Explicit results for wear processes in a Markovian environment[☆]

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Abstract

We consider the reliability of a single-unit system whose cumulative damage over time is a continuous wear process $\{X(t): t \geq 0\}$ that depends on an external environment process $\{Z(t): t \geq 0\}$. We explicitly derive the failure time distribution and moments in terms of Laplace–Stieltjes transforms by analyzing the Markov additive process $\{(X(t), Z(t)): t \geq 0\}$ and demonstrate its applicability on an example problem.
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1. Introduction

In this paper, we investigate a single-unit system that accumulates wear over time due to the influence of a random environment that may be modeled as a continuous time stochastic process $\{Z(t): t \geq 0\}$. In particular, we assume that the wear rate of the system depends explicitly on the state of its random environment. Assuming that repairs do not take place to restore the condition of the system, the cumulative wear up to time t may be characterized by a nondecreasing stochastic process $\{X(t): t \geq 0\}$. We discuss the properties of this process in Section 2.

The system begins its lifetime in perfect working order but experiences wear under the influence of its random environment until the state of the system exceeds a fixed threshold value x, at which time it fails. We denote the time until failure by a random variable T_x and derive explicit transform expressions for the cumulative distribution function and the nth moment ($n \geq 1$) of this random time when the environment process is assumed to be a temporally homogeneous, finite-state Markov process. Our main results are obtained by demonstrating that the joint probability distribution of the Markov additive process (X, Z), conditioned upon the initial state of the environment, satisfies a first-order linear partial differential equation (PDE) which is directly solved via transform methods. Next, using the derived distribution of the failure time, we characterize the Laplace–Stieltjes transform of the failure time moments. The results of this paper are well-suited for practical implementation with existing single- and multi-dimensional Laplace transform inversion algorithms.

The modeling of stochastically deteriorating, single-unit systems has been studied extensively in

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the literature by a number of researchers. The primary approach has been that of stochastic shock and wear models. Shock models can generally be described as follows. A system is subject to shocks of random magnitude at random time intervals. The cumulative wear to the system is characterized by the sum of discrete shock magnitudes accumulated up to some time t. These models have been primarily used for prescribing preventative maintenance procedures, e.g. optimal replacement policies. For an extensive summary of papers addressing this issue, the reader is referred to the excellent survey by Valdez-Flores and Feldman [16]. Gottlieb [6] gave sufficient conditions for the shock process in order for the system failure time distribution to have an increasing failure rate (IFR). Waldmann [17] considered a shock model related to an environment process that describes varying external factors and found an optimal replacement policy under this assumption. More general Markov renewal shock models have been analyzed apart from the optimal replacement problem. For example, the probability distribution of failure time for two distinct Markov renewal models were considered by Shanthikumar and Sumita [13] and later extended by Igaki et al. [7]. Some examples of shock models influencing multi-unit systems include the papers by Ráde [11], Nakagawa [10], and more recently, Stadje and Zuckerman [15].

The seminar paper by Esary et al. [5] considered processes subject to continuous wear, relaxing the shock model assumption that deterioration occurs only at discrete time points and in discrete magnitudes. When the continuous wear process X is assumed to be a nondecreasing Markov process whose resistance to wear decreases with time and wear, it was shown that T_x has an increasing hazard rate average (IHRA) distribution. Çinlar [4] generalized most of the models of [5] by demonstrating that the process (X, Z) is a Markov additive process and gave several examples of such. The first example relevant to this work (cf. [4], p. 199), considered the case when Z is a general Markov process with a finite-state space and X was assumed to increase as a Lévy process. Additionally, random shocks were assumed to occur at each environment transition. In such case, the X process, conditioned on Z follows the same probability law as an increasing Lévy process. The second example (cf. [4], pp. 201–202) is similar to the problem discussed

herein where X is a continuous additive functional of Z and the failure time is a first passage time for the X process. More recently, Abdel-Hameed [3] examined properties of the time to failure when the wear process X was assumed to be an increasing Lévy process. Singpurwalla [14] gave an excellent summary of a variety of stochastic failure time models for systems in a random environment and particularly noted the difficulty of implementing many of these in a practical setting. Liminos and Oprisan [8] provided mathematically rigorous results for several reliability measures for semi-Markov systems; however, the means by which to implement their results is not immediately apparent.

