Optimal Control of a Two-Server Queueing System with Failures

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Abstract

We consider the problem of controlling a two-server Markovian queueing system with heterogeneous servers. The servers are differentiated by their service rates and reliability attributes (i.e., the slower server is perfectly reliable while the faster server is subject to random failures). The aim is to dynamically route customers at arrival, service completion, server failure, and server repair epochs to minimize the long-run average number of customers in the system. Using a Markov decision process model, we prove that it is always optimal to route customers to the faster server when it is available, irrespective of its failure and repair rates, if the system is stable. For the slower server, there exists an optimal threshold policy that depends on the queue length and the state of the faster server. Additionally, we analyze a variant of the main model in which there are multiple unreliable servers with identical service rates, but distinct reliability characteristics. For that case it is always optimal to route customers to idle servers, and the optimal policy is insensitive to the servers' reliability characteristics.

Keywords: Routing control; Markov decision process; threshold policy.
AMS Classification: Primary 60K25; Secondary 90C40

1 Introduction

We consider the problem of dynamically and optimally controlling a two-server queueing system with heterogeneous servers, server failures and a common queue. The two servers are differentiated by their service rates and the fact that the faster server is unreliable while the slower server is reliable. The objective is to route customers either to an available server, or to the queue, at certain event epochs so as to minimize the long-run average number of customers in the system. Specifically, routing decisions are made at each arrival, service completion, active failure, and repair completion epoch; however, because the queue has infinite capacity, customers are not denied admission (i.e., we do not consider admission control). Our main model seeks to answer at least the following two questions: (1) Given the unreliability of the faster server, when should customers be routed to this server, and when should they be routed to the slower reliable one? and (2) Where should customers be routed when active server failures occur (i.e., when a service cycle is interrupted by a failure at the unreliable server)? We answer these questions by way of a Markov decision process (MDP) model. For the main model (described in Section 2), we prove that it is always optimal to route customers to the faster server when it is available, and we establish the optimality of a threshold

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policy for the slow server that depends on the queue length and the status of the fast server. Additionally, using the MDP framework, we analyze a variant of the main model in which there are K ($K \geq 2$) unreliable servers with identical service rates but distinct reliability characteristics. We prove that it is always optimal to route customers to idle servers, and that the optimal policy is insensitive to the reliability attributes of the servers.

The problem we address here for the two-server case is an extension of the so-called "slow server problem" to include failure-prone servers. The original slow server problem seeks to route customers to two heterogeneous (reliable) servers so as to minimize the mean time in system. The problem was first introduced by Larsen [9] who conjectured (but did not prove) that the optimal policy for routing customers to a particular server was a threshold policy, i.e., for server i, there exists a threshold value q_i such that when the queue length is above q_i, it is optimal to route a customer to that server. Subsequently, Larsen and Agrawala [10] proposed a simple approximation to find the optimal threshold value for the slower server, although they did not prove the optimality of the threshold policy. Using an MDP model, Lin and Kumar [11] first established the optimality of a threshold policy for each of the servers. They showed that the threshold value for the fast server is zero (i.e., a customer should be routed to the fast server whenever possible), and the threshold value for the slow server depends on the queue length. Their main results were obtained using value and policy iteration, respectively. Walrand [19] and Koole [8] also proved the optimality of a threshold policy using coupling arguments and value iteration, respectively. Rubinovitch [15] considered the same slow server problem and analyzed the performance of the system. He proposed a numerical algorithm to obtain the optimal threshold value for the slow server. Viniotis and Ephremides [18] first extended the Poisson arrival assumption to the case of Erlang-distributed interarrival times with an arbitrary shape parameter. Then, independent of that work, they considered Erlang-distributed service times for the slow server. Finally, and independently of the prior cases, they considered generally distributed interarrival times, subject to a few constraints. Xu [22] considered admission control for the slow server problem in which the objective is to minimize the total long-run average cost, which includes customer holding costs that are offset by rewards gained at service completion epochs. Xu [22] proved the optimality of a threshold policy for each server, as well as the existence of a threshold queue length for denying admission to the system.

Rykov [16] attempted to generalize the slow server problem to the case of multiple (more than two) heterogeneous and reliable servers that are distinguished by their service rates. Although the author claimed to have proved the optimality of a threshold policy for each server, de Véricourt and Zhou [5] showed that the proofs in [16] are incomplete. Luh and Viniotis [12] proved the optimality of a threshold policy for each server in the multiple server case by formulating the optimal control problem as a linear program and exploiting the problem's structure. They showed that it is optimal to route to the fastest available server when the queue length exceeds a threshold value associated to that server. Subsequently, de Véricourt and Zhou [4] addressed the multiple server case in a call center routing problem assuming a feedback mechanism. Specifically, server i uses server rate μ_i, and a service cycle is successful with probability p_i (and unsuccessful with probability $1 - p_i$). If service is unsuccessful, the customer can be routed back to the queue or to another available server. Under some mild conditions, the authors proved the optimality of a $p\mu$-type rule, i.e., the customer should be routed to the (available) server with the highest $p_i\mu_i$ value, given that it is optimal to route the customer to a server. Additionally, for the two-server case, de Véricourt and Zhou [4] established the optimality of a threshold policy for both servers.

