environment effects. Section 3 presents a degradation model that allows the environment to evolve stochastically over time. Sections 4 develops a methodology for estimating the residual life distribution of a component functioning in a random environment. In Section 5, we assess the influence of a few model parameters on prediction accuracy and illustrate the efficacy of our approach in a case study using real, empirical data. Finally, Section 6 provides some concluding remarks and directions for future work related to dynamic residual lifetime estimation.

2 Relevant Literature

This paper considers two key problems that have historically been investigated in isolation: (1) predicting the lifetime or residual lifetime of a component; and (2) accounting for the environmental conditions and incorporating their effects on the component's degradation process. The problem of estimating lifetime distributions is one that has been studied extensively in the literature. A few

examples include Gebraeel et al. (2005), Van Noortwijk (2009). These models characterize degradation processes using continuous-state stochastic processes (e.g., Wiener or gamma processes). Other models, including Bae and Kvam (2004), Robinson and Crowder (2000), focus on estimating lifetime distributions using random coefficient models to characterize the path followed by the degradation process. A large cross-section of the literature, developed using a reliability testing perspective, assumes the environment is time-invariant. That is, degradation data from a sample of components tested under the same conditions are used to infer and estimate the lifetime distribution of the population. By contrast, the approach presented in this paper is conceptually different and not intended for reliability and degradation testing purposes. Instead, we focus on estimating the residual lifetime distribution of individual, fielded components operating in dynamically-evolving environments.

There are a few common approaches to modeling the degradation of components operating in time-varying environments. One popular approach is the proportional hazards model (PHM), which was first introduced by Cox (1972). A PHM uses known internal and/or external covariates to compute the hazard rate function, or instantaneous risk of failure over time. Some representative examples include Jardine et al. (1997). Two models described in Ghasemi et al. (2010) consider proportional hazards models in which an environmental covariate is driven by a Markov process. While PHMs are useful – particularly when failure time data are available – they require specification of a baseline hazard function that assumes a parametric form (e.g., Weibull baseline hazard). Our framework can be viewed as an alternative to the PHM in that it focuses on modeling the degradation signal $S(t)$, which is influenced by an external environment, to estimate and update the remaining life distribution. A significant amount of information can be ascertained by observing the evolution of $S(t)$ over time, and this information can be leveraged to more accurately estimate the remaining useful lifetime. We aim to model the effects of the environment process on the degradation signal of a particular unit; therefore, this technique may be more appropriate for assessing the residual life distribution of that particular unit.

A second line of literature has focused on modeling the degradation process or its manifestations, such as degradation signals (cf. Si et al. (2011)). Many of these models attempt to estimate a stationary lifetime distribution for a population of components based on a series of accelerated degradation tests (cf. Meeker and Escobar (1998)). Some models estimate the distribution of the residual

lifetimes of individual components, such as those found in Liao and Tian (2013). However, each time the residual lifetime of a component is estimated, the future environment condition is assumed to be constant and equivalent to the current condition. Others have generally modeled two types of random environment effects on the degradation process: (1) random shocks that increase or decrease the degradation instantaneously; or (2) random changes in the degradation rate. The first model to consider a system subject to random shocks is due to Esary and Marshall (1973) who assumed a Poisson shock arrival process. This model was later extended by Abdel-Hameed and Proschan (1973) and Igaki et al. (1995). In all three models, the environment is viewed as a shock process without considering the continuous rate of degradation. Other researchers have investigated the effects of environmental conditions on the degradation rate using general Markov processes. Çinlar (1977) presented a model in which the environment evolves as a Markov process and the degradation evolves according to an increasing Lévy process. Kharoufeh (2003) examined a similar problem wherein the system degrades linearly at a rate that is modulated by a random environment. However, these papers did not account for the possibility that shocks may occur at environment transition epochs.

Our work here is distinguished from existing models in at least two aspects. First, unlike degradation models that focus on estimating the lifetime of a population of components (cf. Doksum and Hóyland (1992)), our primary aim is to estimate the RLD of an individual, fielded component by incorporating its unique degradation signal. As a result, the estimated RLD exploits not only prior information, but also the future environmental profile. Second, unlike typical random shock models (cf. Kharoufeh (2003)), our approach accounts for the reality that environment transitions may induce upward or downward jumps in the amplitude of the degradation signal, depending on the nature of the changes. With the exception of Bian and Gebraeel (2011, 2013), very few papers model the impact of future environmental profiles on the degradation process of a fielded component. However, these two papers account only for the scenario when the future environment is deterministic and known. It is more realistic that components operate in randomly-varying conditions; therefore, the aim of this paper is to provide a novel framework within which the residual lifetime distribution can be estimated considering a randomly-varying environment.

3 Degradation in a Randomly-Evolving Environment

In this section, we present a degradation model and a procedure for estimating the residual lifetime distribution (RLD) of a component via Bayesian updating. Here, we assume the environment evolves stochastically according to a continuous-time Markov chain (CTMC). We begin with the model description and a few preliminaries.

For any time t ($t \geq 0$), let $S(t)$ be the degradation signal at t, and let $s(0)$ be the initial signal observation. Since $s(0)$ is taken to be deterministic, we assume that a population of identical components exhibits the same initial degradation signal. At each t, the component's environment can occupy one, and only one, of the states in a set $\mathcal{S} = \{1, 2, \ldots, m\}$, $m < \infty$. While it is not uncommon to assume the distinct operating regimes are known in advance, the number of distinct states m can be determined in at least two ways. For example, Kharoufeh and Cox (2005) devised a hierarchical clustering technique to estimate the number of states by observing naïve derivative estimates. More recently, Flory et al. (2014) used the classical Bayesian information criterion (BIC) to estimate the number of states. Our model assumes the value m is known in advance; however, it is worth mentioning that we do not preclude potential interactions between environmental states. Therefore, the state space $\mathcal{S}$ consists of all *unique* combinations of environmental conditions and operational settings, where $m \equiv |\mathcal{S}|$. For example, if the unit is subject to two different temperature regimes, denoted t_- and t_+, and two different operating speeds, denoted v_- and v_+, the state space of the governing environment is the Cartesian product of $\mathcal{S}_1 = \{t_-, t_+\}$ and $\mathcal{S}_2 = \{v_-, v_+\}$; that is, $\mathcal{S} = \{(t_-, v_-), (t_-, v_+), (t_+, v_-), (t_+, v_+)\}$, and $m = 4$.

In the spirit of Kharoufeh (2003), we let $\psi(t)$ be the random state of the environment at time t and assume that $\{\psi(t) : t \geq 0\}$ is a CTMC on the finite state space $\mathcal{S}$. The infinitesimal generator matrix of $\{\psi(t) : t \geq 0\}$ is $Q = [q_{i,j}]$, where $q_{i,j}$ is the rate of environment transitions from i to j, $j \neq i$, and $q_i = -q_{i,i}$ is the total rate of leaving state i, where $q_{i,i} = -\sum_{j \neq i} q_{i,j}$. Whenever the environment visits state i, independently of everything else, it occupies that state for an exponentially-distributed time with parameter $q_i > 0$.

Define a rate function $r : \mathcal{S} \to \mathbb{R}_+$ so that $r(\psi(t))$ denotes the component's rate of degradation at time t. That is, whenever $\psi(t) = j \in \mathcal{S}$, the component degrades at rate $r(j)$ where it is assumed that $r(j) > 0$ for each $j \in \mathcal{S}$. Without loss of generality, we assume that $\mathcal{S}$ is completely

ordered such that $r(i) < r(j)$ for $i < j$. We account for the reality that in typical applications, the degradation signal exhibits jumps at environment transition epochs. Therefore, we define a mapping $J : \mathcal{S} \to \mathbb{R}$ so that $J(\psi(t))$ is a function of the jump (either upward or downward) that occurs at time t. For example, in vibration-based signals, it is very common for the vibration signal to experience a downward jump when the load on the unit is reduced. For the model presented herein, the jump magnitude is a deterministic quantity that depends on the environment state just before and just after the transition time of the environment. Finally, the degradation signal at time t is given by

$$S(t) = s(0) + \int_0^t r(\psi(v))\mathrm{d}v + \sum_{j=1}^{N(t)} J(\psi(A_j)) + \gamma W(t)$$

where A_j is the jth transition time of the environment, $N(t)$ is the cumulative number of environment transitions up to time t, $\{W(t) : t \geq 0\}$ is a standard Brownian motion (BM) process, and γ ($\gamma > 0$) is its diffusion parameter, i.e., $\gamma W(t) \sim N(0, \gamma^2 t)$. This term models the noise in the degradation signal that cannot be attributed to the environmental effects. It is assumed that, for each $t \geq 0$,

$$\mathbb{P}\left(\int_0^t |r(\psi(v))|\mathrm{d}v < \infty\right) = 1$$

and $\mathbb{P}(N(t) < \infty) = 1$ to ensure that $S(t)$ is well defined.

Define $U_j = \psi(A_j^+)$, the state of the environment just after the jth transition. Due to the memoryless property of the state dwell times, $\{(U_j, A_j) : j \geq 0\}$ is a Markov renewal process with semi-Markov kernel matrix $\boldsymbol{G}(x) = [G_{i,j}(x)]$ given by

$$G_{i,j}(x) = \mathbb{P}(U_1 = j, A_1 \leq x | U_0 = i) = \begin{cases} \dfrac{q_{i,j}}{q_i}\left(1 - e^{-q_i x}\right), & j \neq i,\ q_i \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Our objective is to provide a framework for dynamically updating the residual lifetime distribution of the component based on discrete observations of the signal and environment processes over time. Suppose the degradation signal is monitored at times $t_0, t_1, \ldots, t_k$ such that $0 = t_0 < t_1 < \cdots < t_k$, and let $s(t_i)$ denote the signal observation at time t_i, $i = 0, 1, \ldots, k$. (We will use lower case letters to denote realizations of random variables.) The set of observations will be represented by a vector $\boldsymbol{s}_k \in \mathbb{R}^{k+1}$ where $\boldsymbol{s}_k = (s(0), s(t_1), \ldots, s(t_k))'$. Additionally, we must observe the magnitude of

jumps occurring at environment transition epochs. Therefore, in addition to the vector $\boldsymbol{s}_k$, we observe the set of environmental conditions $\mathcal{F}_{t_k}$, where $\mathcal{F}_{t_k} = \{(U_j, A_j) : j = 1, 2, \ldots, N(t_k)\}$. Using this convention, the environment maintains state $\psi(A_{j-1})$ over the interval $[A_{j-1}, A_j)$, $j = 1, 2, \ldots, N(t_k)$ with $A_0 = 0$ with probability 1.