The main contributions of this work are explicit results for the probability distribution and all moments of the random failure time for a general, single-unit system that deteriorates continuously and additively due to the influence of a random environment that is modeled as a general, finite-state Markov process. The model focuses on the continuous wear caused by this random environment but ignores any additional shocks. The main results of the paper are compact, Laplace–Stieltjes transform expressions for the failure time distribution and all moments that may be implemented using existing numerical inversion algorithms. The distribution and moments of the random failure time are extremely important in reliability analysis, particularly for the determination of sound maintenance policies in a variety of commercial, governmental, and military settings.

The remainder of the paper is organized as follows. The next section gives the formal model description. In Section 3, we explicitly derive transform results for the failure time distribution for a single-unit system subject to a Markovian random environment. In Section 4 we derive the analytical expressions for the moments of failure time. Finally, in Section 5, a numerical example is presented demonstrating the main results of the paper.

2. Formal model description

Consider a single-unit system subject to continuous and additive deterioration in time due to an explicit dependence on the state of an external random

environment. Under normal operating conditions, the system accumulates wear until the magnitude of its cumulative wear exceeds a fixed threshold value x, at which time the system fails. In particular, the rate of deterioration (or wear rate) of the system at time $t > 0$ is governed by a random environment that is modeled as a finite-state Markov process $\{Z(t): t \geqslant 0\}$. The evolution of the wear process can be described by a continuous-time stochastic process $\{X(t): t \geqslant 0\}$ that assumes values on the nonnegative real line which shall be denoted throughout by $\mathbf{R}^+$. Moreover, we assume that $\{Z(t): t \geqslant 0\}$ and $\{X(t): t \geqslant 0\}$ are both right continuous and have left hand limits everywhere. At times, for the sake of brevity, the processes shall be respectively abbreviated as Z and X with the understanding that both are defined on the parameter set $\mathbf{R}^+$.

The following definitions are needed to describe the mathematical model. Let $Z(t)$ denote the state of the random environment process at time $t \in \mathbf{R}^+$ and define its state space by the set $S \subseteq \mathbf{N}$ where $\mathbf{N}$ denotes the set of natural numbers. For this model, we specifically assume that Z has a finite-state space $S = \{1, \ldots, K\}$, $K \in \mathbf{N}$. Let $R(t)$ be defined as the wear rate of the system at time $t \in \mathbf{R}^+$ and define a nonnegative function $r: S \to \mathbf{R}^+$. The properties of the function $r(\cdot)$ will be dictated by the type of system under consideration and its surrounding environment. Since the wear rate of the system is explicitly dependent on the environment process, the wear rate process $\{R(t): t \geqslant 0\}$ assumes values in the space $\mathscr{D} = \{r(1), \ldots, r(K)\}$.

The evolution of the system can be described as follows. If $Z(t) = i \in S$, then (i) $R(t) := r(Z(t)) = r(i) \in \mathscr{D}$ and (ii) the environment transitions from state $i \in S$ to state $j \in S$ at time $t + \varepsilon$ according to a Markov transition function $\mathbf{P}(t) := [p_{i,j}(t)]$ where $p_{i,j}(t) := P\{Z(t + \varepsilon) = j | Z(t) = i\}$. The environment process Z is a temporally homogeneous, finite-state Markov process so that $p_{i,j}(t)$ does not depend on t for all $i, j \in S$. Denote by $X(t)$, the cumulative wear of the single-unit system up to time $t \in \mathbf{R}^+$. X is a continuous, additive functional of Z, and thus, (X, Z) constitutes a special case of a Markov additive process. In order to compute the probability distribution and moments of the random lifetime of the system, we analyze the bivariate process $\{(X(t), Z(t)): t \geqslant 0\}$.