Kim, et al. [7] considered the multiple server case of the slow server problem assuming two customer classes: primary and secondary. Primary customers have non-preemptive priority over secondary customers and arrive according to a Poisson process. It is assumed that there are infinitely many secondary customers so that the servers are never idle, i.e., if a primary job is not assigned to a server, then the server works on secondary jobs. Their objective is to minimize the long-run average number of primary customers in the system. The authors proved that the optimal policy for each server is of threshold type with respect to the number of primary customers in the queue. This was done by first showing that the individual optimal policy for a primary customer is of threshold type and, subsequently, that the individual optimal policy is also socially optimal. An interesting point in [7] is that the inclusion of a secondary customer class actually simplifies the proofs significantly.

The dynamic control of queueing systems with *unreliable* servers has received only modest attention in the queueing literature. Efrosinin [6] examined the optimal allocation of customers in an $M/M/2$ queueing system with heterogeneous servers differentiated by their service rates and reliability attributes. Specifically, the faster server is subject to partial or complete failures, and the slower server is perfectly reliable. His aim was to prescribe a control policy to minimize the long-run average number of customers in the system. However, as in Rykov [16], some key arguments in Efrosinin [6] related to the optimality of a threshold policy are incomplete, as recently noted by Özkan and Kharoufeh [13]. The dynamic control of queues with unreliable servers has also been studied in the context of flexible manufacturing systems. Such systems are comprised of reconfigurable resources that can be dynamically allocated to compensate for failures. Wu et al. [21] considered a two-station tandem queueing network (with no external arrivals) in which there are dedicated servers at each station, as well as flexible servers that can be used by either station. The servers are differentiated by their service, failure, and repair rates. They minimized the long-run average holding cost by dynamically allocating the flexible servers to the stations and showed that the optimal policy is characterized by switching curves. Those results were extended by Wu et al. [20] to include external arrivals to the system, while some heuristics were developed for the case of more than two stations. Andradóttir et al. [1] examined a system with multiple customer classes and failure-prone servers and maximized the long-run average throughput rate by dynamically assigning servers to customers in the system. They provided heuristic, round-robin policies to maximize the throughput. Andradóttir et al. [2] considered a tandem queueing network with unreliable servers and showed that server unreliability does not change the structure of the optimal policy in some specific cases. They first considered a tandem queue with arbitrary numbers of stations and unreliable servers (which are generalists). It was shown that any non-idling server assignment policy maximizes the long-run average throughput (and the same is true when the servers are reliable). They also considered non-generalist servers in a Markovian two-station system (with two and three servers, respectively). The structures of optimal polices were shown to be insensitive to unreliability of the servers.

The primary objective of the present paper is to examine the slow server problem with two servers in the presence of server failures. It is well known (see Lin and Kumar [11]) that, if both servers are perfectly reliable, then it is always optimal to route customers to the faster server when it is available. Our aim is to determine whether the same policy is optimal when the faster server can fail. Moreover, we seek to characterize the optimal operating policy for the slower server. The main results are as follows. We first show that, if only the faster server is unreliable, then the optimal policy is to route customers to this server when it is idle and not failed. Second, we establish the optimality of a threshold policy for the slower server that depends not only on the queue length, but also on the status of the faster server. That is, the threshold queue length depends on whether the faster server is busy or failed. Additionally, within this framework, we consider a variant of the main model in which there are K ($K \geq 2$) unreliable servers with identical service rates but different reliability characteristics. Specifically, we show that it is optimal to keep all servers busy whenever possible, irrespective of their failure and repair rates. With the exception of [6], to the best of our knowledge, a complete examination of the slow server problem with unreliable servers has not appeared in the applied probability or queueing literature.

The remainder of the paper is organized as follows. Section 2 provides the formal queueing model description and the MDP model formulation. Section 3 presents some preliminary results needed to establish the optimality of threshold policies. Section 4 presents the optimal policies for the slow server and the faster one and shows that these policies are also optimal under the average cost criterion. Section 5 examines an $M/M/K$ variant of the main model. Finally, some concluding remarks and directions for future work are provided in Section 6.

