On a Markov-Modulated Shock and Wear Process

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Abstract

We present transient and asymptotic reliability indices for a single-unit system that is subject to Markov-modulated shocks and wear. The transient results are derived from the (transform) solution of an integro-differential equation describing the joint distribution of the cumulative degradation process and the state of the modulating process. Additionally, we prove the asymptotic normality of a properly centered and time-scaled version of the cumulative degradation at time t. This asymptotic result leads to a simple normal approximation for a properly centered and space-scaled version of the system's lifetime distribution. Two numerical examples illustrate the quality of the normal approximation.

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1 Introduction

In this paper we investigate transient and asymptotic reliability indices for a single-unit system that degrades over time due to normal wear induced by its operating environment and randomly-occurring shocks that cause additional damage to the system. The wear rate and shock arrival rate are both modulated by an external process (the random environment) which is modeled as a continuous-time, regenerative stochastic process $\{Z_t : t \geq 0\}$. At time zero, the device begins its lifetime in perfect working order but degrades over time under the influence of its random environment and shock damage until the cumulative degradation reaches a fixed threshold x, at which time it fails. We denote this random time until failure by T_x. Derived herein are transform expressions for the cumulative distribution function (c.d.f.) of T_x and its nth moment ($n \geq 1$) when the environment is assumed to evolve as a time-homogeneous Markov process. The main transient results are obtained by showing that the joint distribution of the cumulative degradation level and the environment state conditionally satisfies an integro-differential equation. This equation leads to the Laplace-Stieltjes transform (LST) of the lifetime distribution function and the LST of its moments. We additionally investigate a scaled version of the unit's lifetime with the aim of developing simple approximations that allow us to circumvent (numerical) Laplace transform inversion. The asymptotic results are obtained by analyzing the degradation process as a cumulative stochastic process which is shown to obey a central limit theorem.

Systems that incur degradation due to their operating environment and shock damage can be identified in a variety of settings. For instance, low-observable technologies are commonly employed to ensure that weapons systems are difficult to detect, track and engage. These "stealth" systems are usually coated with a radar-absorbing material (RAM) for energy absorption and cancelation. RAM coatings must maintain precise tolerances for an array of electrical properties, most importantly the permeability, and consequently, the near-field reflectivity. However, RAM is extremely sensitive to the ambient environment and suffers wear from normal usage and exposure to the elements (e.g., sun, rain, hail, etc.). Additionally, the coating suffers random strikes (e.g., scratches resulting from aerial refueling, bird strikes and pebble strikes during take-off and landing) that cause damage to the external skin, degrading the aircraft's near-field reflectivity. Assessing the impact of wear and random damage on the near-field reflectivity of the RAM coating is very difficult, but can be done through a detailed and involved inspection process. Once the degradation of the RAM coating reaches or exceeds a significant level, the aircraft must be taken out of service and repaired.

Another example of a device that is subject to a time-varying operating environment and random damage is a cutting tool on manufacturing equipment (e.g., the cutting tool on a lathe). The tool is subject to different wear rates that may depend on several factors including the composition of the work piece, the cutting speed and even variability among machine operators. The cutting tool may also encounter defects in the material that cause excessive tool wear. Once the tool's degradation level reaches a critical threshold, it is unable to maintain engineering specifications and must be replaced.

Stochastic failure models that attempt to capture the impact of a randomly evolving environment have been examined extensively in the applied probability literature. An excellent survey contributed by Singpurwalla [25] presents a number of models with various attributes. Much earlier work, due to Esary, et al. [7] provided several results for both wear and shock processes. That work has been extended in numerous directions by several authors. For instance, Çinlar [3] generalized most of the models of [7] by demonstrating that the joint process of the unit's wear level and the state of its ambient environment may be considered as a Markov-additive process and gave several such examples. The first example considered the case when the random environment is a general Markov process with a finite state space and the wear is assumed to increase as a Lévy process. Additionally, random shocks were assumed to occur at environment transition epochs. The second example (see [3], pp. 201-202) is similar to the problem discussed here where the cumulative wear is a continuous, additive functional of the operating environment, and the first time to failure is a first passage time for the degradation process. However, that model did not include shocks.

Though Råde [23], Shanthikumar and Sumita [24], and Nakagawa [21] provide extensions to the models of [7] they did not incorporate the effect of the unit's operating environment on reliability measures. Ebrahimi [6] investigated survival functions for a cumulative damage shock model with a critical threshold and provided some stochastic ordering results. Li and Luo [19] considered a Markov-modulated shock process wherein the shock inter-arrival times and the random shock damage are both governed by a Markov chain. They obtain reliability bounds when the inter-arrival times have heavy- or light-tailed distributions. Their degradation model does not include a continuous wear component. Mallor and Omey [20] considered a generalized shock process and study some limiting properties. Igaki et al. [12], Skoilakis [26], and Kharoufeh [13] present degradation models that include the influence of a random environment but do not consider random shocks.

Klutke et al. [18] examined the availability of an inspected system whose inter-inspection times and wear rates are random. Subsequently, Klutke and Yang [17] derived an availability result for a system subject to constant linear degradation, shocks and a deterministic inspection policy. Kiessler et al. [16] investigated the limiting average availability of a system whose time-varying wear rates are governed by a continuous-time Markov chain. Kharoufeh et al. [15] extended the model of [16] by including damage-inducing shocks that arrive according to a time-homogeneous Poisson process and deriving the Laplace-Stieltjes transforms of a few transient reliability indices. However, they did not consider the asymptotic behavior of these indices. Ebrahimi [5] considered a system whose degradation is comprised of a continuous wear component as well as jumps. The properties of the model were investigated and bounds were established for the reliability function. However, the random environment in that model corresponds to a shock process that is superimposed on a gamma wear process.

Our work here extends the results of [15], [13] and [16] in several ways. First we consider a model in which the rate of continuous wear and the arrival rate of damage-inducing shocks are both modulated by an external environment. Initially, we focus on characterizing the transient reliability indices (i.e., when the critical degradation threshold is finite) in the spirit of [15, 13]. We provide a characterization of the transient distribution and the lifetime moments of the single-unit system in the form of Laplace-Stieltjes transforms that are amenable to numerical inversion. However, our aim is to go beyond these transform solutions to gain further insight into the behavior of the degradation process (and the resulting lifetime distribution) in an asymptotic sense (as the threshold goes to infinity). To this end, we state and prove several limit theorems related to a time-scaled version of the degradation process and a space-scaled version of the unit's random lifetime. Specifically, we prove the asymptotic normality of these properly scaled quantities. The asymptotic results serve as simple approximations for the lifetime distribution and may be useful for degradation-based reliability models such as those described by Gebraeel et al. [9] and Kharoufeh and Cox [14] among others.

The remainder of the paper is organized as follows. Section 2 provides the model description and the notation used throughout the paper. In section 3, we characterize the lifetime distribution and the nth moment ($n \geq 1$) of the lifetime in a transient sense (i.e., when the degradation threshold x is finite). These characterizations are in the form of Laplace-Stieltjes transforms. Section 4 investigates a time-scaled version of the degradation process and a space-scaled version of the system's lifetime, both of which are shown to obey a central limit theorem. Two illustrative examples are included in section 5, while section 6 provides a few concluding remarks.

2 Model Description

In this section, we describe our mathematical model for a single-unit system subject to environment-induced wear and shocks. All random variables are defined on a common and complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A single-unit system is placed into service at time zero in perfect working order. The system accumulates degradation until a deterministic, critical threshold value x is reached or exceeded, at which time the system is said to be failed. Denote this random first passage time by T_x. We assume throughout that T_x is proper, i.e., for each $x > 0$,

$$\lim_{t \to \infty} \mathbb{P}(T_x \le t) = 1.$$

In fact, the dynamics of our model ensure that T_x is bounded. The degradation accrued over time by the system is attributed to environment-induced wear, as well as shocks that occur at random time intervals. Both the wear and shock arrival rates are modulated by an external stochastic process (the random environment). The wear- and shock-inducing mechanisms are assumed to behave independently of one another; however, they share a common dependence on the state of the random environment.