With regard to the process $\{X(t): t \geqslant 0\}$, the following conditions will normally be satisfied in practice:

(i) For all $t, \varepsilon \geqslant 0$, $X(0) = 0$ and $X(t + \varepsilon) - X(t) \geqslant 0$, w.p. 1.
(ii) The process $\{X(t): t \geqslant 0\}$ satisfies the Markov property,

$$P\{X(t + \varepsilon) \leqslant x | X(\varepsilon) = y, X(u), 0 \leqslant u < y\}$$
$$= P\{X(t + \varepsilon) \leqslant x | X(\varepsilon) = y\}. \qquad (1)$$

In effect, Assumption (i) asserts that the system operates without intervention. In particular, no repairs take place that can restore the unit to a state of lesser deterioration. Assumption (ii) asserts that the future condition of the system can be predicted via the accumulated wear up to time t, irrespective of the history prior to t. If the single-unit system has a known minimum acceptable wear level x, it will fail when the cumulative wear exceeds this level. In this sense, the time until failure for the unit can be considered as a first passage time for the cumulative wear process $\{X(t): t \geqslant 0\}$. We obtain explicit transform expressions for the failure time distribution, as well as all moments of this failure time. In the following section, we first derive the failure time probability distribution.

3. Failure time distribution

The environment process $\{Z(t): t \geqslant 0\}$ is assumed to be a temporally homogeneous Markov process on a finite-state space $S := \{1, \ldots, K\}$. The cumulative wear of the single-unit system up to time $t \in \mathbf{R}^+$ is defined by

$$X(t) = \int_0^t r(Z(u)) \, \mathrm{d}u. \qquad (2)$$

The system fails as soon as the magnitude of its accumulated wear exceeds a fixed threshold value x. Let Ω denote the appropriate sample space for the Markov additive process $\{(X(t), Z(t)): t \geqslant 0\}$. For each sample path, $\omega \in \Omega$, the lifetime of the system is given by the random variable

$$T_x(\omega) = \inf\{t: X(t; \omega) > x\}. \qquad (3)$$

Define the following distribution

$$V_{i,j}(x, t) = P\{X(t) \leqslant x, Z(t) = j | Z(0) = i\}, \qquad (4)$$

where $V_{i,j}(x,t)$ is the joint probability that, at time t, the degradation of the system has not exceeded a value x and the environment process is in state $j \in S$ given that the environment was initially in state $i \in S$. The objective is to find the joint distribution of (X, Z), conditional on i, and apply the dual relationship of (3) to derive an explicit matrix equation for the failure time distribution. The distribution matrix of $X(t)$ is defined as

$$\mathbf{V}(x,t) = [V_{i,j}(x,t)]. \tag{5}$$

Define the matrix transforms

$$\mathbf{V}^*(x,s) = \int_{\mathbf{R}^+} \mathrm{e}^{-st}\mathbf{V}(x,t)\,\mathrm{d}t, \tag{6}$$

the Laplace transform of $\mathbf{V}(x,t)$ with respect to t and

$$\tilde{\mathbf{V}}^*(u,s) = \int_{\mathbf{R}^+} \mathrm{e}^{-ux}\mathbf{V}^*(\mathrm{d}x,s), \tag{7}$$

the Laplace–Stieltjes transform of $\mathbf{V}^*(x,s)$ with respect to x. Let $a_{i,j}(x) := V_{i,j}(x,0)$ and define $\mathbf{a}(x) = [a_{i,j}(x)]$. It is clear that

$$a_{i,j}(x) = I_{\{i=j\}}, \tag{8}$$

where $I_{\{\cdot\}}$ denotes the indicator function. Further define the Laplace–Stieltjes transform of $a_{i,j}(x)$ with respect to x by

$$\tilde{a}_{i,j}(u) = \int_{\mathbf{R}^+} \mathrm{e}^{-ux}V_{i,j}(\mathrm{d}u,0) \tag{9}$$

and its corresponding vector transform

$$\tilde{\mathbf{a}}(u) = \int_{\mathbf{R}^+} \mathrm{e}^{-ux}\mathbf{V}(\mathrm{d}u,0). \tag{10}$$

The double transform $\tilde{\mathbf{V}}^*(u,s)$ is next derived under the assumption that the environment process, $\{Z(t): t \geqslant 0\}$, is a finite-state Markov process.