In our framework, the component's lifetime corresponds to the first time the degradation signal $\{S(t) : t \geq 0\}$ crosses a fixed, deterministic threshold D. Define the true (unknown) lifetime of the component by the random variable $L \equiv \inf\{t > 0 : S(t) \geq D\}$, and let R_k denote the signal's first passage time to level D, given that $\sup_{0 \leq t \leq t_k} S(t) < D$. Then, for each $k \geq 1$, the random variable, $R_k \equiv [L - t_k | L > t_k]$, is the residual lifetime at time t_k given by

$$R_k = \inf \{u > 0 : S(t_k + u) \geq D\},$$

whose conditional cumulative distribution function (c.d.f.) is

$$\mathbb{P}(R_k \leq t - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}) = \mathbb{P}\left(\sup_{t_k < z \leq t} S(z) \geq D \middle| \boldsymbol{s}_k, \mathcal{F}_{t_k} \right), \quad t > t_k, \tag{1}$$

where $(\boldsymbol{s}_k, \mathcal{F}_{t_k})$ represents the information available up to time t_k. Next, we show how to use the signal and environment observations to dynamically update the RLD.

3.1 Bayesian Updating Methodology

This section describes a Bayesian approach for updating the parameters of the signal and environment models, thereby facilitating the use of prior information to estimate and update the RLD of a component functioning in a randomly-varying environment.

3.1.1 Updating the Signal Model

The updating of $S(t)$ requires updating parameters associated with the mappings r and J, the coefficient γ, and the parameters of $\{\psi(t) : t \geq 0\}$. We update the signal model based on real-time observations of degradation signals and environmental conditions. Denote the joint prior distribution of (r, J, γ) by $\pi_s(r, J, \gamma)$, where we suppress the dependence of r and J on the environment state $\psi(t)$ for notational brevity. By monitoring the degradation signal of a fielded component

(using sensors), along with the current state of the environment, the prior distribution π_s will be updated.

To that end, denote the p.d.f. of $\boldsymbol{s}_k$ by $f_s\left(\boldsymbol{s}_k|\mathcal{F}_{t_k}, r, J, \gamma\right)$. In the basic Bayesian framework, the posterior distribution of (r, J, γ), denoted by ν_s, is such that

$$\nu_s(r, J, \gamma|\boldsymbol{s}_k, \mathcal{F}_{t_k}) \propto \pi_s(r, J, \gamma) f_s\left(\boldsymbol{s}_k|\mathcal{F}_{t_k}, r, J, \gamma\right). \tag{2}$$

3.1.2 Updating the Environment Process

In what follows, we describe the method for updating the environment process, i.e., the parameters of $\{\psi(t) : t \geq 0\}$. Specifically, we assume the non-negative, off-diagonal elements of Q are random. The (negative) diagonal elements follow directly since the row sums of Q are all zero.

To simplify notation in what follows, let $\boldsymbol{q} = \{q_{i,j} : i, j \in \mathcal{S},\, j \neq i\}$ be the set of off-diagonal elements of Q. The prior distribution of Q is denoted by $\pi_Q(\boldsymbol{q})$, and the prior distribution of (r, J, γ) is denoted by $\pi_s(r, J, \gamma)$. Consider a time interval $[0, T]$ and suppose the degradation signal has not crossed the threshold D by time T. The signal is sampled at discrete times $t_0, t_1, \ldots, t_k$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_k < T$, and these observations comprise the vector $\boldsymbol{s}_k = (s(0), s(t_1), s(t_2), \ldots, s(t_k))'$. The history of the environment process up to time t_k is $\mathcal{F}_{t_k} = \{(U_j, A_j) : j = 1, \ldots, N(t_k)\}$; however, these observations are not deterministic as A_j, U_j, and $N(t_k)$ are random variables. For $i, j \in \mathcal{S}$, $j \neq i$, let $N_{i,j}(t_k)$ denote the number of environment transitions from state i to state j in the interval $[0, t_k]$, and let $H_i(t_k)$ be the total holding time in state i on this interval. It is well-known (cf. Billingsley (1961)) that for a continuously-monitored CTMC, the likelihood function of $\boldsymbol{q}$ is given by

$$L(\boldsymbol{q}) = \prod_{i=1}^{m} \prod_{j \neq i} q_{i,j}^{n_{i,j}(t_k)} \exp\left(-q_{i,j}\, h_i(t_k)\right),$$

where $n_{i,j}(t_k)$ and $h_i(t_k)$ are realizations of $N_{i,j}(t_k)$ and $H_i(t_k)$, respectively.

Assume that the (i, j)th element of Q has a gamma prior distribution, i.e., for $j \neq i$, $q_{i,j} \sim \Gamma(k_{i,j}, \theta_{i,j})$, where $k_{i,j}$ and $\theta_{i,j}$ are the shape and scale parameters, respectively. It is well known

that the p.d.f. of $q_{i,j}$ is

$$g_{i,j}(x) = \begin{cases} \dfrac{x^{k_{i,j}-1}\exp(-x/\theta_{i,j})}{\Gamma(k_{i,j})\theta_{i,j}^{k_{i,j}}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

We choose a gamma prior distribution for a few pragmatic reasons. First, each element of $\boldsymbol{q}$ is a non-negative, real number; second, the gamma distribution encompasses a number of important distributions (e.g., exponential, Erlang, and chi-square); third, $k_{i,j}$ and $\theta_{i,j}$ can be chosen to model distributions with varying degrees of skewness; and fourth, the gamma prior distribution yields a gamma posterior density function (see Theorem 1) that is easy to use. By applying Bayes' formula, the posterior distribution of $\boldsymbol{q}$, denoted by ν_Q, is such that

$$\nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}) \propto \pi_Q(\boldsymbol{q}) \times L(\boldsymbol{q}).$$

Theorem 1 establishes that the posterior distribution of $q_{i,j}$ is also gamma.

Theorem 1. *Suppose the environment process is observed at times $t_0, t_1, t_2, \ldots, t_k$, $N_{i,j}(t_k) = n_{i,j}(t_k)$, and $H_i(t_k) = h_i(t_k)$ for $i, j \in \mathcal{S}$ such that $j \neq i$. Then the posterior distribution of $q_{i,j}$ is the gamma distribution with parameters $(\tilde{k}_{i,j}, \tilde{\theta}_{i,j})$ where*

$$\tilde{k}_{i,j} = k_{i,j} + n_{i,j}(t_k), \text{ and } \tilde{\theta}_{i,j} = \left[\theta_{i,j}^{-1} + h_i(t_k)\right]^{-1}.$$

Proof. Suppose $N_{i,j}(t_k) = n_{i,j}(t_k)$ and $H_i(t_k) = h_i(t_k)$ during $[0, t_k]$. By applying Bayes' formula and the likelihood function $L(\boldsymbol{q})$, we can write

$$\begin{aligned} \nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}) &\propto \pi_Q(\boldsymbol{q})L(\boldsymbol{q}) \\ &= \left(\prod_{i=1}^{m}\prod_{j\neq i}\frac{q_{i,j}^{k_{i,j}-1}\exp(-q_{i,j}/\theta_{i,j})}{\Gamma(k_{i,j})\theta_{i,j}^{k_{i,j}}}\right)\left(\prod_{j=1}^{m}\prod_{j\neq i} q_{i,j}^{n_{i,j}(t_k)}\exp\left[-q_{i,j}h_i(t_k)\right]\right) \\ &\propto \left(\prod_{i=1}^{m}\prod_{j\neq i} q_{i,j}^{k_{i,j}+n_{i,j}(t_k)-1}\exp(-q_{i,j}(1/\theta_{i,j} + h_i(t_k)))\right). \end{aligned}$$

We normalize $\nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k})$ so that it satisfies the condition of a p.d.f.,

$$\int_{\boldsymbol{q}} \nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}) = 1.$$

Therefore, we obtain

$$\nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}) = \prod_{i=1}^{m} \prod_{j \neq i} \frac{q_{i,j}^{\tilde{k}_{i,j}-1} \exp(-q_{i,j}/\tilde{\theta}_{i,j})}{\Gamma(\tilde{k}_{i,j}) \tilde{\theta}_{i,j}^{\tilde{k}_{i,j}}}$$

where $\tilde{k}_{i,j} = k_{i,j} + n_{i,j}(t_k)$ and

$$\tilde{\theta}_{i,j} = \left[\theta_{i,j}^{-1} + h_i(t_k)\right]^{-1}.$$

That is, the posterior distribution of $q_{i,j}$ is the gamma distribution with parameters $\tilde{k}_{i,j}$ and $\tilde{\theta}_{i,j}$. □

The updated degradation model is united with the environment process to dynamically update the RLD of the component. Given the posterior distributions $\nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k})$ and $\nu_s(r, J, \gamma|\boldsymbol{s}_k, \mathcal{F}_{t_k})$, we estimate the c.d.f. of R_k by conditioning on the posterior parameters. The updated c.d.f. of R_k is then obtained by

$$\mathbb{P}(R_k \leq t - t_k|\boldsymbol{s}_k, \mathcal{F}_{t_k}) = \int_{(r,J,\gamma)} \int_{\boldsymbol{q}} \mathbb{P}(R_k \leq t - t_k|\boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q})\, \nu_s(r, J, \gamma|\boldsymbol{s}_k, \mathcal{F}_{t_k})\, \nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}).$$

In the next section, we propose two approaches for estimating $\mathbb{P}(R_k \leq t - t_k|\boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q})$ and discuss the advantages and disadvantages of both.