2 Main Model and MDP Formulation

Customers arrive to a common, infinite-capacity queue according to a temporally-homogeneous Poisson process with rate λ ($0 < \lambda < \infty$). The queue is served by two parallel servers which are differentiated by their service rates and the fact that the faster server is subject to failures occurring at random intervals. The (uninterrupted) service times at servers 1 and 2 are exponentially distributed with parameters μ_1 and μ_2, respectively, with $0 < \mu_2 \leq \mu_1 < \infty$. While server 2 can be either idle or busy, server 1 can be idle, busy, or failed. (For succinctness, we refer to an up and idle server simply as idle in the remainder of the paper.) Server 1 is subject to failures that occur either when the server is busy (an *active failure*), or when it is idle (an *idle failure*); however, failures cannot occur when the server is under repair. We assume active and idle failures occur according to a Poisson process with the same rate ξ ($0 < \xi < \infty$). A dedicated repair person is assigned to server 1; therefore, when this server fails, its repair begins immediately. Repair times are independent and identically distributed (i.i.d.) exponential random variables with mean $1/\alpha$ ($0 < \alpha < \infty$).

A system controller, who has perfect knowledge of the state of each server and the queue length, routes customers either to one of the servers or to the queue at each arrival, service completion, active failure, and repair completion epoch. The objective is to minimize the long-run average number of customers in the system. The dynamics of the system are as follows. First, at each arrival epoch, the arriving customer joins the queue, and the controller can route one of the customers in the queue (not necessarily the arriving customer) to an idle server. Second, at service or repair completion epochs, if queue is not empty, the controller may choose to route a customer from the queue to an idle server, or do nothing. Finally, when an active failure occurs at server 1, the customer in service is removed from the server and routed either to server 2, if it is idle, or to the queue. Note that whenever a customer is routed to the queue, its position in the queue is inconsequential as our objective is the minimization of long-run average number of customers in the system. Similarly, when a customer is pulled from the queue, it need not be the customer at the head of the line. The maximum throughput rate of this system is

$$\bar{\mu} = \left(\frac{\alpha}{\alpha + \xi}\right)\mu_1 + \mu_2, \tag{1}$$

where $\alpha/(\alpha+\xi)$ is the steady state proportion of time server 1 is not failed. We will assume $\lambda < \bar{\mu}$ for stability. For $t \geq 0$, let $N(t)$ denote the number of customers waiting in the queue (not including those in service), and let $X_1(t)$ denote the status of server 1 at time t so that

$$X_1(t) = \begin{cases} 0, & \text{if server 1 is idle at } t, \\ 1, & \text{if server 1 is busy at } t, \\ -1, & \text{if server 1 is failed at } t. \end{cases}$$

Similarly, $X_2(t) = 0$ ($X_2(t) = 1$) if server 2 is idle (busy) at time t.

The state of the system is described by the stochastic process, $(N, X_1, X_2) \equiv \{(N(t), X_1(t), X_2(t) : t \geq 0\}$, a continuous-time Markov chain (CTMC) on the state space $E = \{0, 1, 2, \dots\} \times \{0, 1, -1\} \times \{0, 1\}$. Therefore, a state $x \in E$ is a triplet of the form,

$$x = (n(x), d_1(x), d_2(x)) \tag{2}$$

where $n(x)$ denotes the number of customers in the queue, and $d_j(x)$ is the status of server j in state $x \in E$, $j = 1, 2$. To simplify notation, we partition the possible server states as follows:

$$\Psi_0(x) = \{j \in \{1, 2\} : d_j(x) = 0 \text{ in state } x \in E\},$$
$$\Psi_1(x) = \{j \in \{1, 2\} : d_j(x) = 1 \text{ in state } x \in E\},$$

and $\Psi_{-1}(x) = \{1\}$ if $d_1(x) = -1$ in state $x \in E$ with $\Psi_{-1}(x) = \emptyset$ otherwise. That is, $\Psi_0(x)$, $\Psi_1(x)$, and $\Psi_{-1}(x)$ denote the sets of idle, busy, and failed servers in state $x \in E$, respectively. The set of admissible actions in state $x \in E$, denoted by $A(x)$, is given by

$$A(x) = \Psi_0(x) \cup \{0\}. \tag{3}$$

Let $a \in A(x)$ be an action in state $x \in E$ such that if $a = j \in \Psi_0(x)$ is chosen, then the customer is routed to the idle server with index j, $j = 1, 2$. On the other hand, the action $a = 0$ means that the customer is routed to the queue. As in [16], we employ shift operators to describe state transitions. Define for each $k \in \{0, 1, 2\}$, e_k as the three-dimensional unit vector with each element equal to zero except the $(k+1)$st element, which is unity. The shift operator applied to vector x under action $a \in A(x)$ is denoted by $S_a x$ and defined by

$$S_a x = \begin{cases} x + e_0, & \text{if } a = 0, \\ x + e_j \, \mathbb{I}_{\{j \in \Psi_0(x)\}}, & \text{if } a = j, \end{cases}$$

where $\mathbb{I}_{\{U\}}$ is the indicator function for event U. For instance, $S_0 x$ shifts vector x by incrementing the number of customers in the queue by unity and leaving the status of the servers unaltered. The operation $S_j x$ transitions an idle server, $j \in \Psi_0(x)$, to the busy state, or it transitions a failed server, $j \in \Psi_{-1}(x)$, to the idle state. We also define an inverse shift operator to account for changes of state resulting from the stochastic evolution of the system (i.e., those not necessarily due to actions). Specifically, for $j \in \{1,2\}$,

$$S_j^{-1}x = x - e_j\,\mathbb{I}_{\{j \notin \Psi_b(x)\}},$$

where, if $j=1$, then $b=-1$; otherwise, $b=0$ and $S_0^{-1}x = x - e_0$. The operator S_j^{-1} alters state x by either (a) making a busy server idle (service completion), or (b) making an idle server failed (idle failure). Whenever we write $S_1^{-2}x$, we mean $S_1^{-1}S_1^{-1}x$, which denotes an active failure of server 1.