Theorem 3.1. *If the Z process, defined on the finite-state space S, is a Markov process with infinitesimal generator matrix,* $\mathbf{Q} = [q_{ij}]$, $i,j \in S$, *then*

$$\tilde{\mathbf{V}}^*(u,s) = [u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q}]^{-1},$$

$$Re(u) > 0, \ Re(s) > 0. \tag{11}$$

where $\mathbf{R}_D = diag(r(1), \ldots, r(K))$.

Proof. Fix $\varepsilon > 0$.

$$V_{i,j}(x,t+\varepsilon)$$

$$= P\{X(t+\varepsilon) \leqslant x, Z(t+\varepsilon) = j | Z(0) = i\}$$

$$= \sum_k P\{Z(t+\varepsilon) = j | X(t+\varepsilon) \leqslant x,$$

$$Z(t) = k, Z(0) = i\}$$

$$\times P\{X(t+\varepsilon) \leqslant x | Z(t) = k, Z(0) = i\}$$

$$\times P\{Z(t) = k\}$$

$$= (1 + \varepsilon q_{jj})V_{i,j}(x - \varepsilon r(j), t)$$

$$+ \sum_{k \in S\setminus\{j\}} \varepsilon q_{kj}V_{i,k}(x - \varepsilon r(k), t) + \mathrm{o}(\varepsilon). \tag{12}$$

Simplifying, dividing by the time increment ε and letting $\varepsilon \downarrow 0$, it is seen that $V_{i,j}(x,t)$ satisfies the partial differential equation

$$\frac{\partial V_{i,j}(x,t)}{\partial t} + \frac{\partial V_{i,j}(x,t)}{\partial x}r(j)$$

$$= \sum_{k \in S} q_{kj}V_{i,k}(x,t), \quad j \in S \tag{13}$$

which may be written in matrix form as

$$\frac{\partial \mathbf{V}(x,t)}{\partial t} + \frac{\partial \mathbf{V}(x,t)}{\partial x}\mathbf{R}_D = \mathbf{V}(x,t)\mathbf{Q}. \tag{14}$$

Next, we employ the transform matrix expressions of (6) and (7). Taking the Laplace transform of both sides of (14) with respect to t, yields

$$s\mathbf{V}^*(x,s) - \mathbf{a}(0) + \frac{\partial \mathbf{V}^*(x,s)}{\partial x}\mathbf{R}_D = \mathbf{V}^*(x,s)\mathbf{Q}.$$

Now, taking the Laplace–Stieltjes transform of the above equation with respect to x gives

$$s\tilde{\mathbf{V}}^*(u,s) - \mathbf{I} + u\tilde{\mathbf{V}}^*(u,s)\mathbf{R}_D = \tilde{\mathbf{V}}^*(u,s)\mathbf{Q}. \tag{15}$$

The theorem follows by rearranging the terms of (15). $\quad\square$

Theorem 3.1 may now be used to derive the explicit expression for failure time distribution of the single-unit system subject to a Markovian environment. Define the joint distribution of T_x and $Z(t)$,

conditional on $Z(0)$, by

$$W_{i,j}(x,t) = P\{T_x \leqslant t, Z(t) = j | Z(0) = i\}.$$

Due to the dual relationship of (3), we obtain the marginal distribution of T_x, conditioned upon $i \in S$ as

$$W_i(x,t) = 1 - \sum_{j \in S} V_{i,j}(x,t). \tag{16}$$

It follows that $G(x,t) := P\{T_x \leqslant t\}$, the unconditional distribution of T_x, is given by

$$G(x,t) = 1 - \alpha \mathbf{V}(x,t)\mathbf{1}, \tag{17}$$

where $\alpha := [\alpha_i]$ with $\alpha_i := P\{Z(0) = i\}$ and $\mathbf{1}$ is a K-dimensional column vector of ones. We now define the following transforms,

$$G^*(x,s) = \int_{\mathbf{R}^+} e^{-st} G(x,t)\,\mathrm{d}t,$$

$$\tilde{G}^*(u,s) = \int_{\mathbf{R}^+} e^{-ux} G^*(\mathrm{d}x,s).$$

Theorem 3.2. *The failure time distribution of a single-unit system in a Markovian environment is*

$$\tilde{G}^*(u,s) = \frac{1}{s} - \alpha \tilde{\mathbf{V}}^*(u,s)\mathbf{1}, \quad Re(u) > 0, \ Re(s) > 0. \tag{18}$$