4 Estimating the Residual Lifetime Distribution

Here, we describe two distinct approaches for estimating the RLD of a component that degrades in a randomly-evolving environment. First, we describe a sample path averaging approach that allows us to generate a large sample of future environment profiles and estimate the RLD for each profile. The second approach makes use of the expected number of visits to each environment state in order to estimate the future environmental profile. Before describing the two approaches, we first show how to obtain the RLD for a deterministic profile, as it forms the basis of both procedures.

4.1 RLD Estimation with Deterministic Future Environment Profiles

When the future environment profile is deterministic, the RLD can be estimated by exploiting boundary crossing probabilities for a standard Brownian motion (BM) process. Unlike first passage times to a fixed threshold, the boundary can be described by a function of time, $d(t)$. For instance, if $d(t)$ is linear in t on the interval $[0, T]$, Siegmund (1986) derived the (conditional) probability that a BM process crosses the linear boundary in this interval. Siegmund's result was extended by Wang and Potzelberger (1997) to the case when $d(t)$ is piecewise linear without jump discontinuities on $[0, T]$. Theorem 2 of this subsection extends Theorem 1 of Wang and Potzelberger (1997) to the case when the boundary function $d(t)$ is piecewise linear with jump discontinuities at finitely many points in $[0, T]$. To set the stage for Theorem 2, let $n(T)$ denote the number of jump discontinuities up to time T, suppose $n(T) = n$ and partition $[0, T]$ so that $[0, T] = \bigcup_{j=1}^{n}[v_{j-1}, v_j)$, where v_j is the time of the jth jump discontinuity. It is important to note that both upward and downward jumps can occur; therefore, for ease of notation, let $m_j \equiv \min\{d_j, d_j^-\}$, where $d_j = d(v_j)$, $d_j^- = d(v_j^-)$, and v_j^- represents the instant of time just before the jth environment transition, $j = 0, 1, \ldots, n$, and let $\boldsymbol{d} = (d_0, d_0^-, d_1, d_1^-, \ldots, d_n, d_n^-)'$. The following result, which was utilized but not proved in Bian and Gebraeel (2011), forms the basis of RLD estimation in a random environment.

Theorem 2. *Let $0 = v_0 < v_1 < \cdots < v_n = T$ denote n fixed jump times and suppose $d(v)$ is linear on $[v_{j-1}, v_j)$, $j = 1, 2, \ldots, n$ with $d(0) > 0$. Then for each $v \in [0, T]$, the complement of the first-passage probability of a Brownian motion process, $\gamma W(v)$, with diffusion parameter γ is given by*

$$\mathbb{P}(\gamma W(v) < d(v) | \boldsymbol{s}_0, \mathcal{F}_{t_0}, r, J, \gamma) = \mathbb{E}\left[h(W(v_1), W(v_2), \ldots, W(v_n); \boldsymbol{d})\right], \tag{3}$$

where

$$h(x_1, x_2, \ldots, x_n; \boldsymbol{d}) = \prod_{j=1}^{n} \mathbf{1}(x_j < m_j/\gamma)\Delta(v_j, v_{j-1}),$$

with

$$\Delta(v_j, v_{j-1}) = 1 - \exp\left[-\frac{2(d_{j-1}/\gamma - x_{j-1})\,(d_j^-/\gamma - x_j)}{v_j - v_{j-1}}\right],$$

and $\mathbf{1}(E)$ is the indicator function for event E.

Proof. The proof of Theorem 2 is provided in the Appendix. □

Suppose the degradation signal is sampled at $k+1$ distinct times $0, t_1, \ldots, t_k$, and the current time is t_k ($t_k < T$). The deterministic process, $\{\psi(t) : t_k < t \leq T\}$, represents the future environmental profile from time t_k up to time T. For some $t \in (t_k, T]$, the deterministic component of the degradation signal is

$$\zeta^k(t) \equiv s(t_k) + \int_{t_k}^{t} r(\psi(v))\mathrm{d}v + \sum_{j \in V_k(t)} J(\psi(v_j)), \tag{4}$$

where $V_k(t) \equiv \{j : v_j \in (t_k, t]\}$ is a strictly ordered set. Hence, the degradation signal $S(t)$ given $(\boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma)$ can be expressed as follows:

$$S(t)\big|(\boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma) = \zeta^k(t) + \gamma[W(t) - W(t_k)] \overset{\mathrm{d}}{=} \zeta^k(t) + \gamma W(t - t_k),$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution. Now the c.d.f. of R_k, the residual lifetime at time t_k, given that the signal has not crossed D up to time t_k, can be obtained via the first-passage probability of $S(t)$ to the boundary $d_k(t)$, where $d_k(t) \equiv D - \zeta^k(t)$, for $t_k < t \leq T$. Applying equation (3), we obtain

$$\begin{aligned}\mathbb{P}(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma) = 1 - \mathbb{E}\Bigg[\prod_{j \in V_k(T)} \mathbf{1}\left(W(v_j - t_k) < \frac{m_j^k}{\gamma}\right) \\ \times \left(1 - \exp\left(-\frac{2[d_k(v_{j-1})/\gamma - W(v_{j-1} - t_k)][d_k(v_j^-)/\gamma - W(v_j - t_k)]}{v_j - v_{j-1}}\right)\right)\Bigg],\end{aligned} \tag{5}$$

where $m_j^k = \min\{d_k(v_j), d_k(v_j^-)\}$ for $j \in V_k(T)$.

The residual c.d.f. described by equation (5) is not easy to compute because it requires multidimensional integration. To circumvent this complication, we employ a Monte Carlo simulation procedure (described in detail in Bian and Gebraeel (2011)) to estimate the right-hand side of (5). In what follows, we utilize Theorem 2 to propose two approaches for estimating a component's RLD when the future environment profile evolves randomly as a CTMC on $\mathcal{S}$.

4.2 Approach I: Sample Path Averaging

Suppose the environment is observed at times $t_0, t_1, \ldots, t_k$. Given $\psi(t_k)$, we simulate the environment process $\{\psi(u) : t_k < u \leq T\}$ up to T. Because each simulated sample path can

be treated as a single, deterministic environmental profile, the RLD can be estimated for each profile independently using Theorem 2. Subsequently, applying the strong law of large numbers, the set of cumulative probability values are averaged to obtain the estimated RLD. The formal procedure is as follows:

Step A.1: Select the number of future environment profiles to simulate, C;

Step A.2: Simulate C sample paths of the environment process on the interval $[t_k, T]$, i.e., given the state at time t_k, $\psi(t_k)$, simulate $\{\psi_i(u) : t_k < u \leq T\}$ for $i = 1, 2, \ldots, C$;

Step A.3: Obtain $\zeta_i^k(T)$ via equation (4) for each $i \in \{1, 2, \ldots, C\}$;

Step A.4: Estimate the RLD at t_k for each sample path i, $\mathbb{P}_i(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q})$, via equation (5). Note that $d_k(T) = D - \zeta_i^k(T)$, $i = 1, 2, \ldots, C$;

Step A.5: For a sufficiently large integer C, estimate the RLD at time t_k by

$$\mathbb{P}(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q}) \approx \frac{1}{C} \sum_{i=1}^{C} \mathbb{P}_i(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q}). \tag{6}$$

Approach I allows us to consider a large number of potential future environmental profiles that the component might encounter. However, this approach is computationally expensive as we must simulate these profiles, as well as the Brownian motion component of the degradation signal. Approach II, described in the next subsection, alleviates this difficulty.

4.3 Approach II: Expected Future Profile

In this approach, we exploit the *expected number of visits* to any state $j \in \mathcal{S}$, given that $\psi(t_k) = i$. To this end, let $N_j(t)$ denote the number of environment transitions to state j up to time t. The Markov renewal function (see Kulkarni (1995)),

$$M_{i,j}(t) \equiv \mathbb{E}(N_j(t) | U_0 = i), \;\; i, j \in \mathcal{S}, \;\; t \geq 0,$$

is the expected number of visits to state j up to time t, given that the process starts in state $i \in \mathcal{S}$. Let $\boldsymbol{M}(t) = [M_{i,j}(t)]$ be the matrix composed of these expectations, and define its Laplace-Stieltjes

transform (LST) by

$$\widetilde{\boldsymbol{M}}(s) \equiv \int_0^\infty e^{-st} \mathrm{d}\boldsymbol{M}(t).$$

The LST of the matrix $\boldsymbol{M}(t)$ can be obtained via the LST of the kernel matrix of the Markov renewal sequence $\{(U_j, A_j) : j \geq 0\}$ as seen in the following proposition.