The random environment is assumed to be a regenerative process, namely an irreducible continuous-time Markov chain (CTMC), $\mathcal{Z} \equiv \{Z_t : t \ge 0\}$, on a finite state space $S \equiv \{1, 2, \ldots, \ell\}$. The CTMC has an infinitesimal generator matrix $\mathbf{Q}$, initial distribution vector $\boldsymbol{\alpha}$ and transition probability functions $\pi_{i,j}(t)$, $i, j \in S$ which comprise the matrix $\mathbf{\Pi}(t) \equiv [\pi_{i,j}(t)]$. The limiting distribution of $\mathcal{Z}$ is $\boldsymbol{\pi}$. Define positive functions $r, \lambda : S \to \mathbb{R}_+$ such that whenever $Z_t = i$, the system wears linearly at a rate $r(i)$, $r(i) > 0$, and shocks occur according to a Poisson process at rate λ_i, $\lambda_i > 0$ for $i = 1, 2, \ldots, \ell$. That is, both the wear rates and shock arrival rates are Markov-modulated. For notational convenience, let $\mathbf{R}_d \equiv \text{diag}(r(1), r(2), \ldots, r(\ell))$ and $\boldsymbol{\lambda} \equiv \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. While the assumption of linear wear appears, on first glance, to be restrictive, non-linear wear paths can be effectively approximated by piece-wise linear paths as demonstrated by Kharoufeh and Cox [14].

Next we describe the total degradation process. The cumulative wear up to time t, denoted by W_t, is

$$W_t = W_0 + \int_0^t r(Z_u)\mathrm{d}u \tag{1}$$

where we assume $W_0 \equiv 0$, and

$$\int_0^t |r(Z_u)|\mathrm{d}u < \infty \quad \text{a.s.}$$

so that W_t is well defined for each $t \ge 0$. The process $\mathcal{W} \equiv \{W_t : t \ge 0\}$, is a cumulative stochastic process (i.e., an additive functional of a regenerative process) as defined in [2, 11, 10, 27]. The system is also damaged by shocks that arrive according to a Poisson process with rate λ_i whenever $Z_t = i$. The damage caused by an individual shock is assumed to be relatively small; however, the cumulative effect of small shocks may be significant, as in the case of fatigue deterioration resulting from mechanical vibrations. Denote by N_t the number of shocks occurring up to time t. The corresponding counting process $\{N_t : t \geq 0\}$ is a Markov-modulated Poisson process (MMPP) with ℓ Poisson arrival rates, $\lambda_1, \lambda_2, \dots, \lambda_\ell$ (see [8] for further details). The damage caused by the nth shock is a (nonnegative) random variable Y_n, and $\{Y_n\}_{n=1}^{\infty}$ is an i.i.d. sequence with proper c.d.f. $F_Y(y) \equiv \mathbb{P}(Y \leq y)$, mean $\mu \equiv \mathbb{E}(Y_1)$ and variance $\sigma_Y^2 \equiv \text{Var}(Y_1)$. We assume throughout that $\mathbb{E}(Y_1) < \infty$ and $\mathbb{E}(Y_1^2) < \infty$. The cumulative damage due to shocks up to time t is a nonnegative random variable,

$$\beta_t = \sum_{n=0}^{N_t} Y_n, \qquad t \geq 0. \tag{2}$$

The total degradation accrued by the system up to time t is the sum of degradation due to wear and that due to shocks given by

$$X_t = \int_0^t r(Z_u)\mathrm{d}u + \sum_{n=0}^{N_t} Y_n, \qquad t \geq 0. \tag{3}$$

The positivity of the degradation rates, $r(1), r(2), \dots, r(\ell)$, ensures that the sample paths of $\mathcal{X} \equiv \{X_t : t \geq 0\}$ are monotonically increasing almost surely (a.s.), and consequently, that events $\{X_t \leq x\}$ and $\{T_x \geq t\}$ are equivalent. The system's random lifetime is given by

$$T_x = \inf\{t > 0 : X_t \geq x\}, \tag{4}$$

or the first time the degradation process $\mathcal{X}$ reaches or exceeds x. Let $G(x,t) \equiv \mathbb{P}(T_x \leq t) = 1 - \mathbb{P}(X_t \leq x)$ denote the unconditional c.d.f. of the unit's lifetime, and let its nth moment be denoted by $\mathbb{E}(T_x^n)$ for $n \geq 1$. This paper is concerned with two primary aims. First, we characterize the transient versions of $G(x,t)$ and $\mathbb{E}(T_x^n)$, $n \geq 1$ and their conditional counterparts. By transient, we mean for all x such that $0 < x < \infty$. Second, we investigate a centered and time-scaled version of X_t as $t \to \infty$, and a centered and space-scaled version of T_x as $x \to \infty$, to construct asymptotic approximations for the transient indices. Sections 3 and 4 provide our main results, and section 5 presents a few numerical illustrations.

3 Transient Analysis

This section provides the transient reliability indices for a single-unit system whose degradation evolution is described by a Markov-modulated shock and wear process. Provided are expressions for the system's lifetime c.d.f., as well as the moments of the lifetime, in the form of Laplace-Stieltjes transforms (LSTs). Let us first introduce the following definitions and notation. The complementary c.d.f. of the lifetime is

$$R(x,t) \equiv \mathbb{P}(X_t \le x) = 1 - G(x,t). \tag{5}$$

Define the joint probability distributions

$$R_{i,j}(x,t) = \mathbb{P}(X_t \le x, Z_t = j | Z_0 = i), \quad i,j \in S. \tag{6}$$

Our first main result characterizes the joint distribution of the process $(\mathcal{X}, \mathcal{Z})$, conditioned on the initial state of the environment. This result intuitively extends Theorem 1 of [15] to the case of a time-varying (modulated) shock arrival rate.

Theorem 1 *For each $i,j \in S$, the distribution function $R_{i,j}(x,t)$ verifies the partial integro-differential equation*

$$\frac{\partial R_{i,j}(x,t)}{\partial t} + \frac{\partial R_{i,j}(x,t)}{\partial x} r(j) = \lambda_j \left(\left[R_{i,j}(\cdot,t) * F_Y \right](x) - R_{i,j}(x,t) \right) + \sum_{k=1}^{\ell} q_{k,j} R_{i,k}(x,t) \tag{7}$$

for $x > 0$, $t \ge 0$ where $(*)$ denotes the convolution operator.

Proof. The proof is similar to that of Theorem 1 in [15]. Considering only the cumulative wear of the system first, let $V_{i,j}(x,t) \equiv \mathbb{P}(W_t \le x, Z_t = j | Z_0 = i)$ and let $\epsilon > 0$ denote a small time increment. Because $\mathcal{Z}$ is a temporally homogeneous Markov process and is independent of the degradation process $\mathcal{X}$, we may write

$$\begin{aligned}
V_{i,j}(x, t+\epsilon) &= \mathbb{P}(W_{t+\epsilon} \le x, Z_{t+\epsilon} = j | Z_0 = i) \\
&= \sum_{k=1}^{\ell} \mathbb{P}(W_{t+\epsilon} \le x, Z_{t+\epsilon} = j | Z_t = k, Z_0 = i)\, \mathbb{P}(Z_t = k) \\
&= \sum_{k=1}^{\ell} \mathbb{P}(Z_{t+\epsilon} = j | Z_t = k, Z_0 = i)\, \mathbb{P}(W_{t+\epsilon} \le x | Z_t = k, Z_0 = i)\, \mathbb{P}(Z_t = k) \\
&= \sum_{k=1}^{\ell} \mathbb{P}(Z_{t+\epsilon} = j | Z_t = k)\, \mathbb{P}(W_{t+\epsilon} \le x, Z_t = k | Z_0 = i) \\
&= \sum_{k=1}^{\ell} \pi_{k,j}(\epsilon)\, \mathbb{P}(W_{t+\epsilon} \le x, Z_t = k | Z_0 = i) \\
&= \sum_{k=1}^{\ell} \pi_{k,j}(\epsilon)\, V_{i,k}(x - r(k)\epsilon, t).
\end{aligned} \tag{8}$$

Next we incorporate the impact of Markov-modulated Poisson shocks occurring at rate λ_{Z_t}. Let N_t denote the random number of shocks up to time t. The transition probability functions for the $\mathcal{Z}$ process, $\pi_{i,j}(\epsilon)$, $i, j \in S$, can be written as

$$\pi_{i,j}(\epsilon) = \delta_{i,j} + \epsilon q_{i,j} + o(\epsilon) \tag{9}$$

where $\delta_{i,j}$ assumes the value 1 when $i = j$ and 0 when $i \neq j$. It is well known that

$$\begin{aligned} \mathbb{P}(N_{t+\epsilon} - N_t = 0 | Z_t = k) &= 1 - \lambda_k \epsilon + o(\epsilon), && \text{(10a)} \\ \mathbb{P}(N_{t+\epsilon} - N_t = 1 | Z_t = k) &= \lambda_k \epsilon + o(\epsilon), && \text{(10b)} \\ \mathbb{P}(N_{t+\epsilon} - N_t \geq 2 | Z_t = k) &= o(\epsilon). && \text{(10c)} \end{aligned}$$

