Monotone Optimal Replacement Policies for a Markovian Deteriorating System in a Controllable Environment

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Abstract

We consider the optimal replacement of a periodically inspected system under Markov deterioration that operates in a controlled environment. Provided are sufficient conditions that characterize an optimal control-limit replacement policy with respect to the system's condition and its environment. The structure of the optimal policy is illustrated by a numerical example.

Keywords: Replacement, Markov decision process, environment, control-limit policy.

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1 Introduction

In this paper, we study a periodically inspected system under Markovian deterioration that operates in a controllable Markovian environment. The failures of the system can only be revealed by an inspection, after which the system is either replaced immediately or allowed to continue operating until the next inspection. Our aim is to determine when to optimally replace the system to minimize the total expected discounted cost of inspection, operation, and replacement over an infinite horizon. To this end, we formulate the resulting optimal replacement problem as a discrete-time, infinite-horizon discounted Markov decision process (MDP). Under a set of reasonable assumptions, we explore the structure of the optimal cost function and provide sufficient conditions that ensure the optimality of a structured replacement policy.

Stochastic models that describe the deterioration of replaceable and/or repairable systems have long been studied in the operations research literature. Extensive surveys on the subject can be found in [14, 15, 18, 20, 23, 25]. Most classical models assume that the system operates in an unchanging environment, and that deterioration is primarily due to internal factors (i.e., normal usage in a fixed environment). Those researchers who have considered the effects of a time-varying operating environment on single- and multi-unit systems have generally focused on modeling the deterioration process to compute reliability or availability indices [4, 5, 7, 8, 12, 19, 22]. The vast majority of these are stochastic shock and/or wear models in a random environment. Singpurwalla [21] provides an extensive review of a variety of stochastic deterioration models for systems that evolve in randomly varying environments.

Despite the extensive literature on systems under Markovian deterioration (i.e., those whose deterioration status evolves as a Markov chain), relatively few studies have considered the problem of prescribing optimal maintenance or replacement policies when the system is influenced by a time-varying, stochastic environment. Waldmann [24] appears to be the first to analyze the structure of an optimal replacement policy for a system subjected to stochastic deterioration in a random environment. Waldmann considers the effects of uncontrollable internal and external factors on the progression of the system's deterioration status in a continuous-time shock model and derives sufficient conditions to establish the optimality of a control-limit policy with respect to the cumulative damage of the system. Özekici [13] models the deterioration of a single-unit system in an uncontrollable environment by its intrinsic age where the environment evolves as a semi-Markov jump process, and the intrinsic age of the system is determined by the total cumulative degradation. It is shown that, if the system's lifetime distribution has an increasing failure rate in each environment state, then a control-limit policy, with respect to the system's status, is optimal; however, for a given system status, the relationship between the optimal actions in different environment states is not explored. Kurt and Maillart [11] examine the optimal replacement of a system that fails by receiving Poisson shocks at a rate that is governed by a discrete-time Markov chain. Unlike the models in [13, 24], they consider a *controllable* Markovian environment and analyze the structure of the resulting optimal cost function with respect to the shock arrival rate and the cumulative number of shocks received.

Our work here differs from other optimal replacement models in a few important ways. First, our model generalizes a number of classical Markov deterioration models by incorporating the effects of varying environmental factors in the formulation. Second, it provides sufficient conditions to establish the optimality of a control-limit policy. These conditions are less stringent than those required by similar models of system's that operate in a fixed environment. Third, it extends the model of Kurt and Maillart [11] by relaxing the assumption that the replacement cost is fixed in all possible deterioration levels and environmental conditions. For the model in [11], this assumption implies that the monotonicity of the optimal cost function is sufficient to establish the existence of an optimal control-limit policy. By contrast, the monotonicity of the optimal cost function is not sufficient to reveal the underlying structure of the optimal policy in our model. Therefore, we state and prove intuitive sufficient conditions to establish the optimality of control-limit replacement policies with respect to both the system's deterioration status and the state of the environment.

The remainder of the paper is organized as follows. In section 2, we describe the model in detail and present the infinite-horizon, discounted MDP formulation. In section 3, we explore the monotonicity of the optimal cost function and derive sufficient conditions to reveal a control-limit structure for the optimal replacement policy. In section 4, as a special case of our model, we examine a fixed environment, which corresponds to the standard Markovian deterioration model, and show that our sufficient conditions are weaker than those for the standard model. Finally, in section 5, we provide a numerical example to illustrate the structure of the optimal replacement policy and the ease with which our sufficient conditions can be met.

2 Model Formulation

Consider a system whose deterioration status evolves as a discrete-time Markov chain (DTMC) on a finite state space $\Psi = \{0, 1, \ldots, S\}$, where the state space is structured in order of increasing deterioration levels, i.e., state 0 means the system is new, and state S means it is failed. The transition probability matrix of this DTMC is governed by an exogenous stochastic process, the environment process, which also evolves as a DTMC on a finite state space $\Upsilon = \{0, 1, \ldots, R\}$. The state of the environment directly influences the evolution of the system's deterioration process by accelerating (or decelerating) its movements toward more degraded states. The environment's state space is ordered such that state 0 has the least amount of influence of the deterioration process, and state R has the greatest influence. It is important to note that, in this model, the deterioration process depends explicitly on the environment process whereas the environment process does not depend on the current level of deterioration (i.e., they are not mutually independent). Independent of the level of deterioration, denote by $Q(r'|r)$ the probability that the environment transitions to state $r' \in \Upsilon$ at epoch $n+1$, given that it was in state $r \in \Upsilon$ at epoch n, and let $\mathbf{Q} = [Q(r'|r)]_{r,r' \in \Upsilon}$ be the one-step transition probability matrix of the environment process. The dependence of the deterioration process on the environment is modeled as follows. Given that the system is in state $s \in \Psi$ (and not replaced) at epoch n, if the environment is in state $r \in \Upsilon$, then $P(s'|s,r)$ denotes the probability that the system next transitions to state $s' \in \Psi$ at epoch $n+1$, with $P(S|S,r) = 1$ for all $r \in \Upsilon$. Therefore, when the environment is in state $r \in \Upsilon$, the transition probability matrix of the system's deterioration status is denoted by $\mathbf{P}(r) = [P(s'|s,r)]_{s,s' \in \Psi}$. Although the environment evolves independently of the system's deterioration process, we assume that it immediately returns to state 0 at the time of a replacement. This scenario is prevalent when the deterioration of the system is dictated by the condition of a supporting system. For example, internal metallic engine components are usually submerged (and operate) in an oil bath that is used to reduce friction and material wear. The condition of the oil has a direct influence on the rate at which the moving components deteriorate. Whenever the system is replaced, a new oil bath is used with the new components in order to provide maximum protection. In this sense, the environment resets to its lowest level at the time of replacement.