Proposition 1. *The Laplace-Stieltjes transform of $\boldsymbol{M}(t)$, denoted by $\widetilde{\boldsymbol{M}}(s)$, is given by*

$$\widetilde{\boldsymbol{M}}(s) = \left[\boldsymbol{I} - \widetilde{\boldsymbol{G}}(s)\right]^{-1} \widetilde{\boldsymbol{G}}(s), \tag{7}$$

where $\boldsymbol{I}$ is the identity matrix and $\widetilde{\boldsymbol{G}}(s) = [\widetilde{G}_{i,j}(s)]$ with

$$\widetilde{G}_{i,j}(s) = \begin{cases} \dfrac{q_{i,j}}{q_i + s}, & j \neq i,\ q_i \neq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

The proof of Proposition 1 can be found in most standard stochastic modeling textbooks (such as Kulkarni (1995)). To obtain $\boldsymbol{M}(t)$, we numerically invert $\widetilde{\boldsymbol{M}}(s)$ in (7) using the stable inversion algorithms of Abate and Whitt (1995). Given that $\psi(t_k) = i$, the expected total number of environment transitions during $(t_k, T]$ is given by

$$\tau_i^k \equiv \mathbb{E}(N(T - t_k) | \psi(t_k) = i) = \sum_{j=1}^{m} M_{i,j}(T - t_k), \quad 0 < t_k < T. \tag{8}$$

By way of a simple example, we next show how to use equations (7) and (8) to assess the expected future profile.

Example. Suppose $\mathcal{S} = \{1, 2, 3\}$, $t_k = 4$, $T = 10$, and $\psi(t_k) = 2$, and inverting (7), it is found that the expected number of visits to states 1, 2, and 3 (after rounding to the nearest integer) are 2, 1, and 1, respectively. Then by (8), we expect four environment transitions during $(4, 10]$. Of those four, two transitions will be to state 1, while states 2 and 3 will be visited once each (in expectation). A total of $4!/(2! \cdot 1! \cdot 1!) = 12$ possible permutations of the state sequence exist; however, we eliminate from consideration any sequence that contains "self-loops" (i.e., one-step transitions from state j to j). After doing so for this example, only the following five sequences

remain: (a) $2 \to 1 \to 2 \to 1 \to 3$; (b) $2 \to 1 \to 3 \to 1 \to 2$; (c) $2 \to 3 \to 1 \to 2 \to 1$; (d) $2 \to 1 \to 2 \to 3 \to 1$; and (e) $2 \to 1 \to 3 \to 2 \to 1$.

More generally, let ϑ ($\vartheta \in \mathbb{N}$) be the remaining number of state sequences after all sequences with self-loops have been eliminated. For each sequence, we estimate each dwell time in state $j \in \mathcal{S}$ by its mean, $1/q_j$. Let $z_n = \{u_0^n, u_1^n, \ldots, u_{\tau_i^k}^n\}$ be the nth state sequence where $u_0^n = i$ for all n, and τ_i^k is given by (8). By the Markov property, the probability that sequence z_n is observed, $\mathbb{P}(z_n)$, is

$$\mathbb{P}(z_n) = \prod_{j=1}^{\tau_i^k} p\left(u_j^n \middle| u_{j-1}^n\right), \quad n = 1, 2, \ldots, \vartheta,$$

where $p(y|x) = \mathbb{P}(U_1 = y | U_0 = x)$. Given the nth state sequence, the RLD at t_k is computed by (5) to obtain $\mathbb{P}_n(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q})$. Finally, the estimated RLD is the weighted average

$$\mathbb{P}(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q}) \approx \sum_{n=1}^{\vartheta} \mathbb{P}_n(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, r, J, \gamma, \boldsymbol{q})\, \mathbb{P}(z_n). \tag{9}$$

Approach II is preferable to Approach I for two important reasons. First, it obviates the need to simulate future environmental profiles because it exploits the expected number of visits to each state and the mean dwell times in those states. Section 5 illustrates that the RLD prediction capability is only slightly diminished using this approximation. Second, if the expected total number of transitions during $(t_k, T]$ is moderate, we need to evaluate only a relatively small number of state sequences.

4.4 An Illustrative Example

Here, we illustrate the degradation model and Bayesian updating framework when the environment evolves as a CTMC. We first use observations of the degradation signal and environment state to update the degradation model as well as the environmental process; subsequently, the RLD is estimated with the updated information. This illustration serves as the basis for the numerical results presented in Section 5.

First, let us assume the prior distribution of $q_{i,j}$ is the gamma distribution with probability

density function

$$\frac{q_{i,j}^{k_{i,j}-1}\exp(-q_{i,j}/\theta_{i,j})}{\Gamma(k_{i,j})\theta_{i,j}^{k_{i,j}}}$$

where $k_{i,j}$ is the shape parameter and $\theta_{i,j}$ is the scale parameter. By Theorem 1, the posterior distribution of Q, given the history of the process up to time t_k, is

$$\nu_Q(\boldsymbol{q}|\mathcal{F}_{t_k}) = \prod_{i=1}^{m}\prod_{j\neq i}\frac{q_{i,j}^{\tilde{k}_{i,j}-1}\exp(-q_{i,j}/\tilde{\theta}_{i,j})}{\Gamma(\tilde{k}_{i,j})\tilde{\theta}_{i,j}^{\tilde{k}_{i,j}}}.$$

$$\pi_Q(\boldsymbol{q}|U(v), 0\leq v\leq t_k) = \prod_{j\neq i}\frac{q_{i,j}^{\bar{k}_{i,j}-1}\exp(-q_{i,j}/\bar{\theta}_{i,j})}{\Gamma(\bar{k}_{i,j})\bar{\theta}_{i,j}^{\bar{k}_{i,j}}}$$

Next, to characterize the signal model, the functions r and J are assumed to be

$$r(\psi(v)) = \alpha\psi(v)+\beta \quad \text{and} \quad J(\psi(V_j)) = \eta[\psi(V_j^+)-\psi(V_j)]$$

where α, β, and η are random parameters. Recall that γ ($\gamma>0$) is the diffusion parameter of $W(t)$. We assume the prior distributions of these parameters to be normal, i.e., $\alpha\sim N(\mu_1,\sigma_1^2)$, $\beta\sim N(\mu_2,\sigma_2^2)$, $\eta\sim N(\mu_3,\sigma_3^2)$, and $\gamma\sim N(\mu_4,\sigma_4^2)$, and they are all mutually independent. Let $\pi_s(\alpha,\beta,\eta,\gamma)$ denote their joint prior distribution. To estimate the posterior distribution of $(\alpha,\beta,\eta,\gamma)$, we need to derive the likelihood function of the degradation model. Recall that the degradation signal observations up to time t_k comprise the vector $\boldsymbol{s}_k$. The likelihood function of $\boldsymbol{s}_k$ is

$$f_s(\boldsymbol{s}_k|\mathcal{F}_{t_k},\alpha,\beta,\eta,\gamma) = \prod_{i=1}^{k}\phi_i(s(t_i)-s(t_{i-1}))$$

where for $i=1,2,\ldots,k$, $\phi_i(\cdot)$ is the p.d.f. of a normal random variable with mean

$$\int_{t_{i-1}}^{t_i}(\alpha\psi(v)+\beta)\mathrm{d}v+\eta[\psi(v_i^+)-\psi(v_i)]$$

and variance $\gamma^2(t_i-t_{i-1})$. The posterior distribution of $(\alpha,\beta,\eta,\gamma)$ is thereby obtained as

$$\nu_s(\alpha,\beta,\eta,\gamma|\boldsymbol{s}_k,\mathcal{F}_{t_k}) = \frac{\pi_s(\alpha,\beta,\eta,\gamma)\times\prod_{i=1}^{k}\phi_i(s(t_i)-s(t_{i-1}))}{\int\limits_{\alpha,\beta,\eta,\gamma}\pi_s(\alpha,\beta,\eta,\gamma)\times\prod_{i=1}^{k}\phi_i(s(t_i)-s(t_{i-1}))},$$

where $\pi_s(\alpha, \beta, \eta, \gamma) = \varphi_1(\alpha)\varphi_2(\beta)\varphi_3(\eta)\varphi_4(\gamma)$, and

$$\varphi_i(x_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(x_i - \mu_i)^2}{2\sigma_i^2}\right], \quad i = 1, 2, 3, 4.$$

Therefore, using the degradation observations up to time t_k, the updated RLD is

$$\mathbb{P}(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}) = \int_{\boldsymbol{q}} \int_{(\alpha,\beta,\eta,\gamma)} \mathbb{P}(R_k \leq T - t_k | \boldsymbol{s}_k, \mathcal{F}_{t_k}, \alpha, \beta, \eta, \gamma, \boldsymbol{q})\, \nu_s(\alpha, \beta, \eta, \gamma | \boldsymbol{s}_k, \mathcal{F}_{t_k})\, \nu_Q(\boldsymbol{q} | \mathcal{F}_{t_k}). \quad (10)$$

The first term in the integrand can be estimated using Equation (6) of Approach I or Equation (9) of Approach II. The second term in the integrand can be estimated using the technique of Markov chain Monte Carlo (MCMC) described in Bian and Gebraeel (2013). The last term is obtained via Theorem 1.

5 Numerical Results

In this section, we illustrate the modeling framework of Section 3 using simulated results and a real case study. For the first experiment, simulated degradation signals subject to a stochastic environment serve as our benchmarks. Specifically, 1000 degradation signals were simulated until they first reach a degradation threshold D, and the first passage time to level D is the component lifetime. The first 500 signals are used to estimate parameters of the prior distributions of $(\alpha, \beta, \eta, \gamma)$, while the remaining 500 are used to illustrate real-time residual lifetime prediction. The prediction error associated with each simulated signal is computed as follows. Let L_i denote the lifetime of the ith simulated signal, and let $\hat{L}_i$ be the estimated lifetime (using our updating procedure). Here, we use the median of the RLD as an estimate of the residual lifetime R_k. Hence, $\hat{L}_i = t_k + \hat{R}_k^i$, where $\hat{R}_k^i$ is the estimated median of the residual lifetime at time t_k for signal i. Then the prediction error for the ith simulated signal (as a percentage) is

$$\delta_i = \frac{\hat{L}_i - L_i}{L_i} \times 100. \quad (11)$$

The sign of δ_i indicates whether the error is due to overestimation or underestimation of the remaining lifetime. Specifically, if $\delta_i > 0$, then the estimate is optimistic and exceeds the true value L_i. On the other hand, if $\delta_i < 0$, then the estimate is conservative and is less than L_i.