We model the system as a Markov decision process (MDP) and first consider the objective of minimizing the discounted expected total number of customers in the system. Subsequently, in Section 4.3, we show that the optimal control policy does not change under the long-run average cost criterion. Let β denote the discount parameter with $\beta > 0$. The discounted expected total number of customers in the system, given the initial state $x_0 \in E$, is

$$V(x_0) = \inf_{\theta\in\Pi}\left\{\limsup_{n\to\infty}\mathbb{E}_\theta\left[\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}} \mathrm{e}^{-\beta t}\left(N(t) + \sum_{j=1}^{2}\mathbb{I}_{\{X_j(t)=1\}}\right)\mathrm{d}t \middle| (N(0),X_1(0),X_2(0)) = x_0\right]\right\},$$

where Π is the set of all admissible policies and t_m denotes the time of the mth decision epoch. For simplicity, we convert the continuous-time MDP model to an equivalent discrete-time version by the method of uniformization (see Chapter 11 of [14] or Chapter 5 of [3] for more additional details about uniformizing continuous-time MDP models). Let

$$\nu = \lambda + \mu_1 + \mu_2 + \xi + \alpha$$

denote the maximum possible rate out of any state in E. Then, the effective discount factor in the uniformized MDP is $\beta' = \nu/(\beta+\nu)$, and the discounted expected total number of customers in the uniformized model, given the initial state $x_0 \in E$, is

$$V(x_0) = \inf_{\theta\in\Pi}\left\{\limsup_{n\to\infty}\mathbb{E}_\theta\left[\sum_{m=0}^{n-1}(\beta')^m L_m \middle| (N(0),X_1(0),X_2(0)) = x_0\right]\right\}, \tag{4}$$

where L_m denotes the number of customers in the system just after the mth transition. From this point forward, we will consider the objective in (4). Let $r(x)$ denote the total number of customers in the system (those in the queue and those being served) in state $x \in E$, then

$$r(x) = n(x) + \sum_{j=1}^{2}\mathbb{I}_{\{j\in\Psi_1(x)\}}.$$

Finally, we define an operator T on an arbitrary function $v : E \to \mathbb{R}$ as

$$Tv(x) \equiv \min_{a\in A(x)} v(S_a x). \tag{5}$$

Bellman's optimality equation in the uniformized model is

$$V(x) = \frac{1}{\nu+\beta}\Bigg\{r(x) + \lambda TV(x) + \sum_{j\in\Psi_1(x)}\mu_j Y_j(V(x)) + \mathbb{I}_{\{1\in\Psi_1(x)\}}\xi TV(S_1^{-2}x)$$
$$+\mathbb{I}_{\{1\in\Psi_{-1}(x)\}}\alpha Z(V(x)) + \mathbb{I}_{\{1\in\Psi_0(x)\}}\xi V(S_1^{-1}x) + p(x)W(V(x))\Bigg\} = BV(x) \tag{6}$$

where B is the operator defined by (6). When $j \in \Psi_1(x)$,

$$Y_j(V(x)) = \begin{cases} TV(S_0^{-1}S_j^{-1}x), & \text{if } n(x) > 0, \\ V(S_j^{-1}x), & \text{if } n(x) = 0, \end{cases} \tag{7}$$

when $1 \in \Psi_{-1}(x)$,

$$Z(V(x)) = \begin{cases} TV(S_0^{-1}S_1x), & \text{if } n(x) > 0, \\ V(S_1x), & \text{if } n(x) = 0, \end{cases} \tag{8}$$

and $p(x)$, the rate of remaining in state $x \in E$, is given by

$$p(x) = \nu - \lambda - \sum_{j \in \Psi_1(x)} \mu_j - \mathbb{I}_{\{1 \in \Psi_1(x)\}}\xi - \mathbb{I}_{\{1 \in \Psi_{-1}(x)\}}\alpha - \mathbb{I}_{\{1 \in \Psi_0(x)\}}\xi. \tag{9}$$

Finally,

$$W(V(x)) = \begin{cases} TV(S_0^{-1}x), & \text{if } n(x) > 0, \\ V(x), & \text{if } n(x) = 0. \end{cases} \tag{10}$$