Because we consider a discrete-time formulation of our problem, both the deterioration process and the environment process transition (or remain in their current states) at times in the set $\{0, \tau, 2\tau, 3\tau, \ldots\}$ for some $\tau > 0$. The system is inspected at these same times, and the time interval between any two successive inspections is called a period. Each inspection costs I units, requires a negligible amount of time (e.g., the inspection can be done remotely via sensing or other technologies), and is assumed to perfectly reveal the system's deterioration status and the state of the environment. If an inspection reveals that the environment and the system are in states $r \in \Upsilon$ and $s \in \Psi$, respectively, then the system may either be replaced with a new (and identical) one at a cost of $C(r,s) < \infty$ units, or left in operation for one more period with an immediate cost of $\ell(r,s) < \infty$ units. However, if an inspection reveals that the system is failed, it must be replaced. All costs are assumed to be incurred at the end of each period and discounted at a rate λ, $\lambda \in (0,1)$. The objective is to minimize the total expected discounted cost of inspection, operation and replacement.

We model this optimal replacement problem as an infinite-horizon, discounted MDP where each decision epoch refers to a time point immediately after an inspection. The state of the MDP is an ordered pair $(r,s) \in \Upsilon \times \Psi$, where $r \in \Upsilon$ denotes the state of the environment and $s \in \Psi$ represents the system's current deterioration status, which we also refer to as the system state. We let $\vartheta(r,s)$ denote the minimum total expected discounted cost, given that the process starts in state $(r,s) \in \Upsilon \times \Psi$ and $a(r,s) \in \{0,1\}$ refers to the corresponding optimal action, where action 0 means waiting for one more period (or doing nothing), and action 1 refers to immediate replacement of the system. To help simplify the notation, we let $w(r,s)$ denote the total expected discounted cost of waiting in state $(r,s) \in \Upsilon \times \Phi$. Note that, for every $r \in \Upsilon$, state (r,S) corresponds to a failed system for which replacement is the only viable action; therefore, for convenience, we let $\Phi = \Psi \setminus \{S\}$ refer to the set of operational system states.

In our model, a replacement in state (r,s) moves the MDP immediately to state $(0,0)$ with probability 1 at a cost of $C(r,s)$ units. Otherwise, the environment moves into state $r' \in \Upsilon$ with probability $Q(r'|r)$ at the next epoch, and the system continues to operate in environment state r at an immediate cost of $\ell(r,s)$ units. It then transitions into state $s' \in \Psi$ with probability $P(s'|s,r)$ at the next epoch. To characterize an optimal replacement policy yielding the minimum total expected discounted cost in each state of the process, we solve the following optimality equations:

$$\vartheta(r,s) = \begin{cases} \min\left\{w(r,s), \vartheta(0,0) + C(r,s)\right\}, & \text{for } (r,s) \in \Upsilon \times \Phi, \\ \vartheta(0,0) + C(r,s), & \text{for } r \in \Upsilon,\ s = S, \end{cases} \tag{1a}$$

where for $(r,s) \in \Upsilon \times \Phi$,

$$w(r,s) = \ell(r,s) + \lambda \sum_{r' \in \Upsilon} \sum_{s' \in \Psi} P(s'|s,r) Q(r'|r) [\vartheta(r',s') + I]. \tag{1b}$$

Note that because all operating and replacement costs are finite and $\lambda < 1$, equations (1a)-(1b) admit a finite, unique solution [17].

Our objective in the remainder of this paper is to explore the structure of the optimal cost function, ϑ, and to derive sufficient conditions that guarantee the existence of an optimal control-limit policy.

3 Structural Results

In this section we analyze the structure of the optimal cost function, ϑ, and the optimal replacement policy, $\{a(r,s)\}$. Under reasonable assumptions, we show that the optimal cost function is monotonically nondecreasing in both r and s. Subsequently, we provide sufficient conditions for the existence of an optimal replacement policy that exhibits a control-limit structure with respect to both system's deterioration status and the state of the environment. In a broader context, control-limit policies have several appealing properties. They facilitate simple implementation in practice and can provide deeper insights into the problem with their intuitive structure. Such policies may also increase the computational tractability of the model by allowing for simpler approximation schemes if the exact solution of the problem is intractable due to the curse(s) of dimensionality [16, 17].

Throughout this section, we use the phrase "rate of deterioration" to refer to the system's likelihood of transitioning to a set of states that are worse than any specific level of deterioration. We begin our analysis by imposing the following reasonable assumptions that are commonly employed in the maintenance optimization literature [15, 18, 20, 23, 25].

Assumption 1 : *The functions $\ell(r,s)$ and $C(r,s)$ are nondecreasing in $s \in \Psi$ for all $r \in \Upsilon$.*

Assumption 2 : *The functions $\ell(r,s)$ and $C(r,s)$ are nondecreasing in $r \in \Upsilon$ for all $s \in \Psi$.*

Assumption 1 asserts that the immediate operating and replacement costs do not decrease in any environment state as the system deteriorates. Similarly, Assumption 2 asserts that the immediate operating and replacement costs do not decrease in any state of deterioration as the environment worsens. Our analysis will make significant use of the notion of an increasing failure rate (IFR) transition probability matrix. The IFR property is a widely used assumption to analyze the structure of the optimal cost/reward functions in reliability and maintenance optimization [15, 23]. The following definition is taken from [2].