The first set of experiments involves simulated degradation signals against which the estimated residual lifetime is compared. The main purpose of these experiments is to examine and illustrate the sensitivity of the Bayesian updating approach to a few of the model's parameters. Subsequently, the details of a validation case study are provided in Section 5.2. The results are compared to a benchmark degradation model presented in Gebraeel et al. (2005), which does not account for the effect of the environment.

5.1 Simulated Degradation Signals

In this subsection, we test online prediction of the RLD under a variety of scenarios in which the degradation model and environment process are updated online. Baseline parameter values were chosen according to Table 1. The degradation threshold value is $D = 350$ units, and we assume $s(0) = 0$. Initially, the environment has two states so that $\mathcal{S} = \{1, 2\}$.

Table 1: Baseline parameter values: Random environment.

Model parameter	Mean of prior p.d.f.	Variance of prior p.d.f.
α	$\mu_1 = 0.3$	$\sigma_1^2 = 0.03$
β	$\mu_2 = 0.5$	$\sigma_2^2 = 0.05$
η	$\mu_3 = 2.0$	$\sigma_3^2 = 0.20$
γ	$\mu_4 = 2.0$	$\sigma_4^2 = 0.20$
$q_{1,2}$	$k_{1,2} = 0.2$	$\theta_{1,2} = 0.1$
$q_{2,1}$	$k_{2,1} = 0.2$	$\theta_{2,1} = 0.1$

For this analysis, we vary one parameter value while holding all others fixed, and we assess the effect of m on RLD prediction. For the sake of parity, we use the same mean dwell time for each environment state. The details of the simulation experiments are provided in what follows:

Group 1: (Assessing the effect of η). Suppose $\mu_3 \in \{1, 5, 10, 15\}$ and set $\sigma_3^2 = 0.1\mu_3$;

Group 2: (Assessing the effect of γ). Suppose $\mu_4 \in \{1, 5, 10, 15\}$ and set $\sigma_4^2 = 0.1\mu_4$;

Group 3: (Assessing the effect of Q). Suppose $m \in \{2, 3, 4, 5\}$. Given m, set $k_{i,j} = 0.2/(m-1)$ and $\theta_{i,j} = 0.1$, for $j \neq i$. Then the prior density of $q_{i,j}$ has mean $\theta_{i,j}k_{i,j} = 0.02/(m-1)$

for $j \neq i$. This implies that, initially, the mean dwell time for each state $i \in \mathcal{S}$ is 1/0.02.

We assess the quality of RLD prediction for different values of η and γ but fix α, β, $s(0)$ and D. These values are fixed because increasing α or β and/or decreasing $D - s(0)$ have the same effect on prediction accuracy as decreasing the parameter γ; hence, we focus on parameters involving γ. For each instance, we simulate degradation signals via the following procedure:

Step B.1: Simulate the environment's evolution.

(a) Generate an m-state generator matrix Q using the prior distributions. That is, for each $i, j \in \mathcal{S}$ such that $j \neq i$, generate a realization of $q_{i,j}$, denoted by $q^*_{i,j}$, from a $\Gamma(k_{i,j}, \theta_{i,j})$ p.d.f. and define $q^*_i = \sum_{j \neq i} q^*_{i,j}$;

(b) Choose $\psi(0)$ randomly (i.e., any state in $\mathcal{S}$ is chosen with probability $1/m$);

(c) If the current state is i, generate an exponential holding time with parameter q^*_i;

(d) The environment next transitions to state j with probability $q^*_{i,j}/q^*_i$;

(e) Return to Step B.1(c) until the total elapsed time reaches $T_{max} = 10,000$. The resulting sample path can be viewed as a deterministic function of t denoted by $u^*(t)$.

Step B.2: Simulate degradation signals subject to the environment process $u^*(t)$. The procedure is identical to the deterministic case described in Bian and Gebraeel (2011).

The component lifetime was estimated by observing the degradation signal up to the 50th ($t_k = 0.5L_i$) and 90th ($t_k = 0.9L_i$) percentiles of the lifetime. The prediction error was then computed using Equation (11). Figure 1 depicts the simulation results for a number of instances using Approach I (sample path averaging) and Approach II (expected future profile) to compute the RLD as in Section 4. Box plots in the column labeled "50%" summarize the prediction error of the residual life distribution when $t_k = 0.5L_i$, and those in the column labeled "90%" summarize prediction error when $t_k = 0.9L_i$. Table 2 summarizes the time needed to estimate the median of the RLD.

From these figures, we deduce the following general conclusions: (1) for large η, jumps caused by environment transitions dominate, i.e., system failure can primarily be attributed to shocks; (2) for large γ, the diffusion term of the degradation signal dominates, and the effect of the environment

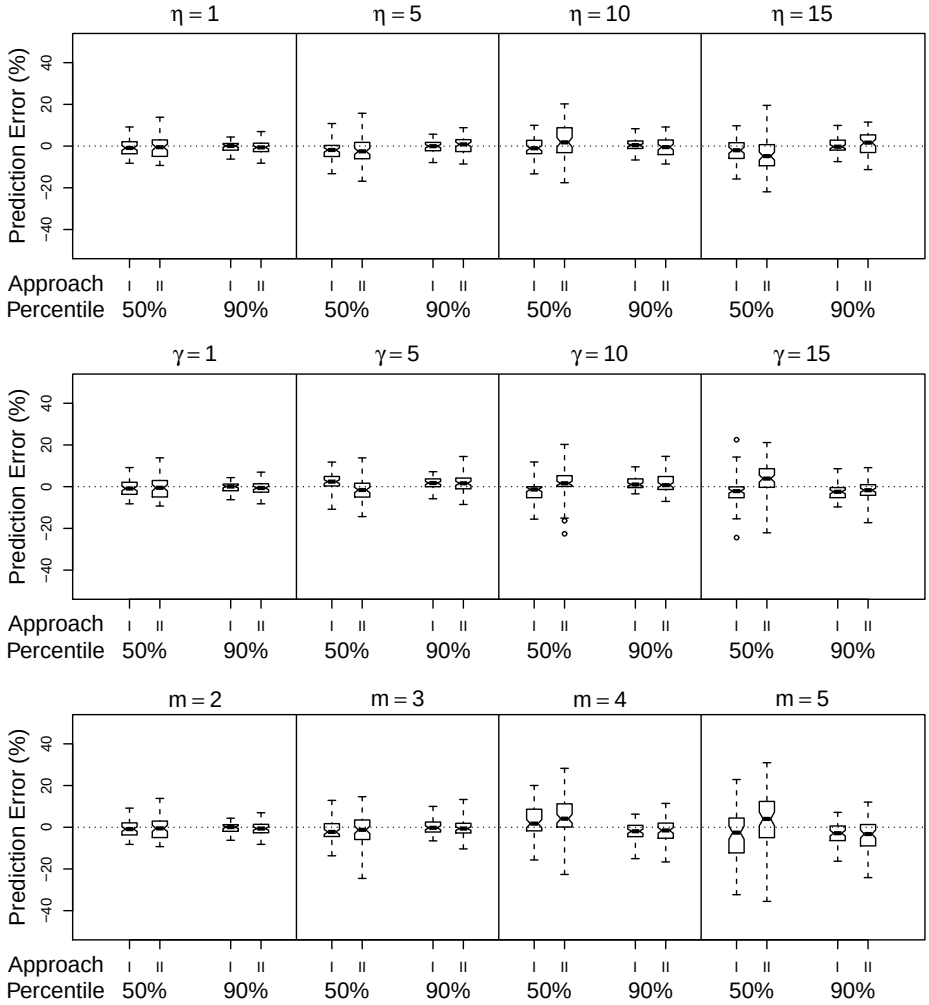

Figure 1: RLD prediction error using Approach I and Approach II.

Table 2: Comparison of computation time (in secs): Approach I versus Approach II.

Parameter	Approach I (50th)	Approach II (50th)	Approach I (90th)	Approach II (90th)
$\eta = 1$	90.17	4.92	21.36	2.37
$\eta = 5$	88.91	5.32	20.27	3.46
$\eta = 10$	89.74	4.77	21.29	2.99
$\eta = 15$	92.98	4.91	21.75	2.83
$\gamma = 1$	88.26	5.23	22.65	2.51
$\gamma = 5$	90.83	5.02	21.73	2.78
$\gamma = 10$	89.12	5.94	22.08	2.27
$\gamma = 15$	90.91	5.66	21.32	3.32
$m = 2$	90.54	5.37	21.25	2.86
$m = 3$	93.32	7.75	22.45	3.98
$m = 4$	93.87	19.14	22.99	7.69
$m = 5$	95.45	51.86	23.48	19.03

can essentially be ignored; and (3) an increase in the number of environment states (m) results in a sharp increase in the mean and variance of the prediction error. Figure 1 also suggests that moderate parameter values, especially the number of environment states, should be chosen for the prior distributions. For example, $\eta \leq 10$, $\gamma \leq 10$, and $m \leq 3$ are reasonable for the experiments we conducted. As compared to using Approach I, Approach II reduces the computational time by eliminating the need to simulate future environmental profiles.

5.2 A Case Study

In this subsection, we present a case study that examines the residual life of ball bearings that operate under stochastically evolving environmental profiles. We use vibration-based degradation signals that are generated from an experimental test-bed designed to perform accelerated degradation tests on a thrust ball bearing using different **load and speed** conditions. These real signals, correlated with the degradation of ball bearings, are used to assess the prediction accuracy of residual life distributions using Approaches I and II. Subsequently, we compare the prediction errors obtained using our methods with those obtained by the degradation model of Gebraeel et al. (2005). That model assumes the degradation signal has an exponential form and leads to a normal approximation of the remaining life distribution. Although the model of Gebraeel et al. (2005) also uses real-time degradation signals to update the degradation model, it does not account for a dynamic environmental condition. Therefore, the latter experiment will assess the value of incorporating this feature when estimating the remaining useful lifetime of the component. Next, we describe the experimental setup before presenting the numerical results.