The magnitude of cumulative damage caused by n-independent shocks is given by

$$\beta_n = \sum_{i=1}^{n} Y_i.$$

Since the shock magnitudes form an i.i.d. sequence with distribution function F_Y, we note that

$$\mathbb{P}(\beta_n \leq y) \equiv F_Y^{(n)}(y),$$

where $F_Y^{(n)}$ denotes the n-fold convolution of F_Y with itself. Conditioning on the number of shocks in the interval $(t, t+\epsilon)$ and the magnitude of damage due to those shocks, we may now write

$$\begin{aligned} R_{ij}(x, t+\epsilon) &= \sum_{k=1}^{\ell} \pi_{i,k}(\epsilon) \sum_{n=0}^{\infty} \mathbb{P}(N_{t+\epsilon} - N_t = n | Z_t = k) \\ &\quad \times \int_0^{\infty} \mathbb{P}(X_{t+\epsilon} \leq x, Z_{t+\epsilon} = j | Z_t = k, Y = y) F_Y^{(n)}(\mathrm{d}y). \end{aligned} \tag{11}$$

Substituting equations (8) and (10) into (11) we have

$$\begin{aligned} R_{i,j}(x, t+\epsilon) = \sum_{k=1}^{\ell} \pi_{k,j}(\epsilon) \times \\ \left(R_{i,k}(x - r(k)\epsilon, t)(1 - \lambda_k \epsilon) + \lambda_k \epsilon \int_0^{\infty} R_{i,k}(x - r(k)\epsilon - y, t) F_Y(\mathrm{d}y) \right) + o(\epsilon). \end{aligned} \tag{12}$$

Using (9) in (12) and simplifying gives

$$\begin{aligned} R_{i,j}(x, t+\epsilon) = (1 - \lambda_k \epsilon) R_{i,j}(x - r(j)\epsilon, t) + \epsilon \sum_{k=1}^{\ell} (1 - \lambda_k \epsilon) q_{k,j} R_{i,k}(x - r(k)\epsilon, t) \\ + \lambda_j \epsilon \int_0^{\infty} R_{i,j}(x - r(j)\epsilon - y, t) F_Y(\mathrm{d}y) \\ + \epsilon^2 \sum_{k=1}^{\ell} \lambda_k q_{k,j} \int_0^{\infty} R_{i,k}(x - r(k)\epsilon - y, t) F_Y(\mathrm{d}y) + o(\epsilon). \end{aligned} \tag{13}$$

Rearranging and simplifying the terms of (13), dividing by the time increment ϵ, and letting $\epsilon \downarrow 0$, shows that

$$\frac{\partial R_{i,j}(x,t)}{\partial t} + \frac{\partial R_{i,j}(x,t)}{\partial x} r(j) = \\ - \lambda_j R_{i,j}(x,t) + \sum_{k=1}^{\ell} q_{k,j} R_{i,k}(x,t) + \lambda_j \int_0^{\infty} R_{i,j}(x-y,t) F_Y(\mathrm{d}y) \quad (14)$$

where the right-most term of (14) is the convolution of the distributions $R_{i,j}$ and F_Y.

■

The differential equation (7) describes the joint (spatial) evolution of the degradation process with the (temporal) evolution of the unit's operating environment. The impact of shocks is captured by the convolution term on the right-hand side of (7).

The system of equations can be written in matrix form to make the solution procedure more transparent. Define the $\ell \times \ell$ matrix $\mathbf{R}(x,t) = [R_{ij}(x,t)]$ and recall that $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. In matrix notation, (7) may be written as

$$\frac{\partial \mathbf{R}(x,t)}{\partial t} + \frac{\partial \mathbf{R}(x,t)}{\partial x} \mathbf{R}_d = \left(\left[\mathbf{R}(\cdot,t) * F_Y \right](x) - \mathbf{R}(x,t) \right) \boldsymbol{\lambda} + \mathbf{R}(x,t)\mathbf{Q} \quad (15)$$

where $\left[\mathbf{R}(\cdot,t) * F_Y \right](x)$ denotes the convolution of each entry of $\mathbf{R}(x,t)$ with F_Y. The initial probability vector of the environment process $\mathcal{Z}$ is $\boldsymbol{\alpha} = [\alpha_i]$ where $\alpha_i \equiv \mathbb{P}(Z_0 = i)$, $i \in S$. Let $\mathbb{P}_i(A) \equiv \mathbb{P}(A|Z_0 = i)$ for $A \in \mathcal{A}$, and let $\mathbb{E}_i(T_x^n) \equiv \mathbb{E}(T_x^n|Z_0 = i)$. The vector $\boldsymbol{e}$ will denote a column vector of ones, and $\boldsymbol{e}_i$ is a column vector whose ith entry is unity and all others are zero. The unconditional c.d.f. of the system's lifetime is given by

$$G(x,t) \equiv \mathbb{P}(T_x \leq t) = 1 - \boldsymbol{\alpha}\mathbf{R}(x,t)\boldsymbol{e}, \quad (16)$$

and the conditional c.d.f. is given by

$$G_i(x,t) \equiv \mathbb{P}_i(T_x \leq t) = 1 - \boldsymbol{e}_i'\mathbf{R}(x,t)\boldsymbol{e} \quad (17)$$

where $\boldsymbol{e}_i'$ denotes the transpose of $\boldsymbol{e}_i$. Let the Laplace-Stieltjes transforms of $G(x,t)$ and $G_i(x,t)$ with respect to x be

$$\widetilde{G}(u,t) \equiv \int_0^{\infty} e^{-ux} G(\mathrm{d}x, t), \qquad \mathrm{Re}(u) > 0,$$

and

$$\widetilde{G}_i(u,t) \equiv \int_0^{\infty} e^{-ux} G_i(\mathrm{d}x, t), \qquad \mathrm{Re}(u) > 0,$$

respectively. Define an $\ell \times \ell$ diagonal matrix, $\widetilde{\mathbf{F}}_d(u)$, with each diagonal entry identically equal to

$$\widetilde{F}_Y(u) \equiv \int_0^\infty e^{-uy} F_Y(\mathrm{d}y), \tag{18}$$

the LST of F_Y with respect to y. The following result generalizes the lifetime distributions of [16], [13] and [15] to the case of a Markov-modulated shock and wear process.

Theorem 2 *The Laplace-Stieltjes transforms of the c.d.f.'s $G(x,t)$ and $G_i(x,t)$, with respect to x are, respectively,*

$$\widetilde{G}(u,t) \quad = \quad 1 - \boldsymbol{\alpha} \exp\left(\mathbf{X}(u)t\right) \boldsymbol{e} \tag{19}$$

and

$$\widetilde{G}_i(u,t) \quad = \quad 1 - \boldsymbol{e}_i' \exp\left(\mathbf{X}(u)t\right) \boldsymbol{e}, \tag{20}$$

where $\mathbf{X}(u) = \mathbf{Q} + (\widetilde{\mathbf{F}}_d(u) - \mathbf{I})\boldsymbol{\lambda} - u\mathbf{R}_d$, $\mathbf{I}$ is the identity matrix and $\mathrm{Re}(u) > 0$.

Proof. Define the matrix LST of $\mathbf{R}(x,t)$ with respect to x as

$$\widetilde{\mathbf{R}}(u,t) = \int_0^\infty e^{-ux} \mathbf{R}(\mathrm{d}x, t). \tag{21}$$

Taking the LST of both sides of (15) with respect to x yields a first order, ordinary differential equation in t,

$$\frac{\mathrm{d}\widetilde{\mathbf{R}}(u,t)}{\mathrm{d}t} + \widetilde{\mathbf{R}}(u,t)\left[u\mathbf{R}_d + \left(\mathbf{I} - \widetilde{\mathbf{F}}_d(u)\right)\boldsymbol{\lambda} - \mathbf{Q}\right] = \mathbf{0}. \tag{22}$$

Applying the initial condition, $\widetilde{\mathbf{R}}(u,0) = \mathbf{I}$ and rearranging terms, this ordinary differential equation has the obvious solution,

$$\widetilde{\mathbf{R}}(u,t) = \exp\left[\left(\mathbf{Q} - (\mathbf{I} - \widetilde{\mathbf{F}}_d(u))\boldsymbol{\lambda} - u\mathbf{R}_d\right) t\right].$$

Thus, the LST of $G(x,t)$ is given by

$$\begin{aligned} \widetilde{G}(u,t) &= 1 - \boldsymbol{\alpha}\left[\int_0^\infty e^{-ux}\mathbf{R}(\mathrm{d}x,t)\right]\boldsymbol{e} \\ &= 1 - \boldsymbol{\alpha}\widetilde{\mathbf{R}}(u,t)\boldsymbol{e}. \end{aligned}$$