Proof. The proof follows directly by taking the Laplace transform of (17) with respect to t and then taking the Laplace–Stieltjes transform with respect to x. □

It should be noted here that the two-dimensional transform of (18) can be inverted numerically using algorithms such as those due to Abate et al. [1] and Moorthy [9]. In Section 4, we use the results of Theorem 3.1 to derive transform expression for all moments of the random lifetime T_x.

4. Analysis of moments

We now turn our attention to the moments of T_x whenever the threshold value $x \in \mathbf{R}^+$ is finite. Define the conditional expectation of the nth moment of T_x by

$$\xi_i^n(x) = E[T_x^n | Z(0) = i] \tag{19}$$

conditioned upon the event that the initial state of the environment Z is $i \in S$. The following lemma will be needed to derive an explicit expression for the (conditional) moments of the random time to failure of the system.

Lemma 4.1. *For $n \geqslant 0$, the nth order partial derivative of $\tilde{\mathbf{V}}^*(u,s)$ w.r.t. s is given by*

$$\frac{\partial^n}{\partial s^n} \tilde{\mathbf{V}}^*(u,s) = (-1)^n n! (u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{-n-1}.$$

Proof. The lemma will be proved by induction on n. By (11), the result clearly holds for $n = 0$. Let $n = 1$ and note that

$$\tilde{\mathbf{V}}^*(u,s)(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q}) = \mathbf{I}$$

also by (11). Differentiating both sides of the above equation with respect to s shows that

$$\frac{\partial}{\partial s} \tilde{\mathbf{V}}^*(u,s) = (-1)\tilde{\mathbf{V}}(u,s)(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{-1},$$

so the result holds for $n = 1$. For the inductive step, assume that

$$\frac{\partial^m}{\partial s^m} \tilde{\mathbf{V}}^*(u,s) = (-1)^m m! (u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{-m-1} \tag{20}$$

for an arbitrary $m \in \mathbf{N}$. Rewriting (20) gives,

$$\frac{\partial^m}{\partial s^m} \tilde{\mathbf{V}}^*(u,s)(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{m+1} = (-1)^m m! \times \mathbf{I}.$$

Differentiating both sides of the above equation and rearranging terms, we have

$$\frac{\partial^{m+1}}{\partial s^{m+1}} \tilde{\mathbf{V}}^*(u,s)(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{m+1}$$

$$= (-1)(m+1)\frac{\partial^m}{\partial s^m} \tilde{\mathbf{V}}^*(u,s)(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^m.$$

Substituting (20) into the above equation, we obtain

$$\frac{\partial^{m+1}}{\partial s^{m+1}} \tilde{\mathbf{V}}^*(u,s)$$

$$= (-1)^{m+1}(m+1)! (u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{-m-2}$$

and the proof is complete. □

We next state our result for the conditional nth moment ($n \geqslant 1$) of the failure time.

Theorem 4.1. *For* $n \geqslant 1$, *the Laplace–Stieltjes transform of* $\xi_i^n(x)$ *is*

$$\tilde{\xi}_i^n(u) = n!(u\mathbf{R}_D - \mathbf{Q})^{-1}\mathbf{1}. \tag{21}$$

Proof. By Eq. (16), the Laplace–Stieltjes transform of $W_i(x,t)$ with respect to t may be written as

$$\tilde{W}_i(x,s) = 1 - \sum_{j \in S} s V_{i,j}^*(x,s), \tag{22}$$

where $V_{i,j}^*(x,s)$ is the Laplace transform of $V_{i,j}(x,t)$ with respect to t. It is well-known that the conditional expectation of (19) may be obtained by evaluating the nth order derivative of (22) at $s = 0$. To that end, let

$$\psi_i^n(x,s)$$

$$:= (-1)^n \frac{\partial^n}{\partial s^n} \tilde{W}_i(x,s) = (-1)^{n+1}$$

$$\times \left(\sum_{j \in S} s \frac{\partial^n}{\partial s^n} V_{i,j}^*(x,s) + n \frac{\partial^{n-1}}{\partial s^{n-1}} V_{i,j}^*(x,s) \right). \tag{23}$$