5.2.1 Experimental Setup

Bearing failures are among the most common and important failure types in rotating and reciprocating machinery, and they have been widely studied in the literature. As noted in Harris (2001), vibration monitoring is among the most popular condition monitoring techniques for bearing applications. This is due to the fact that vibration signals contain distinctive frequencies that are related to various types of bearing defects (see Gebraeel et al. (2005)). The specific experimental test-bed and data acquisition system used to generate the data for this case study have been discussed previously in Bian and Gebraeel (2013), Gebraeel et al. (2005), Gebraeel and Pan (2008).

Therefore, we briefly mention here that the bearing degradation processes were monitored using vibration signals acquired periodically using piezoelectric accelerometers.

In contract to prior studies that focus on *deterministic* future environmental profiles (such as Bian and Gebraeel (2013), Gebraeel and Pan (2008)), this study focuses on testing the prediction accuracy of component RLD when the future environment/operating condition evolves *stochastically*. For instance, in Gebraeel and Pan (2008) it was assumed the future environment state is fixed at the last observed environment state – a significant departure from the model presented herein. Bian and Gebraeel (2013) considered a deterministic, cyclic environment with no jumps in the degradation signal; both of those assumptions are relaxed in our model. Different environmental/operational states affect the degradation rates of bearings, and the state transitions induce jumps in the amplitudes of degradation signals. Figure 2 depicts the evolution of the vibration spectra from a degrading bearing as the rotational speed changes, and it shows that the system experiences a significant shock as the bearing transitions from one operating condition to another.

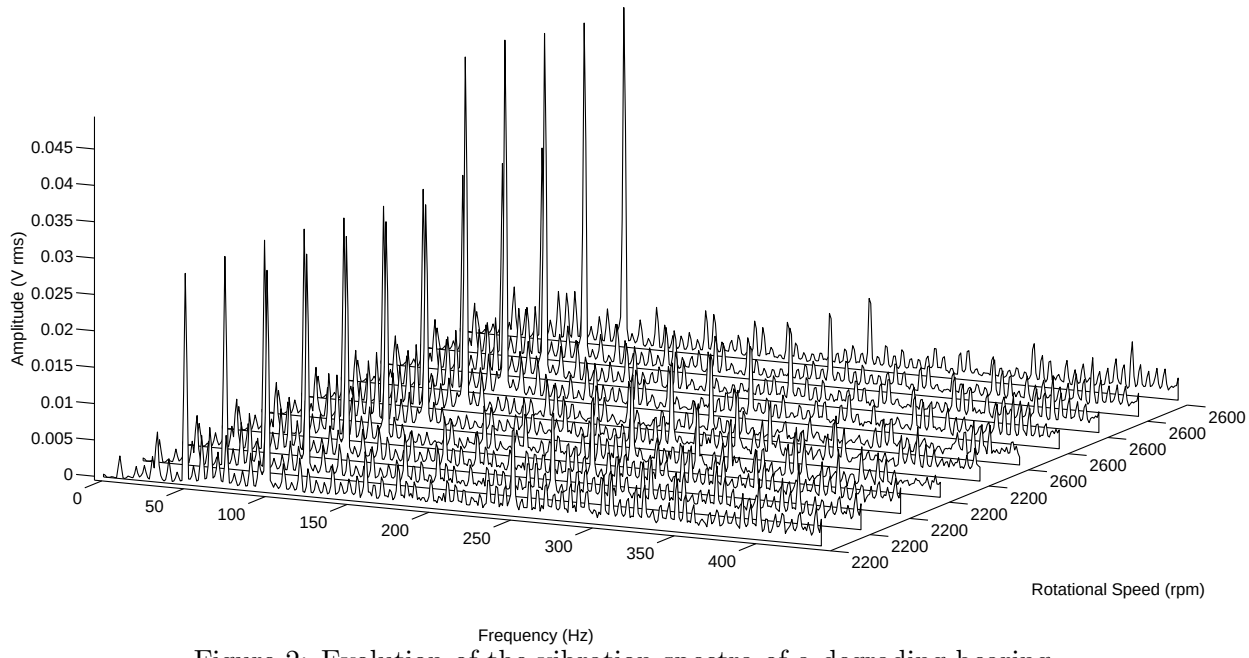

Figure 2: Evolution of the vibration spectra of a degrading bearing.

5.2.2 Constructing Degradation Signals with Stochastic Speed Profiles

Vibration-based degradation signals were constructed using the fact that the vibration amplitudes of bearing-specific defective frequencies are generally correlated with the severity of the

bearing's degradation. Furthermore, we define bearing failure based on the root mean square (RMS) value of the overall vibration of the test rig. According to industrial standards for machinery vibration (e.g., ISO 2372), 2.0–2.2 G (G denotes gravitational acceleration) represents a vibration-based danger level for applications involving general purpose, mid-size machinery (Gebraeel et al. (2005)). Using this standard, we set the failure threshold at $D = 0.025$ Vrms (root mean square volts). We focus on examining the effects of two distinct rotational speeds (2,200 rpm and 2,600 rpm) on the degradation signals of ball bearings. To construct the state space $\mathcal{S} = \{1, 2\}$, we examine the degradation rate in each environmental condition. Thus, state 1 corresponds to the less detrimental condition (2,200 rpm), and state 2 to the more detrimental condition (2,600 rpm). For a scenario with more environmental states, a detailed discussion of experimental tests and the ordering of the degradation rates can be found in Bian and Gebraeel (2013).

Eight degradation tests, labeled Bearings 1–8, were performed, and data from these experiments were used for prior parameter estimation and online validation. In these experiments, Bearings 1 and 2 were run under constant rotational speed, 2,200 rpm and 2,600 rpm, respectively. Bearings 3–8 were run under the speed profile that evolves according to a CTMC. For this two-state CTMC, we assume that $q_{1,2}$ and $q_{2,1}$ follow prior distributions $q_{1,2} \sim \Gamma(3.17 \times 10^{-2}, 1.37)$ and $q_{2,1} \sim \Gamma(4.35 \times 10^{-2}, 0.53)$ as depicted in Figure 3. The prior means of $q_{1,2}$ and $q_{2,1}$ can be computed as $\mathbb{E}(q_{1,2}) = 4.34 \times 10^{-2}$ and $\mathbb{E}(q_{2,1}) = 2.31 \times 10^{-2}$.

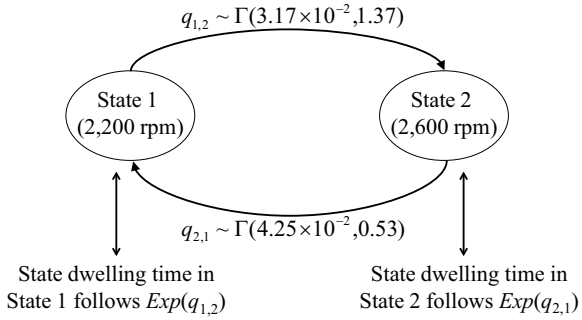

Figure 3: Rate diagram for the CTMC governing rotational speed.

Speed profiles were simulated according to the CTMC process depicted in Figure 3. For each degradation test, we generate realizations of $q_{1,2}$ and $q_{2,1}$ from their prior distributions. The state

dwell times for State 1 and State 2 are generated from exponential populations with parameters $q_{1,2}$ and $q_{2,1}$, respectively. The detailed procedure for generating random environments in provided in Step B.1 of Section 5.1. Degradation signals were observed and monitored throughout each degradation test. Figure 4 depicts the degradation signals from Bearing 8 and the corresponding time-varying speed profiles. The upper plot represents the vibration signal of the bearing, and the lower plot represents the bearing's rotational speed.

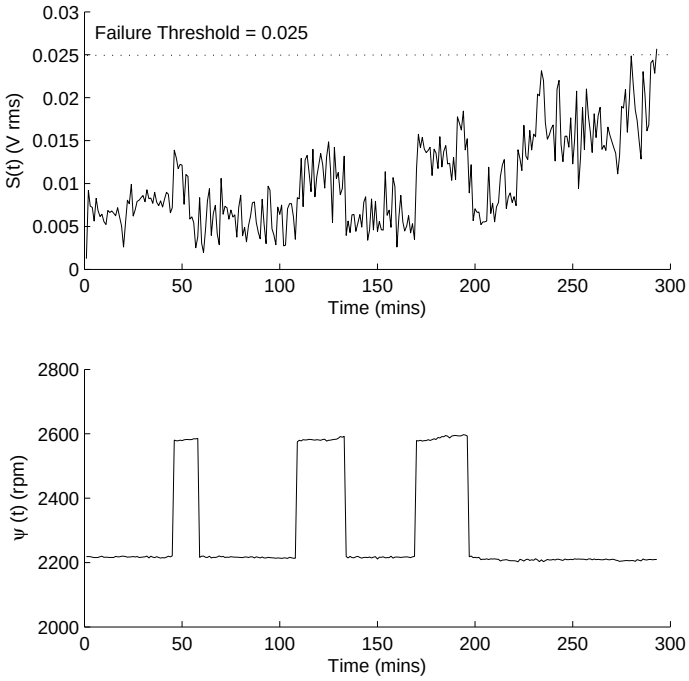

Figure 4: Degradation signals for online validation: Bearing 8.