The LST of the conditional c.d.f., $G_i(x,t)$, is obtained by replacing $\boldsymbol{\alpha}$ by $\boldsymbol{e}_i'$. ∎

We next examine the unconditional and conditional moments of the system lifetime. To this end, let T_x^n be the nth power of the random variable T_x, and respectively denote the unconditional and conditional moments of T_x by $\mathbb{E}(T_x^n)$ and $\mathbb{E}_i(T_x^n)$, $n \geq 1$. Define the LST of $\mathbb{E}(T_x^n)$ with respect to x by

$$\widetilde{\mathbb{E}}(T_u^n) \equiv \int_0^\infty e^{-ux} \mathrm{d}\mathbb{E}(T_x^n),$$

and the LST of $\widetilde{\mathbb{E}}_i(T_x^n)$ as

$$\widetilde{\mathbb{E}}_i(T_u^n) \equiv \int_0^\infty e^{-ux} \mathrm{d}\mathbb{E}(T_x | Z_0 = i).$$

Corollary 1 *For each $n \geq 1$, the Laplace-Stieltjes transform of $\mathbb{E}(T_x^n)$ with respect to x is given by*

$$\widetilde{\mathbb{E}}(T_u^n) = n! \cdot \boldsymbol{\alpha} \left(u\mathbf{R}_d + (1 - \widetilde{F}_Y(u))\boldsymbol{\lambda} - \mathbf{Q} \right)^{-n} \boldsymbol{e}, \tag{23}$$

and

$$\widetilde{\mathbb{E}}_i(T_u^n) = n! \cdot \boldsymbol{e}_i' \left(u\mathbf{R}_d + (1 - \widetilde{F}_Y(u))\boldsymbol{\lambda} - \mathbf{Q} \right)^{-n} \boldsymbol{e} \tag{24}$$

with $\mathrm{Re}(u) > 0$.

Proof. The result can be shown by first defining the Laplace-Stieltjes transform of $\widetilde{G}(u, t)$ (with respect to t) given by

$$\hat{G}(u, s) \equiv \int_0^\infty e^{-st} \widetilde{G}(u, \mathrm{d}t).$$

Apply standard matrix calculus operations (cf. Neuts [22]), we obtain the first result by

$$\widetilde{\mathbb{E}}(T_u^n) = (-1)^n \frac{\partial \hat{G}(u, s)}{\partial s^n} \bigg|_{s=0}.$$

The conditional version (24) is obtained by simply replacing $\hat{G}(u, s)$ by $\hat{G}_i(u, s)$ in the above expression. ■

We note that if, for some constant λ ($\lambda > 0$), $\lambda_i = \lambda$ for all $i \in S$, the results of Theorem 2 and Corollary 1 reduce to the corresponding LSTs of [15]. Furthermore, if $\lambda_i = 0$ for all $i \in S$, the model reduces to a Markov-modulated wear process with no shocks as described in [13] and [16]. The transient results presented here are amenable to numerical Laplace transform inversion; however, numerical inversion can be unstable when x is either very small or very large, or when $|S|$ is large. Alternatively, we may consider the limiting behavior of the degradation process with the aim of constructing asymptotic distribution and moment approximations.

4 Limiting Behavior

We now examine the cumulative degradation process and lifetime distribution in the asymptotic regime (i.e., when $t \to \infty$ and $x \to \infty$, respectively). As before, the row vector, $\boldsymbol{\pi} = [\pi_i]$ for $i = 1, 2, \ldots, \ell$, denotes the stationary distribution of the environment process $\mathcal{Z}$. Let

$$\boldsymbol{\Delta} \equiv (\mathbf{R}_d + \mu\boldsymbol{\lambda})\boldsymbol{e},$$

and note that $\boldsymbol{\pi\Delta}$ is the long-run average rate of degradation (including wear and damage from shocks). Intuitively, we would expect that, when x is large, the average time for the degradation process to first reach level x should be very close to $x/\boldsymbol{\pi}\Delta$. Proposition 1 confirms this intuition by establishing convergence in the mean of the scaled r.v. $x^{-1}T_x$ as $x \to \infty$.

Proposition 1 *As* $x \to \infty$,

$$\frac{\mathbb{E}(T_x)}{x} \to \frac{1}{\boldsymbol{\pi\Delta}}. \tag{25}$$

Proof. To prove this result, we first note that as $u \to 0$

$$\frac{\widetilde{F}_Y(u) - 1}{u} \to \frac{\mathrm{d}}{\mathrm{d}u}\widetilde{F}_Y(u)\bigg|_{u=0} = -\mu,$$

so that

$$\frac{1 - \widetilde{F}_Y(u)}{u} = \mu + \epsilon(u)$$

where $u\epsilon(u) = o(u)$. Therefore, using (23),

$$\begin{aligned}
\lim_{u\to 0} u[u\mathbf{R}_d + (1 - \widetilde{F}_Y(u))\boldsymbol{\lambda} - \mathbf{Q}]^{-1} &= \lim_{u\to 0} u\left[u\left(\mathbf{R}_d + \left(\frac{1 - \widetilde{F}_Y(u)}{u}\right)\boldsymbol{\lambda}\right) - \mathbf{Q}\right]^{-1} \\
&= \lim_{u\to 0} u\left[u(\mathbf{R}_d + \mu\boldsymbol{\lambda}) - \mathbf{Q} + o(u)\boldsymbol{\lambda}\right]^{-1} \\
&= \left(\lim_{u\to 0} u^{-1}\left[u(\mathbf{R}_d + \mu\boldsymbol{\lambda}) - \mathbf{Q} + o(u)\boldsymbol{\lambda}\right]\right)^{-1} \\
&= \left(\lim_{u\to 0} u^{-1}\left[u(\mathbf{R}_d + \mu\boldsymbol{\lambda}) - \mathbf{Q}\right]\right)^{-1} \\
&= \lim_{u\to 0} u\left[u(\mathbf{R}_d + \mu\boldsymbol{\lambda}) - \mathbf{Q}\right]^{-1}.
\end{aligned}$$

Next we make use of the asymptotic properties of the Laplace transform to obtain,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbb{E}(T_x)}{x} &= \lim_{u\to 0} u\widetilde{\mathbb{E}}(T_u) \\
&= \lim_{u\to 0} u\boldsymbol{\alpha}[u\mathbf{R}_d + (1 - \widetilde{F}_Y(u))\boldsymbol{\lambda} - \mathbf{Q}]^{-1}\boldsymbol{e} \\
&= \lim_{u\to 0} \boldsymbol{\alpha}u[u(\mathbf{R}_d + \mu\boldsymbol{\lambda}) - \mathbf{Q}]^{-1}\boldsymbol{e} \\
&= \lim_{u\to 0} \boldsymbol{\alpha}u[u\mathbf{I} - (\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\mathbf{Q}]^{-1}(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\boldsymbol{e}.
\end{aligned}$$

Since $(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}$ is diagonal, $\widehat{\mathbf{Q}} \equiv (\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\mathbf{Q}$ is a generator matrix for a CTMC $\{\widehat{Z}_t : t \geq 0\}$ with transition probability functions $\widehat{\boldsymbol{\Pi}}(t)$ and limiting distribution,

$$\hat{\boldsymbol{\pi}} = \frac{\boldsymbol{\pi}(\mathbf{R}_d + \mu\boldsymbol{\lambda})}{\boldsymbol{\pi}\boldsymbol{\Delta}} \tag{26}$$

since $(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\mathbf{Q}$ is positive recurrent. The transition probability functions have LST

$$\Psi(u) = \int_0^\infty e^{-ut}\mathrm{d}\widehat{\boldsymbol{\Pi}}(t) = u(u\mathbf{I} - \widehat{\mathbf{Q}})^{-1}. \tag{27}$$

Applying the asymptotic properties of the LST gives,

$$\lim_{u\to 0} \Psi(u) = \lim_{t\to\infty} \widehat{\boldsymbol{\Pi}}(t) \equiv \widehat{\boldsymbol{\Pi}}(\infty).$$

Therefore, we may write

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbb{E}(T_x)}{x} &= \lim_{u\to 0} \boldsymbol{\alpha}\Psi(u)(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\boldsymbol{e} \\
&= \boldsymbol{\alpha}\widehat{\boldsymbol{\Pi}}(\infty)(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\boldsymbol{e} \\
&= \frac{1}{\boldsymbol{\pi}\boldsymbol{\Delta}}.
\end{aligned}$$

■

Before generalizing Proposition 1 to the nth moment $(n \geq 2)$, we need the following lemma.