Definition 1 : *A transition probability matrix* $\mathbf{H} = [H(j|i)]$, $i, j = 1, \ldots, n$ *is said to have the increasing failure rate property if* $\sum_{j=k}^{n} H(j|i)$ *is nondecreasing in* i *for all* $k = 1, \ldots, n$.

Along with monotone operating and replacement costs, the IFR property of the environment's transition probability matrix, and that of the system's deterioration matrix in each environment state are critical to establish the monotone behavior of the optimal cost function.

Assumption 3 : *The transition matrix* $\mathbf{P}(r)$ *is IFR for all* $r \in \Upsilon$.

Assumption 3 states that the system's rate of deterioration does not decrease in any environment as it deteriorates. In other words, given two systems operating in the same environment, the system which is found to be in a worse condition at the current inspection epoch is more likely than the other to be found at a worse condition at the next inspection epoch.

Assumption 4 : *The transition matrix* $\mathbf{Q}$ *is IFR.*

In our problem context, the IFR property of $\mathbf{Q}$ implies that the more detrimental the environment is for the system in the current period, the more likely it is to be as such in the next period.

Given two $n \times n$ transition probability matrices, $\mathbf{H}_1 = [h_1(j|i)]$ and $\mathbf{H}_2 = [h_2(j|i)]$, $i, j = 1, \ldots, n$, we write $\mathbf{H}_1 \succeq \mathbf{H}_2$ if $\mathbf{H}_1$ stochastically dominates $\mathbf{H}_2$. Recall that higher indexed environments indicate relatively higher rates of deterioration for the system. Assumption 5 imposes a first-order stochastic dominance relationship among the system's deterioration matrices. Specifically, it asserts that the system's rate of deterioration increases as the environment worsens.

Assumption 5 : *For all* $r \in \Upsilon \setminus \{R\}$, $\mathbf{P}(r+1) \succeq \mathbf{P}(r)$.

Assumptions 1–5 are employed for the results that follow. First, we establish the monotonicity of the optimal cost function in both r and s. Proposition 1 states that the optimal total expected discounted cost is monotonically nondecreasing in the order of deterioration in each environment.

Proposition 1 : *The optimal cost function* $\vartheta(r, s)$ *is nondecreasing in* $s \in \Psi$ *for all* $r \in \Upsilon$.

Proposition 2 states that the optimal total expected discounted cost does not decrease in any system state as the environment worsens.

Proposition 2 : *The optimal cost function* $\vartheta(r, s)$ *is nondecreasing in* $r \in \Upsilon$ *for all* $s \in \Psi$.

We omit the proof of Proposition 1 since, by Assumptions 1 and 3, the proof is similar to that of Proposition 1 in Kurt and Maillart [11]. Similarly, by assumptions 1–5, Proposition 2 can be proved by following the proof of Proposition 2 of [11].

Next, we present the two main results of this paper. First, we derive a sufficient condition for the optimality of a replacement policy that exhibits a control-limit structure with respect to system's deterioration status.

Theorem 1 : *For $r \in \Upsilon$ and $s \in \Phi \setminus \{S-1\}$, let*

$$d(r,s) = \sum_{s'=0}^{s} \big[P(s'|s,r) - P(s'|s+1,r)\big] \;\; \textit{and} \;\; \varphi(r,s) = \frac{\ell(r,s+1) - \ell(r,s)}{1 - \lambda d(r,s) Q(r|r)}.$$

Then, for any $r \in \Upsilon$ and $s \in \Phi \setminus \{S-1\}$, if

$$\begin{aligned} \ell(r,s+1) - \ell(r,s) + \lambda d(r,s) \sum_{r' \in \Upsilon} Q(r'|r) \min\big\{\varphi(r',s), C(r',s+1) - C(r',s)\big\} \\ \geq C(r,s+1) - C(r,s), \end{aligned} \tag{2}$$

$a(r,s) = 1$ implies $a(r,s+1) = 1$.

The proof of Theorem 1 is preceded by Lemma 1 (due to Alagoz et al. [1]) which allows us to provide a lower bound on the difference between the optimal total expected discounted cost in any two successive system states under the same environmental conditions, and in any two successive environments with the same level of deterioration.

Lemma 1 : *Let $\mathbf{H} = \big[H(j|i)\big]$, $i,j = 1,2,\ldots,n$, be an $n \times n$ IFR transition probability matrix and $f : \{1,\ldots,n\} \to \mathbb{R}$ be a nonincreasing function in $i \in \{1,\ldots,n\}$. Then, for $i = 1,2,\ldots,n-1$,*

$$(i) \sum_{j=0}^{i} \big[H(j|i) - H(j|i+1)\big] f(j) \geq f(i) \sum_{j=0}^{i} \big[H(j|i) - H(j|i+1)\big],$$

$$(ii) \sum_{j=i+1}^{n} \big[H(j|i) - H(j|i+1)\big] f(j) \geq f(i+1) \sum_{j=i+1}^{n} \big[H(j|i) - H(j|i+1)\big].$$

Proof of Theorem 1: Before proceeding to the proof of the main result, we first establish the following inequalities which hold for all $r \in \Upsilon$ and $s \in \Phi \setminus \{S-1\}$:

$$(i)\; w(r,s+1) - w(r,s) \geq \ell(r,s+1) - \ell(r,s) + \lambda d(r,s) \sum_{r' \in \Upsilon} Q(r'|r)\big[\vartheta(r',s+1) - \vartheta(r',s)\big], \tag{3a}$$

$$(ii)\; w(r,s+1) - w(r,s) \geq \ell(r,s+1) - \ell(r,s) + \lambda d(r,s) Q(r|r)\big[\vartheta(r,s+1) - \vartheta(r,s)\big], \tag{3b}$$

$$(iii)\; \vartheta(r,s+1) - \vartheta(r,s) \geq \min\big\{\varphi(r,s), C(r,s+1) - C(r,s)\big\}. \tag{3c}$$