We now describe the terms in the bracketed expression of (6) in greater detail. The term $r(x)$ in (6) is the number of customers in the system in state $x \in E$. The next term, $\lambda TV(x)$, represents the transitions due to customer arrivals. When a customer arrives to the system, it immediately joins the queue and, simultaneously, the system controller has the option to route a customer from the queue to an idle server, and the operator T, defined in (5), reflects this option. The next term represents transitions due to service completions. Following a service completion, if the queue is nonempty, the system controller has the option to route a customer from the queue to an idle server, and the operator Y_j, defined in (7), reflects this option. The fourth term represents the transitions due to active failures of server 1. At these failure times, the system controller must route the interrupted customer either to the queue or to server 2, if it is idle. The fifth term represents the transitions due to repairs of server 1. At repair completion times, if the queue is nonempty, then the system controller has the option to route a customer from the queue to an idle server, and operator Z, defined in (8), reflects this option. The sixth term represents the transitions due to idle failures of server 1. If an idle failure occurs, the system controller takes no action. The last term, $p(x)W(V(x))$, represents the fictitious transition epochs. Normally, the state of the uniformized MDP is unchanged at a fictitious transition epoch (cf. [3, 14]). However, by using the operator W defined in (10), we allow actions at the fictitious transition epochs as in [4, 8] to facilitate the proofs. This adjustment is inconsequential because, as will be shown in Proposition 4, the system controller does not change the state of the system at a fictitious transition epoch if the optimal policy is followed. Without loss of generality, we assume $\nu + \beta = 1$, i.e., we re-scale time to simplify the proofs.

2.1 Convergence of the Value Iteration Algorithm

We will make extensive use of the value iteration algorithm (see Puterman [14]) in the proofs that follow. However, because the state space of the MDP model is countably infinite, and the costs per stage are unbounded, the existence of an optimal stationary policy and convergence of the algorithm need to be verified. To this end, we will first prove that our model satisfies Assumptions

6.10.1 and 6.10.2 of [14] (p. 232). Subsequently, by Theorem 6.10.4 of [14] (p. 236), we can conclude the existence of an optimal stationary policy and the convergence of the value iteration algorithm.

In this section, in order to check the Assumptions 6.10.1 and 6.10.2 of [14], we convert the uniformized MDP model described in Section 2 to the format described at Chapter 6 of Puterman [14]. We first convert the three-dimensional state space of the uniformized MDP to a one-dimensional equivalent state space. Define $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, and let $\Delta : E \to \mathbb{N}_0$ be a one-to-one mapping with the following form:

$$\Delta\left((n(x), d_1(x), d_2(x))\right) = \begin{cases} 6\,n(x), & \text{if } d_1(x) = 0,\ d_2(x) = 0, \\ 6\,n(x) + 1, & \text{if } d_1(x) = 1,\ d_2(x) = 0, \\ 6\,n(x) + 2, & \text{if } d_1(x) = -1,\ d_2(x) = 0, \\ 6\,n(x) + 3, & \text{if } d_1(x) = 0,\ d_2(x) = 1, \\ 6\,n(x) + 4, & \text{if } d_1(x) = 1,\ d_2(x) = 1, \\ 6\,n(x) + 5, & \text{if } d_1(x) = -1,\ d_2(x) = 1. \end{cases} \tag{11}$$

Then, the mapping Δ creates a one dimensional equivalent state space for the uniformized MDP model. For example, $\Delta((2, 1, 0)) = 13$, i.e., the state $(2, 1, 0) \in E$ is equivalent to state 13 in the one dimensional state space case. Second, we make the following definitions parallel to Puterman [14]. Let A_s denote the set of actions in state $s \in \mathbb{N}_0$, $\bar{r}(s, a)$ denotes the cost of taking action $a \in A_s$ in state $s \in \mathbb{N}_0$, $p(j|s, a)$ denotes the probability that the system occupies state $j \in \mathbb{N}_0$ after action $a \in A_s$ in state $s \in \mathbb{N}_0$ is taken, D denotes the set of Markovian deterministic decision rules, $\pi = (q_1, \ldots, q_J)$ denotes a J stage policy where $J \in \mathbb{N}$ and $q_k \in D$ for $1 \leq k \leq J$; $P_\pi^J(j|s)$ denotes the probability of occupying state j after J steps by using policy π starting from state $s \in \mathbb{N}_0$. In this new model formulation, the action process can be described as follows. In case of customer arrivals to the system, the arriving customer first joins the queue and subsequently the system controller takes a customer from the queue an routes it to an idle server or back to queue. In case of an active failure in server 1, we can assume that the customer whose service is interrupted is first routed to the queue and subsequently the system controller decides to route the same customer either back to queue or to an idle server. At all other decision epochs, the system controller takes a customer from the queue and routes it either to queue or an idle server. Since cost per stage is the number of customers in the system and an action moves a customer among the queue and the idle servers, taking any action in any state has the same cost which is the number of customers in the system in the corresponding state. In other words, $\bar{r}(s, a)$ is constant with respect to any a, $a \in A_s$ in each state $s \in \mathbb{N}_0$ and equal to number of customers in the system in state s. Therefore, by (11), for all $a \in A_s$,