Lemma 1 *Let $f(\cdot)$ be a function of exponential order on the positive real line such that $f^{(m)}(t) \to \infty$ as $t \to \infty$ for $m = 0, 1, \ldots, n-1$, and let $\widetilde{f}(s)$ denote its Laplace-Stieltjes transform. Then*

$$\lim_{s\to 0} s^n \widetilde{f}(s) = n! \lim_{t\to\infty} \frac{f(t)}{t^n}.$$

Proof. It is well known that

$$\mathcal{L}\left(\frac{\mathrm{d}^{n+1}f}{\mathrm{d}t^{n+1}}\right)(s) = s^{n+1}f^*(s) - \sum_{k=1}^{n+1} s^{k-1}f^{(n-k+1)}(0),$$

where $f^*(s)$ denotes the Laplace transform of f. Letting $s \to 0$ on both sides of the above expression, we have

$$\begin{aligned}
\lim_{s\to 0} s^{n+1}f^*(s) - f^{(n)}(0) &= \lim_{s\to 0} \mathcal{L}\left(\frac{\mathrm{d}^{n+1}f}{\mathrm{d}t^{n+1}}\right)(s) \\
&= \lim_{s\to 0} \int_0^\infty e^{-st} f^{(n+1)}(t)\mathrm{d}t \\
&= \lim_{a\to\infty} \int_0^a f^{(n+1)}(t)\mathrm{d}t \\
&= \lim_{a\to\infty} f^{(n)}(a) - f^{(n)}(0).
\end{aligned}$$

That is,

$$\lim_{s\to 0} s^n \widetilde{f}(s) = \lim_{s\to 0} s^{n+1} f^*(s) = \lim_{t\to\infty} f^{(n)}(t) = n! \lim_{t\to\infty} \frac{f(t)}{t^n},$$

where the last equality is obtained by n applications of l'Hospital's rule. ∎

Proposition 2 *For each $n \geq 2$, as $x \to \infty$,*

$$\frac{\mathbb{E}(T_x^n)}{x^n} \to \frac{1}{(\boldsymbol{\pi}\boldsymbol{\Delta})^n}. \tag{28}$$

Proof. Lemma 1 gives us

$$\lim_{u\to 0} u^n \widetilde{\mathbb{E}}(T_u^n) = n! \lim_{x\to\infty} \frac{\mathbb{E}(T_x^n)}{x^n}.$$

That is,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbb{E}(T_x^n)}{x^n} &= \frac{1}{n!} \lim_{u\to 0} u^n \widetilde{\mathbb{E}}(T_u^n) \\
&= \lim_{u\to 0} \boldsymbol{\alpha} \left[u(u\mathbf{I} - (\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\mathbf{Q}(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1})\right]^n \boldsymbol{e} \\
&= \boldsymbol{\alpha} \left[\widehat{\mathbf{\Pi}}(\infty)(\mathbf{R}_d + \mu\boldsymbol{\lambda})^{-1}\right]^n \boldsymbol{e} \\
&= \frac{1}{(\boldsymbol{\pi}\boldsymbol{\Delta})^n}.
\end{aligned}$$

∎

Proposition 2 provides some insight as to the behavior of $x^{-1}T_x$ as the degradation threshold x gets large. In particular, we see that as $x \to \infty$,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{T_x}{x} - \frac{1}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right)^n\right] = \mathbb{E}\left[\sum_{k=0}^{n} \binom{n}{k} \left(\frac{T_x}{x}\right)^{n-k} \left(\frac{-1}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right)^k\right] &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbb{E}\left[\left(\frac{T_x}{x}\right)^{n-k}\right] \left(\frac{1}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right)^k \\
&\to \left(\frac{1}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.
\end{aligned}$$

However, a stronger result can be proved. Note that the environment process $\{Z_t : t \geq 0\}$ is regenerative with regeneration epochs $S_0 = 0$ and

$$S_{k+1} = \min\{t > S_k : Z_t = Z_0,\ Z_{t^-} \neq Z_0\}, \qquad k \geq 0.$$

Let $\bar{r} \equiv \boldsymbol{\pi}\mathbf{R}_d\boldsymbol{e}$ denote the long-run average wear rate, let $\kappa \equiv 1/\mathbb{E}(S_1)$ be the average cycle frequency and $\bar{\lambda} \equiv \boldsymbol{\pi}\boldsymbol{\lambda}\boldsymbol{e}$ is the long-run average shock arrival rate. Using these quantities, we now prove that $x^{-1}T_x$ converges almost surely to a fixed value.

Theorem 3 *As $x \to \infty$,*

$$\frac{T_x}{x} \to \frac{1}{\boldsymbol{\pi}\boldsymbol{\Delta}} \quad \text{a.s.} \tag{29}$$

Proof. Let $U_0 \equiv W_{S_0} = 0$ and $U_{k+1} \equiv W_{S_{k+1}} - W_{S_k}$ for $k \geq 0$. We note that $\{U_k\}_{k=1}^{\infty}$ is an i.i.d. sequence of random variables. Thus, by the strong law of large numbers (SLLN) and the renewal reward theorem,

$$\frac{W_{S_n}}{S_n} = \frac{n}{S_n} \cdot \frac{1}{n} \sum_{k=1}^{n} U_k \to \kappa \mathbb{E}(W_{S_1}) = \bar{r} \quad \text{a.s.}$$

We see also by the SLLN that

$$\frac{\beta_t}{t} = \frac{N_t}{t} \cdot \frac{1}{N_t} \sum_{k=1}^{N_t} Y_k \to \bar{\lambda}\mu \quad \text{a.s.}$$

Combining the shock and wear components, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} \frac{W_t + \beta_t}{t} &= \lim_{n\to\infty} \frac{W_{S_n}}{S_n} + \lim_{t\to\infty} \frac{\beta_t}{t} \\ &= \bar{r} + \bar{\lambda}\mu \quad \text{a.s.} \\ &= \boldsymbol{\pi\Delta} \quad \text{a.s.} \end{aligned}$$

Now note that $X_{T_x} = x < X_{T_x+1}$, and so

$$\liminf_{x\to\infty} \frac{x}{T_x} = \lim_{x\to\infty} \frac{X_{T_x}}{T_x} = \boldsymbol{\pi\Delta} = \lim_{x\to\infty} \frac{X_{T_x+1}}{T_x} \geq \limsup_{x\to\infty} \frac{x}{T_x} \quad \text{a.s.}$$

Therefore,

$$\lim_{x\to\infty} \frac{x}{T_x} = \boldsymbol{\pi\Delta} \quad \text{a.s.},$$

which implies our result. ∎

Next our aim is to characterize centered and scaled versions of X_t and T_x as $t \to \infty$ and $x \to \infty$, respectively. To this end, define the random variable

$$\psi_1(\bar{r}) = \int_0^{S_1} [r(z_u) - \bar{r}] \, \mathrm{d}u = W_{S_1} - \bar{r} S_1,$$

and let $\sigma_W^2 \equiv \mathrm{Var}(\psi_1(\bar{r}))$ denote the cyclic wear variance. In the results that follow, $N(0,1)$ denotes a standard normal random variable (i.e., one with mean 0 and variance 1), and $\Rightarrow$ means convergence in distribution. Before stating our main results, it will be helpful to review an important result due to Glynn and Whitt [11] which applies to both the shock and wear processes.

Theorem 4 *(Glynn and Whitt [11]). There exist constants a and b such that*

$$t^{-1/2}(W_t - at) \Rightarrow bN(0,1) \quad \textit{as } t \to \infty$$

if and only if $\mathbb{E}(\psi_1(a)) = 0$ *and* $\mathrm{Var}(\psi_1(a)) = \mathbb{E}(S_1) b^2$.

This result establishes necessary and sufficient moment conditions for the shock and wear processes to converge to a standard normal variable when properly scaled.

Proposition 3 *As $t \to \infty$,*

$$t^{-1/2}(X_t - \boldsymbol{\pi}\boldsymbol{\Delta}t) \Rightarrow \sigma N(0,1),$$

where $\sigma = \sqrt{\kappa\sigma_W^2 + \bar{\lambda}\sigma_Y^2}$.