We will consider the proofs of (i) and (ii) together.
(i)-(ii) Consider an arbitrary $r \in \Upsilon$ and $s \in \Phi \setminus \{S-1\}$. We have,

$$\begin{aligned} & w(r,s+1) - w(r,s) \\ & = \ell(r,s+1) - \ell(r,s) + \lambda \Bigg[\sum_{s'=0}^{s} \Bigg(\big[P(s'|s+1,r) - P(s'|s,r)\big] \sum_{r' \in \Upsilon} Q(r'|r)\vartheta(r',s') \Bigg) \\ & \qquad + \sum_{s'=s+1}^{S} \Bigg(\big[P(s'|s+1,r) - P(s'|s,r)\big] \sum_{r' \in \Upsilon} Q(r'|r)\vartheta(r',s') \Bigg) \Bigg]. \end{aligned} \tag{4}$$

For notational convenience, let $Z(s) = \sum_{r' \in \Upsilon} Q(r'|r)\vartheta(r',s)$ for $s \in \Psi$. By Proposition 1, recall that $\vartheta(r',s)$ is nondecreasing in $s \in \Psi$ for all $r' \in \Upsilon$. Therefore, $-Z(s)$ is nonincreasing in $s \in \Psi$. Then,

$$\begin{aligned} \sum_{s'=0}^{s} \Bigg(\big[P(s'|s+1,r) - P(s'|s,r)\big] \sum_{r' \in \Upsilon} Q(r'|r)\vartheta(r',s') \Bigg) &= \sum_{s'=0}^{s} \big[P(s'|s+1,r) - P(s'|s,r)\big] Z(s') \\ &= \sum_{s'=0}^{s} \big[P(s'|s,r) - P(s'|s+1,r)\big]\big[-Z(s')\big] \geq -Z(s)d(r,s). \end{aligned} \tag{5}$$

Note that because $\mathbf{P}(r)$ is IFR and $-Z(s')$ is nonincreasing in $s' \in \Psi$, the inequality in (5) follows from Lemma 1 part (i). Likewise, by Lemma 1 part (ii) and the identity $\sum_{s'=s+1}^{S} \big[P(s'|s,r) - P(s'|s+1,r)\big] = -d(r,s)$, we see that

$$\sum_{s'=s+1}^{S} \Bigg(\big[P(s'|s+1,r) - P(s'|s,r)\big] \sum_{r' \in \Upsilon} Q(r'|r)\vartheta(r',s') \Bigg) \geq Z(s+1)d(r,s). \tag{6}$$

By the definition of the function Z, (4), (5) and (6) imply

$$w(r,s+1) - w(r,s) \geq \ell(r,s+1) - \ell(r,s) + \lambda d(r,s) \sum_{r' \in \Upsilon} Q(r'|r)\big[\vartheta(r',s+1) - \vartheta(r',s)\big]. \tag{7}$$

By Proposition 1, recall that $\vartheta(r',s+1) \geq \vartheta(r',s)$ for all $r' \in \Upsilon$. Therefore,

$$\sum_{r' \in \Upsilon} Q(r'|r)\big[\vartheta(r',s+1) - \vartheta(r',s)\big] \geq Q(r|r)\big[\vartheta(r,s+1) - \vartheta(r,s)\big]. \tag{8}$$

Since $\mathbf{P}(r)$ is IFR, we also have $d(r,s) \geq 0$. By (8), this yields:

$$d(r,s) \Bigg(\sum_{r' \in \Upsilon} Q(r'|r)\big[\vartheta(r',s+1) - \vartheta(r',s)\big] \Bigg) \geq d(r,s)Q(r|r)\big[\vartheta(r,s+1) - \vartheta(r,s)\big]. \tag{9}$$

Then, by (7), inequality (9) implies

$$w(r,s+1) - w(r,s) \geq \ell(r,s+1) - \ell(r,s) + \lambda d(r,s)Q(r|r)\big[\vartheta(r,s+1) - \vartheta(r,s)\big].$$

Next, we will consider the proof of (iii).
(iii) For fixed $r \in \Upsilon$ and $s \in \Phi \setminus \{S-1\}$, consider the following possible cases for $\vartheta(r,s+1) - \vartheta(r,s)$.

1. If $\vartheta(r, s+1) = \vartheta(0,0) + C(r, s+1)$, then

$$\vartheta(r, s+1) - \vartheta(r, s) \geq C(r, s+1) - C(r, s) \geq \min\big\{\varphi(r, s), C(r, s+1) - C(r, s)\big\}, \tag{10}$$

where the first inequality in (10) is implied by $\vartheta(r, s) \leq \vartheta(0,0) + C(r, s)$.

2. If $\vartheta(r, s+1) = w(r, s+1)$, then

$$\vartheta(r, s+1) - \vartheta(r, s) \geq w(r, s+1) - w(r, s) \tag{11a}$$
$$\geq \ell(r, s+1) - \ell(r, s) + \lambda d(r, s) Q(r|r)\big[\vartheta(r, s+1) - \vartheta(r, s)\big], \tag{11b}$$

where (11a) is implied by $\vartheta(r, s) \leq w(r, s)$, and (11b) follows from (3b). Because $Q(r|r) \leq 1$, $\lambda < 1$ and $d(r, s) \leq 1$, by (11a) and (11b), after rearranging the terms we obtain

$$\vartheta(r, s+1) - \vartheta(r, s) \geq \frac{\ell(r, s+1) - \ell(r, s)}{1 - \lambda d(r, s) Q(r|r)} = \varphi(r, s) \geq \min\big\{\varphi(r, s), C(r, s+1) - C(r, s)\big\}.$$

Now, we will prove the main result. Suppose (2) holds for some $\widehat{r} \in \Upsilon$ and $\widehat{s} \in \Phi \setminus \big\{S-1\big\}$. First, we will establish the following relation:

$$\vartheta(\widehat{r}, \widehat{s}+1) - \vartheta(\widehat{r}, \widehat{s}) \geq C(\widehat{r}, \widehat{s}+1) - C(\widehat{r}, \widehat{s}). \tag{12}$$

Consider the following possible cases for $\vartheta(\widehat{r}, \widehat{s}+1) - \vartheta(\widehat{r}, \widehat{s})$.