$$\bar{r}(s, a) = \begin{cases} \lfloor s/6 \rfloor, & \text{if } (s \mod 6) = 0 \text{ or } 2, \\ \lfloor s/6 \rfloor + 1, & \text{if } (s \mod 6) = 1 \text{ or } 3 \text{ or } 5, \\ \lfloor s/6 \rfloor + 2, & \text{if } (s \mod 6) = 4, \end{cases} \tag{12}$$

where $\lfloor \cdot \rfloor$ is the floor function. For example, since $\Delta((2, 1, 0)) = 13$, then $A_{13} = \{0, 2\}$, i.e., the action set in state 13 is to first take a customer from queue and then route it to either queue or server 2, and $\bar{r}(13, a) = 3$ for all $a \in A_{13}$.

Assumption 6.10.1 of [14] states that there exists a constant $u < \infty$ and a positive real valued function on $\mathbb{N}_0$, say w, satisfying $\inf_{s\in\mathbb{N}_0} w(s) > 0$ such that for all $s \in \mathbb{N}_0$,

$$\sup_{a\in A_s} |\bar{r}(s,a)| \leq u\, w(s). \tag{13}$$

Let $w(s) = \lfloor s/6 \rfloor + 2$ for all $s \in \mathbb{N}_0$ and $u = 1$. Then $w : \mathbb{N}_0 \to \mathbb{R}$ is nondecreasing and $\inf_{s\in\mathbb{N}_0} w(s) \geq 2$, and by (12), it is clear that these particular w and u satisfy (13).

Assumption 6.10.2.a of [14] states that there exists a constant $\kappa \in [0, +\infty)$ such that for all $a \in A_s$ and $s \in \mathbb{N}_0$,

$$\sum_{j\in\mathbb{N}_0} p(j|s,a)w(j) \leq \kappa\, w(s). \tag{14}$$

Since at each action, the system controller can relocate a single customer (by routing it to the queue or an idle server), by taking an action in state $s \in \mathbb{N}_0$, the next state j is an element of the set $\{s-11, \ldots, s+11\}$ by (11). For example, $\Delta((2,1,0)) = 13$ and $A_{13} = \{0,2\}$, i.e., the action set in state 13 is to first take a customer from queue and then route it to either queue or server 2. If the system controller chooses action 0, then the new state will be $\Delta((2,1,0)) = 13 \in \{2,\ldots,24\}$, on the other hand if the system controller chooses action 2, then the new state will be $\Delta((1,1,1)) = 10 \in \{2,\ldots,24\}$. Then,

$$\begin{aligned}\sum_{j\in\mathbb{N}_0} p(j|s,a)w(j) = \sum_{j=s-11}^{s+11} p(j|s,a)w(j) \leq \sum_{j=s-11}^{s+11} w(j) \leq 23w(s+11) \\ = 23\left(\left\lfloor \frac{s+11}{6} \right\rfloor + 2\right) \leq 23\left(\left\lfloor \frac{s}{6} \right\rfloor + 4\right) \leq 46\left(\left\lfloor \frac{s}{6} \right\rfloor + 2\right) = 46w(s).\end{aligned}$$

Hence, if $\kappa = 46$, then (14) is satisfied, so does Assumption 6.10.2.a of [14].

Lastly, Assumption 6.10.2.b of [14] states that for each $\gamma \in [0,1)$, there exists a $\rho \in [0,1)$ and an integer J_0 such that

$$\gamma^{J_0} \sum_{j\in\mathbb{N}_0} P_\pi^{J_0}(j|s)w(j) \leq \rho w(s) \tag{15}$$

for all $\pi = (q_1, \ldots, q_{J_0})$ where $q_k \in D$, $1 \leq k \leq J_0$. Let γ be an arbitrary number in $[0,1)$, $\rho = \gamma$, and J be a positive integer. Then,

$$\begin{aligned}\gamma^J \sum_{j\in\mathbb{N}_0} P_\pi^J(j|s)w(j) &= \gamma^J \sum_{j=s-11J}^{s+11J} P_\pi^J(j|s)w(j) \leq \gamma^J \sum_{j=s-11J}^{s+11J} w(j) \\ &\leq \gamma^J (22J+1)w(s+11J) = \gamma^J 23J\left(\left\lfloor \frac{s+11J}{6} \right\rfloor + 2\right) \\ &\leq \gamma^J 23J\left(\left\lfloor \frac{s}{6} \right\rfloor + 2J + 2\right) \leq 23\gamma^J J(J+1)\left(\left\lfloor \frac{s}{6} \right\rfloor + 2\right).\end{aligned}$$