Proof. To prove this result, we can consider the damage due to shocks and the wear degradation separately. For the shock damage, note that $t^{-1}N_t \to \bar{\lambda}$ a.s., so we may conclude by Theorem 2.1 of [11] that

$$t^{-1/2}(\beta_t - \mu\bar{\lambda}t) \approx t^{-1/2}\sum_{n=0}^{N_t}(Y_n - \mu) \Rightarrow N(0,\ \bar{\lambda}\sigma_Y^2) \tag{30}$$

as $t \to \infty$. The environment process $\mathcal{Z}$ is regenerative with the first regeneration epoch having duration S_1. Therefore, the renewal reward theorem shows that

$$\mathbb{E}(\psi_1(\bar{r})) = \mathbb{E}\left(\int_0^{S_1}[r(Z_u) - \bar{r}]\mathrm{d}u\right) = 0.$$

Furthermore, we have that

$$\mathrm{Var}(\psi_1(\bar{r})) = \kappa\sigma_W^2\mathbb{E}(S_1).$$

Because $\psi(\bar{r})$ satisfies the moment conditions outlined in [11], we conclude that

$$t^{-1/2}(W_t - \bar{r}t) \Rightarrow N(0,\ \kappa\sigma_W^2) \tag{31}$$

as $t \to \infty$. The result is finally obtained by combining equations (30) and (31) and noting that $\boldsymbol{\pi}\boldsymbol{\Delta} = \bar{r} + \mu\bar{\lambda}$. ∎

Proposition 3 provides the means by which to analyze a centered and space-scaled version of the lifetime distribution in the asymptotic regime.

Theorem 5 *As $x \to \infty$,*

$$\left(\frac{x}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right)^{-1/2}\left(T_x - \frac{x}{\boldsymbol{\pi}\boldsymbol{\Delta}}\right) \Rightarrow \frac{\sigma}{\boldsymbol{\pi}\boldsymbol{\Delta}}N\left(0,1\right).$$

where $\sigma = \sqrt{\kappa\sigma_W^2 + \bar{\lambda}\sigma_Y^2}$.

Proof. The equivalence of events $\{T_x > t\}$ and $\{X_t \le x\}$ shows that

$$\mathbb{P}(T_x > t) = \mathbb{P}(X_t \le x) = \mathbb{P}\left(\frac{X_t - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}} \le \frac{x - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}}\right).$$

Let $t, x \to \infty$ in a nontrivial way, i.e., such that

$$\frac{x - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}} \to c,$$

where c is a real constant. By Theorem 3, this equivalently means that

$$\frac{t - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}} \to -c,$$

as $t, x \to \infty$. Since

$$\mathbb{P}(T_x > t) = \mathbb{P}\left(\frac{T_x - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}} > \frac{t - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}}\right),$$

we have

$$\lim_{t,x\to\infty} \mathbb{P}\left(\frac{T_x - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}} > \frac{t - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}}\right) = \lim_{t,x\to\infty} \mathbb{P}\left(\frac{X_t - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}} \le \frac{x - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}}\right).$$

Denote by $\Phi(\cdot)$ the standard normal distribution function. Then we may write,

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{T_x - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}} > -c\right) = \lim_{t\to\infty} \mathbb{P}\left(\frac{X_t - (\boldsymbol{\pi\Delta})t}{\sigma\sqrt{t}} \le c\right) = \Phi(c),$$

by Proposition 3. Thus,

$$\lim_{x\to\infty} \mathbb{P}\left(\frac{T_x - x/(\boldsymbol{\pi\Delta})}{\sqrt{\sigma^2 x/(\boldsymbol{\pi\Delta})^3}} \le c\right) = 1 - \Phi(-c) = \Phi(c),$$

which implies our result. ∎

Theorem 5 provides a simple normal approximation to the lifetime distribution that may be used as a surrogate for equations (19) and (20). This approximation, along with the asymptotic moment results, can be used in a variety of ways. For instance, the standard reliability indices (e.g., mean time-to-failure or the reliability function) can be compared for identical, single-unit systems that operate in different environments. Moreover, the asymptotic normality of the lifetime distribution can be used in the context of sequential decision making, particularly for adaptive maintenance planning. By inspecting and updating the unit's degradation level periodically, one can prescribe dynamic optimal maintenance actions that evolve temporally with the unit's remaining useful lifetime distribution. The remaining lifetime distribution can be easily computed (via look-up tables) using the asymptotic normal distribution, whereas it is very cumbersome to repeatedly compute its transient counterpart. The next section illustrates the quality of the asymptotic approximations through two numerical examples.

5 Numerical Illustrations

This section provides two numerical examples that illustrate the usefulness of the simple asymptotic approximations of section 4. To this end, let $\gamma \equiv 1/\boldsymbol{\pi\Delta}$, and define the random variable

$$\Gamma_x \equiv (\gamma x)^{-1/2} (T_x - \gamma x).$$

Theorem 4 asserts that $\Gamma_x \Rightarrow \gamma\sigma N(0,1)$ as $x \to \infty$. Define the (conditional) c.d.f. of Γ_x, given that the initial state of the environment is $i \in S$, as

$$\begin{aligned} F_i(x,t;\gamma) &\equiv \mathbb{P}(\Gamma_x \leq t | Z_0 = i) \\ &= \mathbb{P}((\gamma x)^{-1/2} (T_x - \gamma x) \leq t | Z_0 = i) \\ &= G_i(x, t\sqrt{\gamma x} + \gamma x). \end{aligned} \tag{32}$$

Next denote by $\hat{\Phi}(t;\gamma,\sigma)$ the c.d.f. of a normal random variable with mean zero and variance $(\gamma\sigma)^2$. Furthermore, let $\mathcal{T} \subseteq \mathbb{R}_+ \equiv [0,\infty)$, and denote the maximum absolute deviation between $F_i(x,t;\gamma)$ and $\hat{\Phi}(t;\gamma,\sigma)$ by

$$\sup_{t\in\mathcal{T}} |F_i(x,t;\gamma) - \hat{\Phi}(t;\gamma,\sigma)|, \qquad x > 0.$$

By Theorem 5, we can assert that $\sup_{t\in\mathcal{T}} |F_i(x,t;\gamma) - \hat{\Phi}(t;\gamma,\sigma)| \to 0$ as $x \to \infty$. However, the analytical rate of convergence is unavailable because the distribution function $F_i(x,t;\gamma)$ can only be obtained numerically via the inverse Laplace transform,

$$F_i(x,t;\gamma) = \mathcal{L}^{-1}\left(u^{-1}\widetilde{G}_i(u, t\sqrt{\gamma x} + \gamma x)\right) \tag{33}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator. This inverse transform will serve as the baseline distribution against which we will compare the asymptotic normal approximation $\hat{\Phi}(t;\gamma,\sigma)$ on the fixed set $\mathcal{T}$ for several (increasing) degradation thresholds.

Additionally, we would like to assess the quality of the asymptotic moment approximations by comparing them to the transient moments given by

$$\mathbb{E}(T_x^n) = \mathcal{L}^{-1}\left(u^{-1}\mathbb{E}(T_u^n)\right), \qquad n \geq 1. \tag{34}$$

The asymptotic approximation of the nth moment of T_x is given by

$$\mathbb{E}(T_x^n) \approx \frac{x^n}{(\boldsymbol{\pi\Delta})^n}, \qquad n \geq 1. \tag{35}$$

The transient c.d.f. values of (33), and the transient moments of (34), were obtained by coding the inversion algorithms of Abate and Whitt [1] in the MATLAB computing environment.

5.1 An Alternating Environment

The first example is a simple environment which alternates between two distinct states ($S = \{1, 2\}$). Thus, the environment may be viewed as an alternating renewal process (or up-down machine) wherein the up and down times are mutually independent. The (positive recurrent) generator matrix of the process is

$$\mathbf{Q} \;=\; \begin{bmatrix} -8.0 & 8.0 \\ 6.0 & -6.0 \end{bmatrix},$$

and its stationary distribution is

$$\boldsymbol{\pi} = [\ 0.4286 \quad 0.5714\ ].$$

The diagonal matrix of wear rates is $\mathbf{R}_d = \operatorname{diag}(1.0, 0.25)$, the diagonal matrix of Poisson shock arrival rates is $\boldsymbol{\lambda} = \operatorname{diag}(0.25, 2.0)$, and the vector Δ is given by

$$\Delta = \begin{bmatrix} 1.0625 \\ 0.7500 \end{bmatrix}.$$

The Laplace-Stieltjes transform of the shock-magnitude distribution function is

$$\widetilde{F}_Y(u) = \frac{4.0}{4.0 + u}, \qquad \operatorname{Re}(u) > 0,$$

i.e., the damage induced by each shock is an exponential random variable with rate parameter 4.0. Using these input parameters, and assuming the environment is initially in state 1 with probability 1.0, we can easily compute the following:

$$\gamma = \frac{1}{\boldsymbol{\pi\Delta}} \approx 1.13131; \quad \bar{\lambda} = \boldsymbol{\pi\lambda e} \approx 1.25000;$$

$$\sigma_W^2 \approx 0.02778; \quad \sigma_Y^2 = 0.06250;$$

$$\kappa = 1/\mathbb{E}(S_1) \approx 3.4286; \quad \sigma = \sqrt{\kappa\sigma_W^2 + \bar{\lambda}\sigma_Y^2} \approx 0.41637.$$

First we compare the lower (first and second) moments of T_x using equations (34) and (35). The results are summarized in Table 1.