1. If $\vartheta(\widehat{r}, \widehat{s}+1) = \vartheta(0,0) + C(\widehat{r}, \widehat{s}+1)$, then because $\vartheta(\widehat{r}, \widehat{s}) \leq \vartheta(0,0) + C(\widehat{r}, \widehat{s})$, we have

$$\vartheta(\widehat{r}, \widehat{s}+1) - \vartheta(\widehat{r}, \widehat{s}) \geq C(\widehat{r}, \widehat{s}+1) - C(\widehat{r}, \widehat{s}).$$

2. If $\vartheta(\widehat{r}, \widehat{s}+1) = w(\widehat{r}, \widehat{s}+1)$, then

$$\vartheta(\widehat{r}, \widehat{s}+1) - \vartheta(\widehat{r}, \widehat{s}) \geq w(\widehat{r}, \widehat{s}+1) - w(\widehat{r}, \widehat{s}) \tag{13a}$$
$$\geq \ell(\widehat{r}, \widehat{s}+1) - \ell(\widehat{r}, \widehat{s}) + \lambda d(\widehat{r}, \widehat{s}) \sum_{r' \in \Upsilon} Q(r'|\widehat{r})\big[\vartheta(r', \widehat{s}+1) - \vartheta(r', \widehat{s})\big], \tag{13b}$$

where (13a) is implied by $\vartheta(\widehat{r}, \widehat{s}) \leq w(\widehat{r}, \widehat{s})$, and (13b) follows from (3a). By (3c), we have

$$\sum_{r' \in \Upsilon} Q(r'|\widehat{r})\big[\vartheta(r', \widehat{s}+1) - \vartheta(r', \widehat{s})\big] \geq \sum_{r' \in \Upsilon} Q(r'|\widehat{r}) \min\big\{\varphi(r', \widehat{s}), C(r', \widehat{s}+1) - C(r', \widehat{s})\big\}.$$

Because $d(\widehat{r}, \widehat{s}) \geq 0$, this implies:

$$d(\widehat{r}, \widehat{s}) \left(\sum_{r' \in \Upsilon} Q(r'|\widehat{r})\big[\vartheta(r', \widehat{s}+1) - \vartheta(r', \widehat{s})\big] \right)$$
$$\geq d(\widehat{r}, \widehat{s}) \sum_{r' \in \Upsilon} Q(r'|\widehat{r}) \min\big\{\varphi(r', \widehat{s}), C(r', \widehat{s}+1) - C(r', \widehat{s})\big\}. \tag{14}$$

Then, since (2) is satisfied for the pair $(\widehat{r}, \widehat{s})$, (13a), (13b) and (14) imply

$$\vartheta(\widehat{r}, \widehat{s}+1) - \vartheta(\widehat{r}, \widehat{s}) \geq C(\widehat{r}, \widehat{s}+1) - C(\widehat{r}, \widehat{s}).$$

Now, let $a(\widehat{r}, \widehat{s}) = 1$ implying that $\vartheta(\widehat{r}, \widehat{s}) = \vartheta(0,0) + C(\widehat{r}, \widehat{s})$. By (12), this implies $\vartheta(\widehat{r}, \widehat{s}+1) \geq \vartheta(0,0) + C(\widehat{r}, \widehat{s}+1)$. By the definition of $\vartheta(\widehat{r}, \widehat{s}+1)$, we also have $\vartheta(\widehat{r}, \widehat{s}+1) \leq \vartheta(0,0) + C(\widehat{r}, \widehat{s}+1)$. These yield $\vartheta(\widehat{r}, \widehat{s}+1) = \vartheta(0,0) + C(\widehat{r}, \widehat{s}+1)$, which implies $a(\widehat{r}, \widehat{s}+1) = 1$. □

Note that our discounted MDP model can be equivalently represented as an undiscounted MDP with a costless absorbing state which can be reached from every other state with probability $1-\lambda$. Therefore, the term $\lambda Q(r|r)d(r,s)$ in condition (2) can be interpreted as the decrease in the probability of remaining in the same environment r but moving to a system state which is at least as good as state s, when the system moves from state s to the next worse state.

In Theorem 2, we provide a sufficient condition to relate the optimal replacement actions of two systems that are operating with the same level of deterioration in different environments.

Theorem 2 : *For $r \in \Upsilon \setminus \{R\}$ and $s \in \Phi$, let*

$$q(r) = \sum_{r'=0}^{r} \big[Q(r'|r) - Q(r'|r+1)\big],$$

and

$$\eta(r,s) = \frac{\ell(r+1,s) - \ell(r,s) + \lambda q(r) P(S|s,r)\big[C(r+1,S) - C(r,S)\big]}{1 - \lambda q(r) P(s|s,r)}.$$

Then, for any $r \in \Upsilon \setminus \{R\}$ and $s \in \Phi$, if

$$\ell(r+1,s) - \ell(r,s) + \lambda q(r)\Bigg(\sum_{s' \in \Phi} P(s'|s,r) \min\big\{\eta(r,s'), C(r+1,s') - C(r,s')\big\} + P(S|s,r)\big[C(r+1,S) - C(r,S)\big]\Bigg) \geq C(r+1,s) - C(r,s), \quad (15)$$

$a(r,s) = 1$ implies $a(r+1,s) = 1$.

Proof : Similar to the proof of Theorem 1, we first establish a set of auxiliary results. Specifically, we will show that, for all $r \in \Upsilon \setminus \{R\}$ and $s \in \Phi$, the following inequalities hold:

$$(i)\ w(r+1,s) - w(r,s) \geq \ell(r+1,s) - \ell(r,s) + \lambda q(r) \sum_{s' \in \Psi} P(s'|s,r)\big[\vartheta(r+1,s') - \vartheta(r,s')\big], \quad (16a)$$

$$(ii)\ w(r+1,s) - w(r,s) \geq \ell(r+1,s) - \ell(r,s) + \lambda q(r)\Bigg(P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] + P(S|s,r)\big[C(r+1,S) - C(r,S)\big]\Bigg), \quad (16b)$$

$$(iii)\ \vartheta(r+1,s) - \vartheta(r,s) \geq \min\big\{\eta(r,s), C(r+1,s) - C(r,s)\big\}. \quad (16c)$$

We will consider the proofs of (i)-(ii) together.