Since $\gamma \in [0,1)$, there exists $J_0 \in \mathbb{N}$ such that $\gamma^{J_0-1}(J_0^2 + J_0) \leq 1/23$ because γ^{J_0-1} decreases exponentially whereas $(J_0^2 + J_0)$ increases with polynomial degree of 2. Hence, for all $\gamma \in [0,1)$ and $\pi = (q_1, \ldots, q_{J_0})$ where $q_k \in D$, $1 \leq k \leq J_0$ and $\gamma^{J_0-1}(J_0^2 + J_0) \leq 1/23$;

$$\gamma^{J_0} \sum_{j\in\mathbb{N}_0} P_\pi^{J_0}(j|s)w(j) \leq 23\gamma^{J_0}(J_0^2 + J_0)w(s) \leq \rho w(s).$$

Then inequality (15) is satisfied, and so is Assumption 6.10.2.b of Puterman [14]. Therefore, by Theorem 6.10.4 of [14] (p. 236), there exists an optimal stationary policy, and the value iteration algorithm converges.

3 Preliminaries

In this section, we prove some properties of the operators T and B introduced in Section 2. Let $v : E \to \mathbb{R}$ be a real-valued function, and let Θ be the set of all v satisfying

$$v(x) \le v(S_0 x), \quad \forall\, x \in E, \tag{16a}$$
$$v(x) \le v(S_j x), \quad \forall\, x \in E \text{ such that } j \in \Psi_0(x), \tag{16b}$$
$$v(S_1 x) \le v(S_2 x), \quad \forall\, x \in E \text{ such that } \Psi_0(x) = \{1, 2\}, \tag{16c}$$
$$v(S_1 x) \le v(S_0 x), \quad \forall\, x \in E \text{ such that } 1 \in \Psi_0(x). \tag{16d}$$

Note that $\Theta \neq \emptyset$ because any constant function $v : E \to \mathbb{R}$ is in Θ. We first present the following lemma to which we will appeal frequently in our proofs.

Lemma 1 *Let $\{y_1, \ldots, y_n\}$ and $\{z_1, \ldots, z_m\}$ be two sets of real numbers for some $n, m \in \mathbb{N}$. If for each $i \in \{1, \ldots, m\}$ there exists some $j \in \{1, \ldots, n\}$ such that $y_j \le z_i$, then $\min\{y_1, \ldots, y_n\} \le \min\{z_1, \ldots, z_m\}$.*

Next, we prove that the operator T preserves the properties of functions belonging to Θ (inequalities (16a) – (16d)).

Proposition 1 *If $v \in \Theta$, then $Tv \in \Theta$, where the operator T is defined by (5).*

Proof. Let us first consider the preservation of property (16a) under T. Considering (3) and the fact that $\Psi_0(x) = \Psi_0(S_0 x)$,

$$\begin{aligned} Tv(x) - Tv(S_0 x) &= \min_{a \in A(x)} v(S_a x) - \min_{a \in A(S_0 x)} v(S_a S_0 x) \\ &= \min\{v(S_0 x), v(S_i x) : i \in \Psi_0(x)\} \\ &\qquad - \min\{v(S_0^2 x), v(S_i S_0 x) : i \in \Psi_0(x)\} \\ &\le 0 \end{aligned}$$

by (16a) and Lemma 1. Now, by (3), (16b), and Lemma 1,

$$\begin{aligned} Tv(x) - Tv(S_j x) &= \min\{v(S_0 x), v(S_i x) : i \in \Psi_0(x)\} \\ &\qquad - \min\{v(S_0 S_j x), v(S_i S_j x) : i \in \Psi_0(x) \setminus \{j\}\} \\ &\le 0. \end{aligned}$$

By (3), (16d), and Lemma 1,

$$Tv(S_1 x) - Tv(S_2 x) = \min\{v(S_0 S_1 x), v(S_2 S_1 x)\} - v(S_1 S_2 x) \le 0.$$

Finally, by (3), (16c), (16d), and Lemma 1, it is easy to see that

$$Tv(S_1 x) - Tv(S_0 x) = \min\{v(S_0 S_1 x), v(S_i S_1 x) : i \in \Psi_0(S_1 x)\} - v(S_1 S_0 x) \le 0.$$

■

It is also important to show that the operator B preserves inequalities (16a) through (16d), which will play a crucial role in the proof of the optimal policies for both servers.

Proposition 2 *If $v \in \Theta$ then $Bv \in \Theta$, where B is defined by* (6).