Table 1: $\mathbb{E}(T_x^n|Z_0=1)$, $n=1,2$, when $\mathcal{Z}$ has two states.

	$\mathbb{E}(T_x\|Z_0=1)$		$\mathbb{E}(T_x^2\|Z_0=1)$	
Threshold (x)	Transient	Asymptotic	Transient	Asymptotic
0.25	3.41992E-01	2.82828E-01	1.35629E-01	7.99918E-02
0.50	6.45565E-01	5.65657E-01	4.74599E-01	3.19967E-01
1.00	1.21845E+00	1.13131E+00	1.64830E+00	1.27987E+00
5.00	5.74431E+00	5.65657E+00	3.40842E+01	3.19967E+01
10.00	1.14009E+01	1.13131E+01	1.32223E+02	1.27987E+02
20.00	2.27140E+01	2.26263E+01	5.20481E+02	5.11948E+02
50.00	5.66534E+01	5.65657E+01	3.22110E+03	3.19967E+03
100.00	1.13219E+02	1.13131E+02	1.28416E+04	1.27987E+04
200.00	2.26350E+02	2.26263E+02	5.12807E+04	5.11948E+04
500.00	5.65744E+02	5.65657E+02	3.20182E+05	3.19967E+05

As seen in Table 1, the asymptotic approximations track closely with their transient counterparts. The discrepancy between the values is diminishing as x increases. In general, the asymptotic approximation underestimates the time to reach level x and can, therefore, be viewed as a conservative approximation.

Next we compared the asymptotic normal lifetime distribution $\hat{\Phi}(t;\gamma,\sigma)$ with the numerically inverted lifetime c.d.f. using (33). Figure 1 displays three different transient distributions and the asymptotic approximation. For this illustration, we note that the asymptotic approximation performs well, even when x is small.

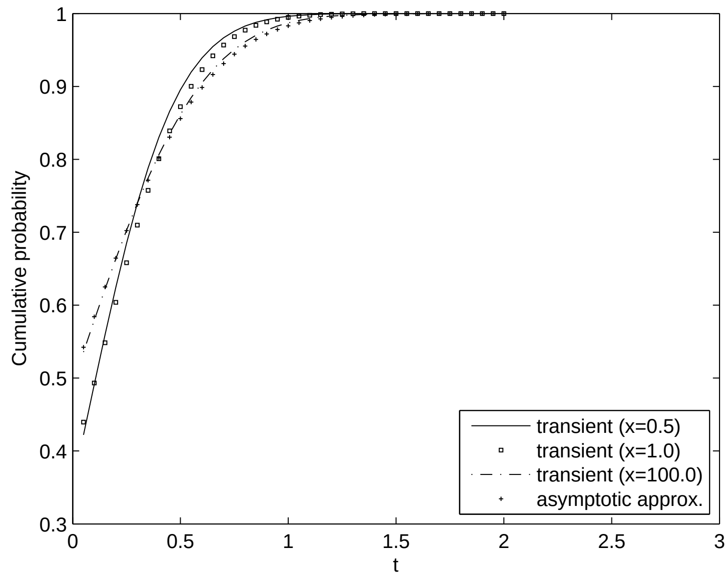

Figure 1: Transient versus asymptotic lifetime c.d.f.s.

To further illustrate the quality of the c.d.f. approximations, we compared the transient and asymptotic distributions using the maximum absolute deviation in probability. This error measure is plotted against values of x ranging from 1.0 to 100.0 in increments of 5.0 units in Figure 2.

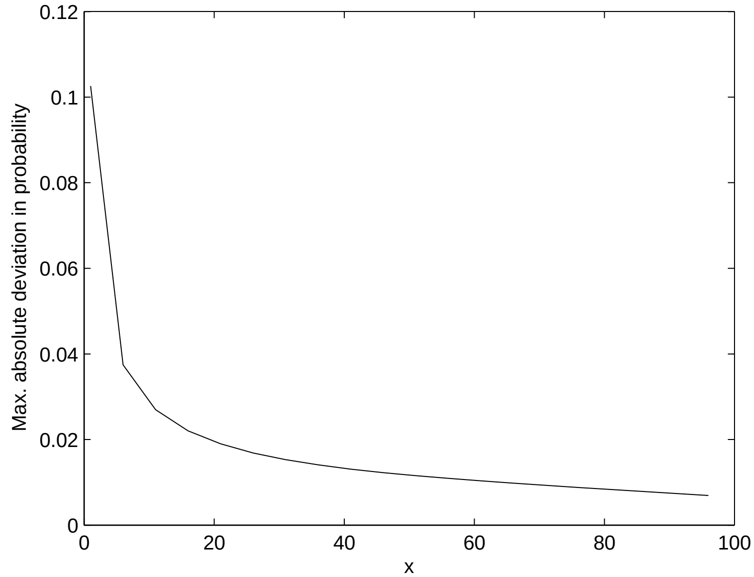

Figure 2: Maximum absolute deviation in probability as $x \to \infty$.

Figure 2 indicates a possibly exponential rate of convergence to the limiting distribution; however, this cannot be verified analytically. For this example, the deviation is very small, even with $x < 10.0$. Next we study a more diverse environment with ten distinct states.

5.2 General CTMC Environment

The second example considers an environment that transitions between ten states, i.e., the environment $\{Z_t : t \geq 0\}$ has state space $S = \{1, 2, \ldots, 10\}$. The 10×10 infinitesimal generator matrix is given by

$$\mathbf{Q} = \begin{bmatrix} -3.86 & 0.06 & 0.49 & 0.51 & 0.07 & 0.49 & 0.41 & 0.93 & 0.37 & 0.53 \\ 0.39 & -3.37 & 0.04 & 0.45 & 0.38 & 0.50 & 0.56 & 0.26 & 0.25 & 0.55 \\ 0.25 & 0.46 & -3.93 & 0.33 & 0.37 & 0.84 & 0.27 & 0.20 & 0.92 & 0.28 \\ 0.35 & 0.86 & 0.33 & -4.67 & 0.48 & 0.81 & 0.78 & 0.05 & 0.63 & 0.37 \\ 0.74 & 0.86 & 0.90 & 0.89 & -6.18 & 0.86 & 0.39 & 0.61 & 0.88 & 0.06 \\ 0.65 & 0.47 & 0.31 & 0.76 & 0.34 & -4.30 & 0.03 & 0.55 & 0.64 & 0.54 \\ 0.94 & 0.79 & 0.25 & 0.88 & 0.25 & 0.57 & -5.41 & 0.10 & 0.80 & 0.84 \\ 0.83 & 0.66 & 0.43 & 0.46 & 0.58 & 0.61 & 0.56 & -4.71 & 0.44 & 0.15 \\ 0.47 & 0.00 & 0.84 & 0.80 & 0.52 & 0.10 & 0.20 & 0.44 & -3.55 & 0.17 \\ 0.63 & 0.13 & 0.18 & 0.13 & 0.16 & 0.16 & 0.09 & 0.07 & 0.10 & -1.65 \end{bmatrix}$$

The stationary distribution of this chain is

$$\boldsymbol{\pi} = [\ 0.1257 \quad 0.1004 \quad 0.0894 \quad 0.0984 \quad 0.0497 \quad 0.0992 \quad 0.0574 \quad 0.0683 \quad 0.1179 \quad 0.1937\ ].$$

The Markov-modulated wear rates and shock arrival rates are summarized in Table 2.

Table 2: Summary of shock and wear rates for example 2.