Eq. (23) implies that

$$\xi_i^n(x) = \psi_i^n(x,0)$$

$$= (-1)^{n+1} n \sum_{j \in S} \frac{\partial^{n-1}}{\partial s^{n-1}} V_{i,j}^*(x,s) \Bigg|_{s=0.} \tag{24}$$

In order to solve the differential equation (24), transform methods are again employed. The Laplace–Stieltjes transform of $\xi_i^n(x)$ with respect to x is

$$\tilde{\xi}_i^n(u) = \int_{\mathbf{R}^+} e^{-ux} \xi_i^n(\mathrm{d}x).$$

By taking the Laplace–Stieltjes Transform of Eq. (24) on both sides,

$$\tilde{\xi}_i^n(u) = (-1)^{n+1} n \sum_{j \in S} \frac{\partial^{n-1}}{\partial s^{n-1}} \tilde{V}_{i,j}^*(u,s) \Bigg|_{s=0}$$

$$= (-1)^{n+1} n \frac{\partial^{n-1}}{\partial s^{n-1}} \tilde{\mathbf{V}}^*(u,s) \Bigg|_{s=0} \mathbf{1}, \tag{25}$$

where $\tilde{\mathbf{V}}^*(u,s)$ is the matrix transform of (11) and $\mathbf{1}$ is a K-dimensional column vector of ones. Finally,

applying Lemma 4.1 and setting $s = 0$, we have that

$$\tilde{\xi}_i^n(u) = n!(u\mathbf{R}_D - \mathbf{Q})^{-1}\mathbf{1}$$

and the proof is complete. $\quad \square$

Let $\xi^n(x) := E[T_x^n]$ be the unconditional nth moment of the failure time. The following corollary follows directly from Theorem 4.1.

Corollary 4.1. *Suppose Z has initial probability distribution* α. *Then,*

$$\tilde{\xi}^n(u) = n!\alpha(u\mathbf{R}_D - \mathbf{Q})^{-1}\mathbf{1}. \tag{26}$$

The above transform can be easily inverted numerically by a number of one-dimensional inversion algorithms. The reader is referred to Abate and Whitt [2] for one such approach. In the following section, the main results of the paper will be demonstrated on an example problem.

5. Numerical example

In this section, we demonstrate the applicability of the main results of this paper via a numerical example. In particular, we consider the propagation of a crack in metallic materials. For a more thorough treatment of fatigue crack dynamics, the reader is referred to the paper by Ray and Tangirala [12] and references contained therein. Assume that the metallic component is placed in operation in perfect working order (i.e., there is no crack in the material initially). However, due to normal wear and fatigue, a crack is initiated and continues to grow in time.

Let $X(t)$ denoted the length of the crack at time t and assume that the (linear) rate at which the crack grows is subject to its random environment (applied stress, ambient conditions, and other factors). It is assumed that these environmental factors can be characterized by $\{Z(t): t \geqslant 0\}$, a temporally homogeneous Markov process that alternates between two distinct states such that its finite-state space is given by $S = \{1, 2\}$ (i.e., Z is a simple on/off process). Whenever the environment is in state $i \in S$, the crack grows at rate $r(i)$ units per unit time, $i \in S$. The initial distribution of the environment process is arbitrarily chosen to be $\alpha = [1 \ 0]$,