(i)-(ii) Choose an arbitrary $r \in \Upsilon \setminus \{R\}$ and $s \in \Phi$. We have

$$w(r+1,s) - w(r,s) = \ell(r+1,s) + \lambda \sum_{r' \in \Upsilon} \sum_{s' \in \Psi} Q(r'|r+1) P(s'|s,r+1) \vartheta(r',s') - \ell(r,s) - \lambda \sum_{r' \in \Upsilon} \sum_{s' \in \Psi} Q(r'|r) P(s'|s,r) \vartheta(r',s')$$

$$\geq \ell(r+1,s) + \lambda \sum_{r' \in \Upsilon} \sum_{s' \in \Psi} Q(r'|r+1) P(s'|s,r) \vartheta(r',s') - \ell(r,s) - \lambda \sum_{r' \in \Upsilon} \sum_{s' \in \Psi} Q(r'|r) P(s'|s,r) \vartheta(r',s') \tag{17a}$$

$$= \ell(r+1,s) - \ell(r,s) + \lambda \Bigg[\sum_{r'=0}^{r} \Bigg(\big[Q(r'|r+1) - Q(r'|r)\big] \sum_{s' \in \Psi} P(s'|s,r) \vartheta(r',s') \Bigg) + \sum_{r'=r+1}^{R} \Bigg(\big[Q(r'|r+1) - Q(r'|r)\big] \sum_{s' \in \Psi} P(s'|s,r) \vartheta(r',s') \Bigg) \Bigg]. \tag{17b}$$

By Proposition 1 recall that $\vartheta(r',s')$ is nondecreasing in $s' \in \Psi$ for all $r' \in \Upsilon$. Then, because $\mathbf{P}(r+1) \succeq \mathbf{P}(r)$, Lemma 4.7.2 of [17] implies that $\sum_{s' \in \Psi} P(s'|s,r+1)\vartheta(r',s') \geq \sum_{s' \in \Psi} P(s'|s,r)\vartheta(r',s')$ for all $r' \in \Upsilon$, and this yields the inequality (17a). Now, for notational convenience, for $r' \in \Upsilon$, let $W(r') = \sum_{s' \in \Psi} P(s'|s,r)\vartheta(r',s')$. By Proposition 2, recall that $\vartheta(r',s')$ is nondecreasing in $r' \in \Upsilon$ for all $s' \in \Psi$. Therefore, $-W(r')$ is nonincreasing in $r' \in \Upsilon$. Then,

$$\sum_{r'=0}^{r} \Bigg(\big[Q(r'|r+1) - Q(r'|r)\big] \sum_{s' \in \Psi} P(s'|s,r) \vartheta(r',s') \Bigg) = \sum_{r'=0}^{r} \big[Q(r'|r+1) - Q(r'|r)\big] W(r')$$

$$= \sum_{r'=0}^{r} \big[Q(r'|r) - Q(r'|r+1)\big]\big[-W(r')\big] \geq -W(r) q(r). \tag{18}$$

Note that because $-W(r')$ is nonincreasing in $r' \in \Upsilon$, by the IFR property of $\mathbf{Q}$, the inequality in (18) follows from Lemma 1 part (i). Similarly, by the identity $\sum_{r'=r+1}^{R} \big[Q(r'|r) - Q(r'|r+1)\big] = -q(r)$ and Lemma 1 part (ii), we obtain

$$\sum_{r'=r+1}^{R} \Bigg(\big[Q(r'|r+1) - Q(r'|r)\big] \sum_{s' \in \Psi} P(s'|s,r) \vartheta(r',s') \Bigg) \geq W(r+1) d(r). \tag{19}$$

Then, by the definition of the function W, (17a), (17b), (18) and (19) imply that:

$$w(r+1,s) - w(r,s) \geq \ell(r+1,s) - \ell(r,s) + \lambda q(r) \sum_{s' \in \Psi} P(s'|s,r) \big[\vartheta(r+1,s') - \vartheta(r,s')\big]. \tag{20}$$

Note that

$$\sum_{s' \in \Psi} P(s'|s,r) \big[\vartheta(r+1,s') - \vartheta(r,s')\big]$$

$$\geq P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] + P(S|s,r)\big[\vartheta(r+1,S) - \vartheta(r,S)\big] \tag{21a}$$

$$= P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] + P(S|s,r)\big[C(r+1,S) - C(r,S)\big], \tag{21b}$$

where (21a) is implied by the fact that $\vartheta(r+1,s') \geq \vartheta(r,s')$ for all $s' \in \Psi$ $\big($by Proposition 2$\big)$ and (21b) follows from the definitions of $\vartheta(r,S)$ and $\vartheta(r+1,S)$. Now, because $\mathbf{Q}$ is IFR, $q(r) \geq 0$. By (21a) and (21b), this implies:

$$\begin{aligned} &q(r)\left(\sum_{s'\in\Psi} P(s'|s,r)\big[\vartheta(r+1,s') - \vartheta(r,s')\big]\right) \\ &\qquad \geq q(r)\bigg(P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] + P(S|s,r)\big[C(r+1,S) - C(r,S)\big]\bigg). \end{aligned} \tag{22}$$

By (20), (22) yields:

$$\begin{aligned} w(r+1,s) - w(r,s) \geq \ell(r+1,s) - \ell(r,s) + \lambda q(r)\bigg(&P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] \\ &+ P(S|s,r)\big[C(r+1,S) - C(r,S)\big]\bigg). \end{aligned}$$

(iii) Choose an arbitrary $r \in \Upsilon \setminus \{R\}$ and $s \in \Phi$, consider the following possible cases for $\vartheta(r+1,s) - \vartheta(r,s)$.

1. If $\vartheta(r+1,s) = \vartheta(0,0) + C(r+1,s)$, then

$$\vartheta(r+1,s) - \vartheta(r,s) \geq C(r+1,s) - C(r,s) \geq \min\big\{\eta(r,s), C(r+1,s) - C(r,s)\big\}, \tag{23}$$

   where the first inequality in (23) is implied by $\vartheta(r,s) \leq \vartheta(0,0) + C(r,s)$.