5.2.3 Numerical Findings

We divided the resulting degradation signals into two groups. The signals of Bearings 1-5 were treated as training data used to estimate the parameters of the degradation model, whereas Bearings 6–8 were treated as testing data used to assess the quality of RLD prediction. The estimated model parameters obtained from the training data are summarized in Table 3.

Next, the validation Bearings 6–8 were used to evaluate the accuracy of online prediction of the RLDs. Specifically, degradation signals from the validation bearings observed at different life percentiles were used to update the stochastic parameters of the model and update the RLDs. The updated RLDs were evaluated at the pth percentile of the lifetime, for $p \in \{20, 30, 40, 50, 60, 70, 80, 90\}$,

Table 3: Estimated model parameters using signals of Bearings 1–5.

Parameter	Prior mean	Prior variance
α	$\mu_1 = 7.62 \times 10^{-4}$	$\sigma_1^2 = 1.62 \times 10^{-6}$
β	$\mu_2 = -3.44 \times 10^{-4}$	$\sigma_2^2 = 4.39 \times 10^{-6}$
η	$\mu_3 = 4.81 \times 10^{-3}$	$\sigma_3^2 = 9.25 \times 10^{-6}$
γ	$\mu_4 = 1.02 \times 10^{-3}$	$\sigma_4^2 = 5.18 \times 10^{-6}$

using Approaches I and II of Section 4. The prediction errors of the estimated lifetimes are summarized in Table 4.

Table 4: Lifetime prediction error (%) for the validation data.

Observed	Bearing 6		Bearing 7		Bearing 8	
Up To	I	II	I	II	I	II
30th Percentile	20.43	27.82	22.79	29.23	-18.10	25.98
60th Percentile	9.89	13.32	12.73	16.11	-8.88	-13.37
90th Percentile	-9.11	-10.01	9.94	11.98	-7.93	-10.38

The results of Table 4 reveal a few important findings. First, we note that the magnitude of the prediction error is similar for both methods. This is a significant finding because far less computational effort is required using Approach II as compared to Approach I. Second, as more real-time information is revealed (i.e., as p increases), the prediction error decreases monotonically. For instance, in the case of Bearing 6, the error is nearly 28% when only 30% of the bearing's lifetime has been observed; however, this error drops to only 10% when 90% of the lifetime has been observed. That is, as the true failure time approaches, the ability to predict this failure time improves. For components with very long lifetimes, these errors may be acceptable.

5.3 Benchmarking Performance

In this subsection, we compare the prediction errors of our techniques with those of a benchmark model presented in Gebraeel et al. (2005) that characterizes degradation signals without considering the effects of the environment. This model was chosen as a benchmark for two main reasons. First, the model in Gebraeel et al. (2005) utilizes real-time degradation signals to update the degradation status and obtain the corresponding posterior RLD for an operating component. Therefore, we are able to evaluate the prediction accuracy of both models at the same sampling times using the

same degradation signals. Second, the model in Gebraeel et al. (2005) is similar to our model in that it includes a Brownian motion signal noise term; however, it does not account explicitly for time-varying environment effects or signal jumps. For these reasons, any potential performance improvements using our methods can be attributed to the incorporation of environmental effects.

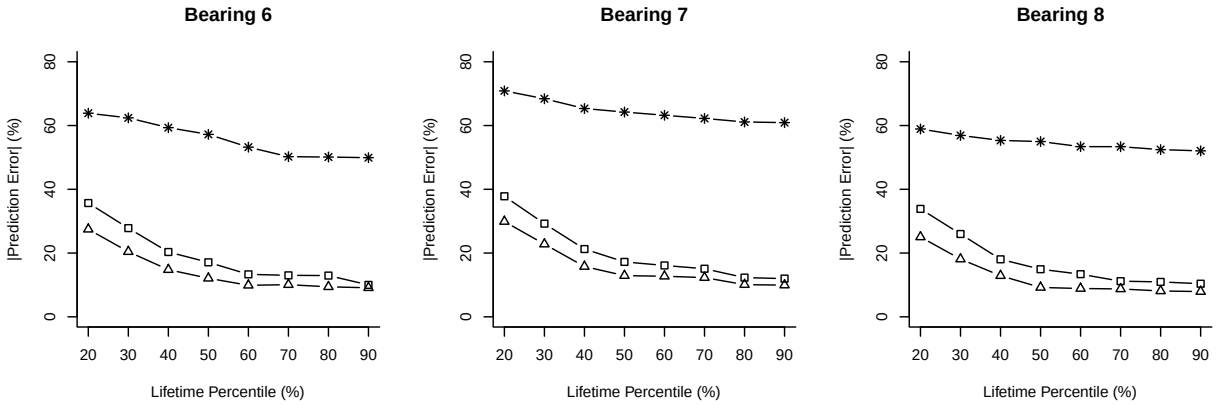

Figure 5: Absolute prediction error: Approach I ($\triangle$), Approach II ($\square$), Benchmark ($*$).

The degradation signal model in Gebraeel et al. (2005) assumes the exponential form

$$S(t) = \phi + \theta \exp\left(\beta t + \gamma W(t) - \gamma^2 t/2\right), \quad t \geq 0, \tag{12}$$

where ϕ is a constant, θ is a lognormal random variable, $\beta \sim N(\mu_1, \sigma_1^2)$ and $\gamma W(t) \sim N(0, \gamma^2 t)$. Using a simple transformation, Gebraeel et al. (2005) derived an expression for the RLD at time t_k, which can be expressed as

$$\mathbb{P}(R_k \leq t - t_k | \boldsymbol{s}_k) = \Phi\left(\frac{\tilde{\mu}(t) - D}{\sqrt{\tilde{\sigma}^2(t)}}\right), \tag{13}$$

where $\tilde{\mu}(t)$ and $\tilde{\sigma}^2(t)$ denote the posterior mean and variance of $S(t)$, respectively, D is the failure threshold and Φ is the standard normal c.d.f.

Equation (13) was used to estimate the RLD of each test bearing, and the absolute prediction errors ($|\delta_i|$) were computed via Equation (11) for each case. These absolute error percentages are graphed in Figure 5, along with the errors obtained using Approaches I and II of Section 4. It is seen that the prediction error is drastically reduced using Approaches I and II when the effect of a time-varying environment is considered in the model.

It is also noteworthy that, although the performance of the benchmark method improves over time (as degradation information is progressively revealed), the magnitude of the prediction error is more than double those of our techniques, even at 90% of the lifetime. Figure 5 also highlights the close correspondence between Approaches I and II and corroborates the assertion that performance is only slightly diminished when using the expected number of visits to each environment state as opposed to sample path averaging.

6 Conclusions

This paper has presented a stochastic degradation modeling framework for estimating the RLD of partially-degraded components that are subject to time-varying environment or operating conditions. Our approach uses historical and real-time signals related to environment conditions, as well as the underlying physical degradation process. Unique to our model is the use of potential future environment profiles for estimating the RLD. The case study involving empirically-generated, vibration-based degradation signals lends significant credibility to the usefulness of our approach. It was shown that the Bayesian updating scheme provides reasonable lifetime prediction results, especially as information is progressively revealed over time.

The modeling framework and numerical results are encouraging; however, this research has raised a few interesting and important questions that are worthy of further consideration. First, it will be important to devise a scheme for aggregating (or clustering) similar environment conditions. The present model assumes that the number of distinct environment states (m) is known *a priori*, but estimating the number of distinct states is an important and challenging next step. Aggregating states that have similar effects on the degradation signal will reduce the number of environment states, thereby improving computational efficiency. Second, in case the dwell time in each environment state is not exponentially distributed, it may be possible to apply phase-type approximations, such as those proposed in Kharoufeh et al. (2010). The revised stochastic process describing the environment would retain the Markov property, but at the expense of an enlarged state space.

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References

Abate, J. and Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal on Computing*, 7:36–43.

Abdel-Hameed, M. and Proschan, F. (1973). Nonstationary shock models. *Stochastic Processes and their Applications*, 1(4):383–404.

Bae, S. and Kvam, P. (2004). A nonlinear random-coefficients model for degradation testing. *Technometrics*, 46(4):460–469.

Bian, L. and Gebraeel, N. (2011). A stochastic methodology for prognostics under time-varying environmental future profiles. In *Proceedings of the 2011 Conference on Intelligent Data Understanding*.

Bian, L. and Gebraeel, N. (2013). Stochastic methodology for prognostics under continuously varying environmental profiles. *Statistical Analysis and Data Mining*, 6(3):260–270.

Billingsley, P. (1961). *Statistical Inference for Markov Processes*. University of Chicago Press.

Bu, S., Yu, F., and Liu, P. (2011). Stochastic unit commitment in smart grid communications. In *IEEE INFOCOM 2011 Workshop on Green Communications and Networking*, pages 307 –312.

Çinlar, E. (1977). Shock and wear models and Markov additive processes. In Shimi, I. and Tsokos, C., editors, *Theory and Applications of Reliability*, pages 193–214. Academic Press.

Cha, J. and Mi, J. (2007). Study of a stochastic failure model in a random environment. *Journal of applied probability*, 44(1):151–163.

Cox, D. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society. Series B (Methodological)*, 34(2):187–220.

Doksum, K. and Hóyland, A. (1992). Models for variable-stress accelerated life testing experiments based on wiener processes and the inverse gaussian distribution. *Technometrics*, 34(1):74–82.

Esary, J. and Marshall, A. (1973). Shock models and wear processes. *The Annals of Probability*, 1(4):627–649.

Flory, J., Kharoufeh, J., and Gebraeel, N. (2014). A switching diffusion model for lifetime estimation in randomly-varying environments. *To appear in IIE Transactions*.

Gebraeel, N., Lawley, M., Li, R., and Ryan, J. (2005). Residual-life distributions from component degradation signals: A Bayesian approach. *IIE Transactions*, 37(6):543–557.