Table 1
$P\{T_x \leqslant t\}$ when Z is an on/off process

t	Analytical	Simulation	t	Analytical	Simulation
1.197	1.2589E-01	1.2880E-01	2.018	9.6765E-01	9.6980E-01
1.288	2.3732E-01	2.4300E-01	2.109	9.8299E-01	9.8290E-01
1.379	3.7196E-01	3.7040E-01	2.200	9.9149E-01	9.9140E-01
1.470	5.1284E-01	5.1400E-01	2.291	9.9584E-01	9.9570E-01
1.562	6.4366E-01	6.4840E-01	2.382	9.9819E-01	9.9800E-01
1.653	7.5388E-01	7.5010E-01	2.474	9.9912E-01	9.9940E-01
1.744	8.3960E-01	8.3600E-01	2.565	9.9950E-01	9.9960E-01
1.835	9.0102E-01	9.0020E-01	2.656	9.9989E-01	9.9960E-01
1.926	9.4205E-01	9.4240E-01	2.747	9.9997E-01	1.0000E+00

i.e., the environment starts in state 1 with probability 1. Moreover, the infinitesimal generator matrix is given by

$$\mathbf{Q} = \begin{bmatrix} -b & b \\ a & -a \end{bmatrix},$$

while the diagonal matrix of wear rates is

$$\mathbf{R}_D = \begin{bmatrix} r(1) & 0 \\ 0 & r(2) \end{bmatrix}.$$

Now, the partial differential equation describing the probability distribution of $X(t)$ is given by

$$\frac{\partial V_{i,j}(x,t)}{\partial t} + \frac{\partial V_{i,j}(x,t)}{\partial x} r(j)$$

$$= \sum_{k=1}^{2} q_{kj} V_{i,k}(x,t), \quad j = 1, 2$$

which can be written in matrix form as

$$\frac{\partial \mathbf{V}(x,t)}{\partial t} + \frac{\partial \mathbf{V}(x,t)}{\partial x} \mathbf{R}_D = \mathbf{V}(x,t)\mathbf{Q},$$

where $\mathbf{V}(x,t) = [V_{i,j}(x,t)]$. Now applying Eqs. (11) and (18), the transform $\tilde{G}^*(u,s)$ is obtained by

$$\tilde{G}^*(u,s) = \frac{1}{s} - \alpha(u\mathbf{R}_D + s\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}$$

$$= \frac{1}{s} - [1 \quad 0](\delta\gamma - ab)^{-1} \begin{bmatrix} \delta & b \\ a & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\delta = r(2)u + s + a$ and $\gamma = r(1)u + s + b$ are scalars. The above two-dimensional LST is converted to a two-dimensional Laplace transform by

Table 2
Comparison of $\xi^1(x)$ when Z is an on/off process

x	Analytical	Simulated
0.25	3.6094E-01	3.6139E-01
0.50	7.3594E-01	7.3617E-01
1.00	1.4859E+00	1.4852E+00
5.00	7.4859E+00	7.4875E+00
10.00	1.4986E+01	1.4979E+01
20.00	2.9986E+01	2.9987E+01
50.00	7.4986E+01	7.4983E+01

pre-multiplying both sides of the equation by u^{-1}. The inversion algorithm of Moorthy [9] was implemented to compute the analytical cumulative probability values at several points and compared with simulation results.

The specific problem parameters chosen for this example are as follows. The threshold value $x = 1.0$ and the values comprising the generator matrix are $a = b = 25/3$. The state-dependent crack growth rates are $r(1)=1.0833$ and $r(2)=0.250$. For comparison purposes, an empirical cumulative distribution function was constructed via Monte Carlo simulation techniques. The cumulative probabilities were compared at several values of t and the results summarized in Table 1.

In the following example, we demonstrate computation of the first two moments of the failure time using identical problem parameters. In this case, the one-dimensional inversion algorithm of Abate and Whitt [2] was implemented and found to perform extremely well. Tables 2 and 3 contain a summary of

Table 3
Comparison of $\xi^2(x)$ when Z is an on/off process

x	Analytical	Simulated
0.25	1.4462E-01	1.4495E-01
0.50	5.7353E-01	5.7360E-01
1.00	2.2751E+00	2.2751E+00
5.00	5.6388E+01	5.6407E+01
10.00	2.2528E+02	2.2537E+02
20.00	9.0056E+02	9.0038E+02
50.00	5.6264E+03	5.6260E+03

the numerical values as compared with Monte Carlo simulation results.

Acknowledgements

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