2. If $\vartheta(r+1,s) = w(r+1,s)$, then

$$\vartheta(r+1,s) - \vartheta(r,s) \geq w(r+1,s) - w(r,s) \tag{24a}$$

$$\begin{aligned} \geq \ell(r+1,s) - \ell(r,s) + \lambda q(r)\bigg(&P(s|s,r)\big[\vartheta(r+1,s) - \vartheta(r,s)\big] \\ &+ P(S|s,r)\big[C(r+1,S) - C(r,S)\big]\bigg), \end{aligned} \tag{24b}$$

   where (24a) is implied by $\vartheta(r,s) \leq w(r,s)$, and (24b) follows from (16b). Because $q(r) \leq 1$, $P(s|s,r) \leq 1$ and $\lambda < 1$, by rearranging terms in (24a) and (24b), we obtain

$$\begin{aligned} \vartheta(r+1,s) - \vartheta(r,s) &\geq \frac{\ell(r+1,s) - \ell(r,s) + \lambda q(r) P(S|s,r)\big[C(r+1,S) - C(r,S)\big]}{1 - \lambda q(r) P(s|s,r)} \\ &= \eta(r,s) \geq \min\big\{\eta(r,s), C(r+1,s) - C(r,s)\big\}. \end{aligned}$$

Now, we will prove the main result. Suppose (2) holds for some $\widehat{r} \in \Upsilon \setminus \{R\}$ and $\widehat{s} \in \Phi$. To establish the result, it is sufficient to show that $\vartheta(\widehat{r}+1,\widehat{s}) - \vartheta(\widehat{r},\widehat{s}) \geq C(\widehat{r}+1,\widehat{s}) - C(\widehat{r},\widehat{s})$. Consider the following possible cases for $\vartheta(\widehat{r}+1,\widehat{s}) - \vartheta(\widehat{r},\widehat{s})$.

1. If $\vartheta(\widehat{r}+1,\widehat{s}) = \vartheta(0,0) + C(\widehat{r}+1,\widehat{s})$, then because $\vartheta(\widehat{r},\widehat{s}) \leq \vartheta(0,0) + C(\widehat{r},\widehat{s})$, we have

$$\vartheta(\widehat{r}+1,\widehat{s}) - \vartheta(\widehat{r},\widehat{s}) \geq C(\widehat{r}+1,\widehat{s}) - C(\widehat{r},\widehat{s}).$$

2. If $\vartheta(\widehat{r}+1,\widehat{s}) = w(\widehat{r}+1,\widehat{s})$, then

$$\vartheta(\widehat{r}+1,\widehat{s}) - \vartheta(\widehat{r},\widehat{s}) \geq w(\widehat{r}+1,\widehat{s}) - w(\widehat{r},\widehat{s}) \tag{25a}$$

$$\geq \ell(\widehat{r}+1,\widehat{s}) - \ell(\widehat{r},\widehat{s}) + \lambda q(\widehat{r}) \sum_{s'\in\Psi} P(s'|s,r)\big[\vartheta(\widehat{r}+1,s') - \vartheta(\widehat{r},s')\big] \tag{25b}$$

$$= \ell(\widehat{r}+1,\widehat{s}) - \ell(\widehat{r},\widehat{s}) + \lambda q(\widehat{r})\Bigg(\sum_{s'\in\Phi} P(s'|\widehat{s},\widehat{r})\big[\vartheta(\widehat{r}+1,s') - \vartheta(\widehat{r},s')\big] + P(S|\widehat{s},\widehat{r})\big[C(\widehat{r}+1,S) - C(\widehat{r},S)\big]\Bigg) \tag{25c}$$

$$\geq \ell(\widehat{r}+1,\widehat{s}) - \ell(\widehat{r},\widehat{s}) + \lambda q(\widehat{r})\Bigg(\sum_{s'\in\Phi} P(s'|\widehat{s},\widehat{r}) \min\big\{\eta(\widehat{r},s'), C(\widehat{r}+1,s') - C(\widehat{r},s')\big\} + P(S|\widehat{s},\widehat{r})\big[C(\widehat{r}+1,S) - C(\widehat{r},S)\big]\Bigg) \tag{25d}$$

$$\geq C(\widehat{r}+1,\widehat{s}) - C(\widehat{r},\widehat{s}), \tag{25e}$$

where (25a) is implied by $\vartheta(\widehat{r},\widehat{s}) \leq w(\widehat{r},\widehat{s})$, (25b) follows from (16a), and (25c) is implied by the definitions of $\vartheta(\widehat{r},S)$ and $\vartheta(\widehat{r}+1,S)$. Then, (25d) and (25e) follow from (16c) and satisfying (15) for the pair $(\widehat{r},\widehat{s})$, respectively.

□

Note that the term $\lambda P(s|s,r)q(r)$ in condition (15) can be interpreted in a way that is analogous to the term $\lambda Q(r|r)d(r,s)$ in (2). It is also clear that satisfying conditions (2) and (15) for all possible pairs implies that the optimal replacement policy can be characterized as a monotone switching curve policy.

4 Special Case

When the environment is invariant, (i.e., $|\Upsilon| = 1$), our model reduces to the MDP formulation of a well-known optimal replacement problem introduced by Derman [3] and extended by Klein [9] and Kolesar [10], among others. For this special case, Kawai et al. [6] propose a sufficient condition for the optimality of a control-limit policy. Along with the IFR property of the system's deterioration matrix and Assumption 1, Kawai et al. [6] requires the following condition to be satisfied for all $s \in \Phi \setminus \{S-1\}$:

$$\ell(0,s+1) - \ell(0,s) \geq C(0,s+1) - C(0,s). \tag{26}$$

Note that because the environment is invariant in this special case, $R = 0$ or equivalently $Q(0|0) = 1$. Therefore, after some simplification, our condition (2) can be written as

$$\ell(0,s+1) - \ell(0,s) \geq [1 - \lambda d(0,s)]\,[C(0,s+1) - C(0,s)]\,. \tag{27}$$

It is clear that $d(0,s) \leq 1$, and by the IFR property of $\mathbf{P}(0)$, $d(0,s) \geq 0$. Then, since $\lambda < 1$, we have that $\lambda d(0,s) \leq 1$. Therefore, by Assumption 1, the right-hand side of (27) is less than or equal to the right-hand side of (26). Hence, our condition is weaker than that of Kawai et al. [6]. In particular, for systems with deterioration matrices with large diagonal entries, inequality (27) represents a much weaker condition than (26).