Gebraeel, N. and Pan, J. (2008). Prognostic degradation models for computing and updating residual life distributions in a time-varying environment. *IEEE Transactions on Reliability*, 57(4):539–550.

Ghasemi, A., Yacout, S., and Ouali, M. (2010). Evaluating the reliability function and the mean residual life for equipment with unobservable states. *IEEE Transactions on Reliability*, 59(1):45–54.

Harris, T. (2001). *Rolling Bearing Analysis*. John Wiley & Sons, Inc., 4 edition.

Igaki, N., Sumita, U., and Kowada, M. (1995). Analysis of Markov renewal shock models. *Journal of Applied Probability*, 32(3):821–831.

Jardine, A., Banjevic, D., and Makis, V. (1997). Optimal replacement policy and the structure of software for condition-based maintenance. *Journal of Quality in Maintenance Engineering*, 3(2):109–119.

Kharoufeh, J. (2003). Explicit results for wear processes in a Markovian environment. *Operations Research Letters*, 31(3):237–244.

Kharoufeh, J. and Cox, S. (2005). Stochastic models for degradation-based reliability. *IIE Transactions*, 37(6):533–542.

Kharoufeh, J., Solo, C., and Ulukus, M. (2010). Semi-Markov models for degradation-based reliability. *IIE Transactions*, 42(8):599–612.

Kiessler, P., Klutke, G., and Yang, Y. (2002). Availability of periodically inspected systems subject to Markovian degradation. *Journal of Applied Probability*, 39(4):700–711.

Kulkarni, V. (1995). *Modeling and Analysis of Stochastic Systems*. Chapman and Hall/CRC.

Lee, M., Whitmore, G., Laden, F., Hart, J., and Garshick, E. (2004). Assessing lung cancer risk in railroad workers using a first hitting time regression model. *Environmetrics*, 15(5):501–512.

Liao, C. and Tseng, S. (2006). Optimal design for step-stress accelerated degradation tests. *IEEE Transactions on Reliability*, 55(1):59–66.

Liao, H. and Tian, Z. (2013). A framework for predicting the remaining useful life of a single unit under time-varying operating conditions. *IIE Transactions*, 45(9):964–980.

Meeker, W. and Escobar, L. (1998). *Statistical Methods for Reliability Data*. Wiley & Sons, Inc., New York, NY.

Özekici, S. (1995). Optimal maintenance policies in random environments. *European Journal of Operational Research*, 82(2):283–294.

Robinson, M. and Crowder, M. (2000). Bayesian methods for a growth-curve degradation model with repeated measures. *Lifetime Data Analysis*, 6(4):357–374.

Si, X., Wang, W., Hu, C., and Zhou, D. (2011). Remaining useful life estimation: A review on the statistical data driven approaches. *European Journal of Operational Research*, 213(1):1–14.

Siegmund, D. (1986). Boundary crossing probabilities and statistical applications. *The Annals of Statistics*, 14(2):361–404.

Teng, X. and Pham, H. (2006). A new methodology for predicting software reliability in the random field environments. *IEEE Transactions on Reliability*, 55(3):458–468.

Van Noortwijk, J. (2009). A survey of the application of Gamma processes in maintenance. *Reliability Engineering and System Safety*, 94(1):2–21.

Wang, L. and Potzelberger, K. (1997). Boundary crossing probability for Brownian motion and general boundaries. *Journal of Applied Probability*, 34(1):54–65.

Zhu, L., Yu, F., Ning, B., and Tang, T. (2011). Stochastic charging management for plug-in electric vehicles in smart microgrids fueled by renewable energy sources. In *Proceedings of the IEEE Online Conference on Green Communications*, pages 7–12.

Appendix: Proof of Theorem 2

Proof. For a single linear boundary on the interval $[0, T]$ of the form $d(v) = av+b$, Siegmund (1986) proved that the (conditional) probability that a standard BM process does not cross the boundary in this interval is given by

$$\mathbb{P}(W(v) < av + b, v < T | W(T) = x) = 1 - \exp\left[-\frac{2b(aT + b - x)}{T}\right].$$

For our model, we have

$$\begin{aligned}\mathbb{P}(\gamma W(v) < d(v), v \leq T) &= \mathbb{P}\left(W(v) < \frac{d(v)}{\gamma}, v \leq T\right) \\ &= \int_{-\infty}^{m_1/\gamma} \mathbb{P}\left(W(v) < \frac{d(v)}{\gamma}, v < v_1 | W(v_1) = x_1\right) \\ &\quad \times \mathbb{P}\left(W(v) < \frac{d(v)}{\gamma}, v > v_1 | W(v_1) = x_1\right) \mathrm{d}\mathbb{P}_{v_1}(x_1), \end{aligned} \tag{14}$$

where

$$\mathrm{d}\mathbb{P}_v(x) = \frac{1}{\sqrt{2\pi v}} \exp\left(-x^2/2v\right) \mathrm{d}x,$$

i.e., $\mathrm{d}\mathbb{P}_v(x)/\mathrm{d}x$ is the probability density function of $W(v)$. The product of (14) holds since $\{W(t) : t \geq 0\}$ possesses the strong Markov property. Using the results of Siegmund (1986), the first term in the integrand of (14) is given by

$$\mathbb{P}\left(W(v) < \frac{d(v)}{\gamma}, v < v_1 | W(v_1) = x_1\right) = 1 - \exp\left[-\frac{2\frac{d_0}{\gamma}(\frac{d_1^-}{\gamma} - x_1)}{v_1}\right]$$

Owing to the fact that a standard BM process has stationary and independent increments, provided that $W(v_1) = x_1$, it is clear that $W(v + v_1) - x_1$ is also a BM process starting from the origin. Arguing along the same lines as Wang and Potzelberger (1997), for any $v \in (v_1, T]$, the second

term in the integrand of (14) is

$$\begin{aligned}
\mathbb{P}\left(W(v) < \frac{d(v)}{\gamma} | W(v_1) = x_1\right) &= \mathbb{P}\left(W(v) < \frac{d(v+v_1)}{\gamma} - x_1, v \le T - v_1\right) \\
&= \int_{-\infty}^{(m_2 - x_1)/\gamma} \left(1 - \exp\left[-\frac{2(\frac{d_1}{\gamma} - x_1)(\frac{d_2^-}{\gamma} - x_1 - x_2)}{(v_2 - v_1)}\right]\right) \\
&\times \mathbb{P}\left(W(v) < \frac{d(v+v_1)}{\gamma} - x_1, v \in (v_2 - v_1, T - v_1) | W(v_2 - v_1) = x_2\right) \mathrm{d}\mathbb{P}_{v_2 - v_1}(x_2) \\
&= \int_{-\infty}^{m_2/\gamma} \left(1 - \exp\left[-\frac{2(\frac{d_1}{\gamma} - x_1)(\frac{d_2^-}{\gamma} - x_2)}{(v_2 - v_1)}\right]\right) \\
&\times \mathbb{P}\left(W(v) < \frac{d(v+v_1)}{\gamma} - x_1, v \in (v_2 - v_1, T - v_1) | W(v_2 - v_1) = x_2 - x_1\right) \mathrm{d}\mathbb{P}_{v_2 - v_1}(x_2 - x_1) \\
&= \int_{-\infty}^{m_2/\gamma} \left(1 - \exp\left[-\frac{2(d_1/\gamma - x_1)(d_2^-/\gamma - x_2)}{(v_2 - v_1)}\right]\right) \\
&\times \mathbb{P}\left(W(v) < \frac{d(v+v_2)}{\gamma} - x_2, v \le T - v_2\right) \mathrm{d}\mathbb{P}_{v_2 - v_1}(x_2 - x_1).
\end{aligned}$$

Now, similar steps can be followed to obtain the probability

$$\mathbb{P}(W(v) < d(v+v_2)/\gamma - x_2,\, v \le T - v_2) d\mathbb{P}_{v_2 - v_1}(x_2 - x_1).$$

Repeating these steps in an inductive manner, one obtains

$$\begin{aligned}
&\mathbb{P}\left(W(v) < \frac{d(v)}{\gamma}, v_{n-1} < v \le T\right) \\
&\qquad = \int_{-\infty}^{m_n/\gamma} \left(1 - \exp\left[-\frac{2(\frac{d_{n-1}}{\gamma} - x_{n-1})(\frac{d_n^-}{\gamma} - x_n)}{v_n - v_{n-1}}\right]\right) \mathrm{d}\mathbb{P}_{v_n - v_{n-1}}(x_n - x_{n-1}).
\end{aligned}$$

where for $j = 1, 2, \ldots, n$ and $x_0 = 0$,

$$\mathrm{d}\mathbb{P}_{v_j - v_{j-1}}(x_j - x_{j-1}) = \frac{1}{\sqrt{2\pi(v_j - v_{j-1})}} \exp\left[-\frac{(x_j - x_{j-1})^2}{2(x_j - x_{j-1})}\right] \mathrm{d}x_j$$

since $\{W(t) : t \geq 0\}$ is a BM process. Finally, due to the independent increments property, we obtain

$$\mathbb{P}(W(v) < d(v)/\gamma,\, v \leq T) = \int_{-\infty}^{\frac{m_1}{\gamma}} \cdots \int_{-\infty}^{\frac{m_n}{\gamma}} \prod_{j=1}^{n} \left(1 - \exp\left[\frac{(d_{j-1}/\gamma - x_{j-1})(d_j^-/\gamma - x_j)}{v_j - v_{j-1}}\right]\right) g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{15}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)'$, and

$$g(\boldsymbol{x}) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(v_j - v_{j-1})}} \exp\left[-\frac{(x_j - x_{j-1})^2}{2(v_j - v_{j-1})}\right].$$

The result follows directly by noting that $g(\boldsymbol{x})$ is the probability density function of the random vector $(W(v_1), W(v_2), \ldots, W(v_n))$. □