State (i)	Wear Rate ($r(i)$)	Shock Arrival Rate (λ_i)
1	1.0000	0.2359
2	2.0000	0.7117
3	3.0000	0.5264
4	4.0000	0.4782
5	5.0000	0.5673
6	6.0000	0.5698
7	7.0000	0.9437
8	8.0000	0.6053
9	9.0000	0.9887
10	10.0000	0.0794

We assume that the damage induced by each shock follows an Erlang distribution with parameters 2 and 8.0. That is, $\mathbb{E}(Y_1) = 2/8.0 = 0.25$. The Laplace-Stieltjes transform of the shock-magnitude c.d.f. is therefore

$$\widetilde{F}_Y(u) = \left(\frac{8.0}{8.0+u}\right)^2, \qquad \mathrm{Re}(u) > 0.$$

The remaining parameter values are as follows:

$$\gamma = \frac{1}{\boldsymbol{\pi\Delta}} \approx 0.16937; \quad \bar{\lambda} = \boldsymbol{\pi\lambda e} \approx 0.50731;$$

$$\sigma_W^2 = 13.30623; \quad \sigma_Y^2 = 0.03125;$$

$$\kappa = 1/\mathbb{E}(S_1) \approx 0.48523; \quad \sigma = \sqrt{\kappa\sigma_W^2 + \bar{\lambda}\sigma_Y^2} \approx 2.5411.$$

As before, we first assess the quality of the asymptotic moment approximations. The results are summarized in Table 3, and they consistently show that the asymptotic approximation improves as the degradation threshold x increases. As in the first example, the asymptotic approximation tends to underestimate the transient moments.

Table 3: $\mathbb{E}(T_x^n|Z_0 = 1)$, $n = 1, 2$, when $\mathcal{Z}$ has ten states.

	$\mathbb{E}(T_x\|Z_0 = 1)$		$\mathbb{E}(T_x^2\|Z_0 = 1)$	
Threshold (x)	Transient	Asymptotic	Transient	Asymptotic
10.00	1.97116E+00	1.69365E+00	4.15994E+00	2.86846E+00
25.00	4.50836E+00	4.23413E+00	2.10670E+01	1.79279E+01
40.00	7.03183E+00	6.77461E+00	5.07335E+01	4.58954E+01
55.00	9.54684E+00	9.31509E+00	9.30839E+01	8.67709E+01
70.00	1.20536E+01	1.18556E+01	1.48036E+02	1.40555E+02
85.00	1.45522E+01	1.43961E+01	2.15506E+02	2.07246E+02
100.00	1.70425E+01	1.69365E+01	2.95412E+02	2.86846E+02
115.00	1.95247E+01	1.94770E+01	3.87671E+02	3.79354E+02
130.00	2.19988E+01	2.20175E+01	4.92203E+02	4.84770E+02
145.00	2.44647E+01	2.45580E+01	6.08924E+02	6.03094E+02
160.00	2.69226E+01	2.70984E+01	7.37755E+02	7.34326E+02

Next we consider the lifetime distribution approximations. In Figure 3 we plot three distinct transient distributions along with the asymptotic normal approximation. The discrepancy between the c.d.f.s is significant when x is small (namely when $x = 1.0$), but improves dramatically as x increases. When $x = 100.0$, the asymptotic approximation is very accurate as expected.

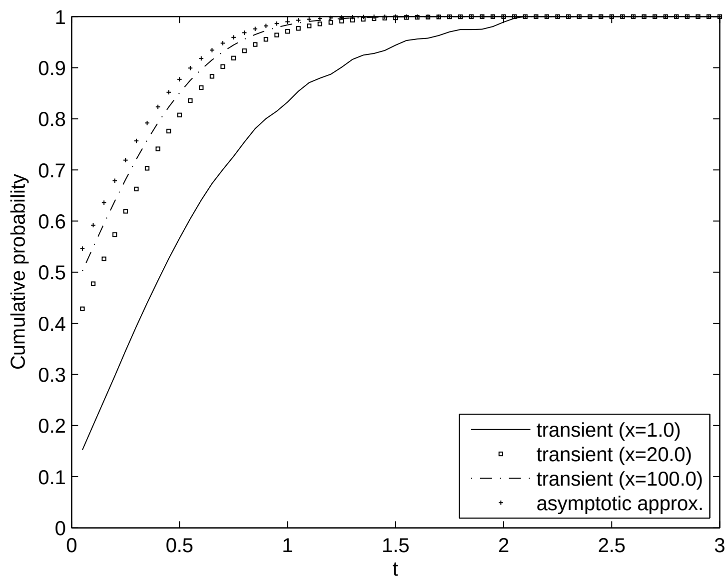

Figure 3: Transient versus asymptotic lifetime c.d.f.s.

As for the two-state example, we next plotted the maximum absolute deviation in probability as a function of the degradation threshold, x. Consistent with the two-state case, Figure 4 shows that the worst-case deviation is monotonically decreasing (to zero) as $x \to \infty$.

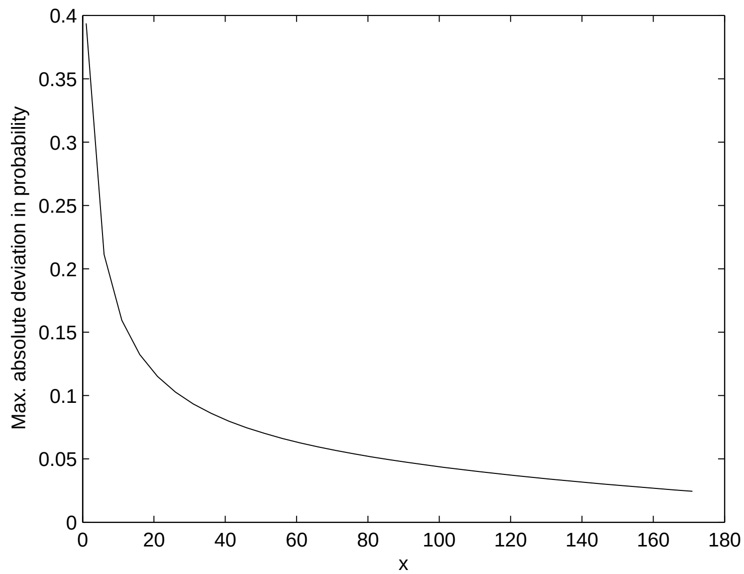

Figure 4: Transient versus asymptotic lifetime c.d.f.s.

The two numerical examples illustrate the usefulness of the asymptotic approximations, namely that they allow one to circumvent the task of numerically inverting Laplace transforms. The next section summarizes our main contributions and provides some concluding remarks.

6 Conclusions

In this paper we have presented both transient and asymptotic reliability indices for a single-unit system whose degradation process may be modeled as a Markov-modulated shock and wear process. This analytical framework provides a great deal of modeling flexibility, particularly in the context of degradation-based reliability modeling, by allowing for a wide range of wear dynamics and the inclusion of damage-inducing shocks that are dictated by the device's random operating environment. These characteristics make it appealing for modeling real systems when degradation can be directly observed, or when a direct mapping between the environmental conditions and the degradation can be accurately modeled (e.g., by employing specific physics-of-failure models).

While the models are mathematically sound and easy to implement, in their current form they lack the flexibility to account for environment state sojourn times, or shock inter-arrival times, that are not exponentially distributed. In this sense, the models are somewhat restrictive. Moreover, the models assume that future wear and damage are independent of the history of the degradation process – an assumption that may be very difficult to justify in practice. Nevertheless, the main results provide a framework for simple, asymptotically accurate approximations that can serve as a starting point for applications requiring real-time updating of remaining lifetime distributions (as in a condition-based maintenance environment). Specifically, real-time observations of the evolution of the environment, or the degradation level, can be used to update the parameters of the modulating process, the state-dependent wear rates, and the shock arrival rates, using the techniques described in Kharoufeh and Cox [14]. Tracking the occurrence of environment transitions, and the number of shocks occurring over a (sufficiently long) time interval will provide reasonable statistical estimates, so long as the number of environment states is not too large. The existence of a minimal representation of the modulating Markov chain that leads to accurate distribution estimates was posited in [14]. Such a representation indicates that the results of this work can be implemented in a realistic setting.

In the future it will be instructive to investigate similar approximation schemes that allow for a generalized environment process that does not evolve as a continuous-time Markov chain. If the models are to be of any practical value, it will be necessary to accurately estimate the functions r and λ describing the evolution of wear and damage as a function of the environment. For this purpose, real degradation data is required, as is the guidance and experience of subject matter experts, to ensure the wear and damage models are appropriate for the application. An examination of multi-unit systems is also warranted. However, for a multi-unit system in a random environment, it may not be possible to treat each unit independently because, although the unit lifetimes may be statistically independent, they share a common dependence on the operating environment; therefore, under certain conditions, the unit lifetimes are associated to one another as shown by Çinlar et al. [4]. Nevertheless, our approach allows for a multi-unit system to be viewed as a single-unit system if the aggregate influence of the environment can be discerned. Finally, an important issue that we did not consider in this paper is the rate of convergence of the lifetime moments and distributions to their respective limits. Our empirical results indicate that the rate of convergence may be exponential, though this conjecture cannot be confirmed analytically using the transform results. It will be instructive to derive error bounds for our approximations in future work.

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