5 Numerical Illustration

In this section, we present a numerical example to show that the conditions that we present by Theorems 1 and 2 are not hard to satisfy. We also present the resulting control-limit structure of the optimal replacement policy.

We assume $\Upsilon = \Psi = \{0, 1, 2, \ldots, 9\}$, where $s = 0$ refers to a new unit and $s = 9$ corresponds to a failed unit. Likewise, environment state $r = 0$ is least detrimental while state $r = 9$ is most detrimental. We define the immediate operating costs for each state as follows:

$$\ell(r, s) = \begin{cases} s + 1 & \text{for } r = 0 \text{ and } s \in \Phi, \\ \ell(r-1, s) + 3 & \text{for } r \in \Upsilon \setminus \big\{0\big\} \text{ and } s \in \Phi; \end{cases}$$

Likewise, the replacement costs are given by

$$C(r, s) = \begin{cases} 1500 + 10s & \text{for } r = 0 \text{ and } s \in \Phi, \\ 2000 & \text{for } r = 0 \text{ and } s = S, \\ C(r-1, s) + 200 & \text{for } r \in \Upsilon \setminus \big\{0\big\} \text{ and } s \in \Psi. \end{cases}$$

The IFR transition probability matrix of the environment ($\mathbf{Q}$) is given by

$$Q(r'|r) = \begin{cases} 0.99 & \text{for } r = r' \in \Upsilon, \\ 0.01 & \text{for } r = 0 \text{ and } r' = 1, \text{ or } r = 9 \text{ and } r' = 8, \\ 0.005 & \text{for } r \in \Upsilon \setminus \big\{0, 9\big\} \text{ and } r' \in \big\{r-1, r+1\big\}, \\ 0 & \text{otherwise.} \end{cases}$$

To define the transition matrix of the system's deterioration process, $\mathbf{P}(r)$, $r \in \Upsilon$, we define two functions, $\rho : \Upsilon \to \mathbb{R}_+$ and $\kappa : \Phi \to \mathbb{R}_+$, and a stochastic matrix $\mathbf{K} = \big[K(s', s)\big]_{s,s' \in \Phi}$ such that $\rho(r) = 1 + 0.05r$ for $r \in \Upsilon$ and $\kappa(s) = 0.01 + 0.005s$ for $s \in \Phi$. Then, we define the matrix $\mathbf{K}$ as:

$$K(s', s) = \begin{cases} 0.99 & \text{for } s = s', \\ 0.01 & \text{for } s = 0 \text{ and } s' = 1, \text{ or } s = 8 \text{ and } s' = 7, \\ 0.005 & \text{for } s \in \Phi \setminus \big\{0, 8\big\} \text{ and } s' \in \big\{s-1, s+1\big\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using ρ, κ and $\mathbf{K}$, we define the transition probability matrices, $\mathbf{P}(r)$, $r \in \Upsilon$, as follows:

$$P(s'|s, r) = \begin{cases} \kappa(s)\rho(r) & \text{for } s \in \Phi,\ s' = S, \text{ and } r \in \Upsilon, \\ \big[1 - \kappa(s)\rho(r)\big]K(s', s) & \text{for } s, s' \in \Phi \text{ and } r \in \Upsilon, \\ 1 & \text{for } s = s' = S \text{ and } r \in \Upsilon, \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily verified that these parameters satisfy Assumptions 1–5 as well as conditions (2) and (15). Using Theorems 1 and 2, and assuming ties are broken arbitrarily in favor of replacement, the optimal replacement policy is expected to follow a control-limit structure in both r and s.

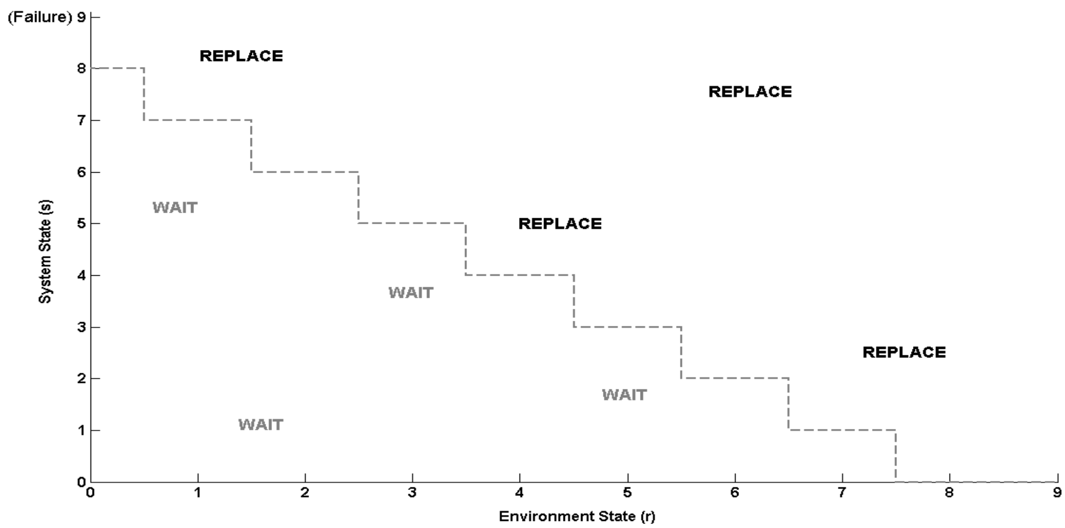

Figure 1: The optimal replacement policy for the numerical example.

Figure 1 depicts the resulting optimal policy. The regions corresponding to wait and replacement actions are separated by a monotone switching curve (the curve is included in the replacement region). Therefore, the decision maker is more likely to replace the system as its deterioration status worsens and/or the environment becomes more detrimental to the system. For instance, if the environment is in state $r = 1$, replacement is optimal if the system's deterioration status corresponds to state $s = 7$ or worse, whereas replacement is optimal, regardless of the system's status, when the environment is in the most detrimental state.

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