Proof. We will show that B preserves inequalities (16a), (16b), (16c), and (16d) separately. We first show that if $v \in \Theta$, then $Bv(x) \leq Bv(S_0x)$ for all $x \in E$. It is clear that $r(x) - r(S_0x) = -1$ and $p(x) = p(S_0x)$ by (9). Moreover, by (7), (8), (10), and Proposition 1, it is clear that, if $v \in \Theta$ and $n(x) > 0$, $Y_j(v(x)) \leq Y_j(v(S_0x))$ for all $j \in \Psi_1(x)$, $Z(v(x)) \leq Z(v(S_0x))$ when $1 \in \Psi_{-1}(x)$, and $W(v(x)) \leq W(v(S_0x))$ for all $x \in E$. When $n(x) = 0$, for all $j \in \Psi_1(x)$,

$$\begin{aligned} Y_j(v(x)) - Y_j(v(S_0x)) &= v(S_j^{-1}x) - Tv(S_j^{-1}x) \\ &= v(S_j^{-1}x) - \min\{v(S_0S_j^{-1}x), v(S_iS_j^{-1}x) : i \in \Psi_0(S_j^{-1}x)\} \\ &\leq 0 \end{aligned}$$

by (16a), (16b), and Lemma 1. In a similar way, when $n(x) = 0$, we can show that $W(v(x)) \leq W(v(S_0x))$, and if $1 \in \Psi_{-1}(x)$, $Z(v(x)) \leq Z(v(S_0x))$ by (16a) and (16b). Then, by the fact that $\Psi_i(x) = \Psi_i(S_0x)$ for all $i \in \{-1, 0, 1\}$, (6), (16a), and Proposition 1,

$$\begin{aligned} Bv(x) - Bv(S_0x) &= r(x) - r(S_0x) + \lambda(Tv(x) - Tv(S_0x)) \\ &\quad + \sum_{j \in \Psi_1(x)} \mu_j(Y_j(v(x)) - Y_j(v(S_0x))) + \mathbb{I}_{\{1 \in \Psi_{-1}(x)\}}\alpha(Z(v(x)) - Z(v(S_0x))) \\ &\quad + \mathbb{I}_{\{1 \in \Psi_0(x)\}}\xi(v(S_1^{-1}x) - v(S_0S_1^{-1}x)) + \mathbb{I}_{\{1 \in \Psi_1(x)\}}\xi(Tv(S_1^{-2}x) - Tv(S_0S_1^{-2}x)) \\ &\quad + p(x)(W(v(x)) - W(v(S_0x))) \\ &\leq 0. \end{aligned}$$

Next, we show that if $v \in \Theta$, then $Bv(x) \leq Bv(S_jx)$ for all $x \in E$ such that $j \in \Psi_0(x)$. For $j = 1$, by (6),

$$\begin{aligned} Bv(x) - Bv(S_1x) = r(x) - r(S_1x) &+ \lambda\left[Tv(x) - Tv(S_1x)\right] \\ &+ \sum_{k \in \Psi_1(x)} \mu_k\left[Y_k(v(x)) - Y_k(v(S_1x))\right] - \mu_1 Y_1(v(S_1x)) \\ &+ \xi\left[v(S_1^{-1}x) - Tv(S_1^{-1}x)\right] + p(x)W(v(x)) - p(S_1x)W(v(S_1x)). \end{aligned}$$

Now, note that by (7) and (10),

$$W(v(x)) = Y_j(v(S_jx)), \quad \forall j \in \Psi_0(x). \tag{17}$$

After some algebraic manipulation by considering Proposition 1, (7), and (9), we obtain

$$\begin{aligned} Bv(x) - Bv(S_1x) &\leq \xi\left[v(S_1^{-1}x) - Tv(S_1^{-1}x)\right] + \mu_1\left[W(v(x)) - Y_1(v(S_1x))\right] \\ &\qquad + p(S_1x)\left[W(v(x)) - W(v(S_1x))\right] \\ &\leq \xi\left[v(S_1^{-1}x) - \min\{v(S_0S_1^{-1}x), v(S_iS_1^{-1}x) : i \in \Psi_0(S_1^{-1})\}\right] \\ &\leq 0 \end{aligned}$$

by (10), (17), (16a), (16b), and Lemma 1. Next, we consider the case $j = 2$. Again, after some algebraic manipulation and by Proposition 1, (7), (9), (10), (16b), and (17), we see that, when $1 \in \Psi_{-1}(x)$,

$$\begin{aligned}Bv(x) - Bv(S_2x) \leq \mu_2 \left[W(v(x)) - Y_2(v(S_2x))\right] + \alpha \left[Z(v(x)) - Z(v(S_2x))\right] \\ + p(S_2x) \left[W(v(x)) - W(v(S_2x))\right] \\ \leq 0,\end{aligned}$$

when $1 \in \Psi_0(x)$,

$$\begin{aligned}Bv(x) - Bv(S_2x) \leq \mu_2 \left[W(v(x)) - Y_2(v(S_2x))\right] + \xi \left[v(S_1^{-1}x) - v(S_2S_1^{-1}x)\right] \\ + p(S_2x) \left[W(v(x)) - W(v(S_2x))\right] \\ \leq 0,\end{aligned}$$

and when $1 \in \Psi_1(x)$,

$$\begin{aligned}Bv(x) - Bv(S_2x) \leq \mu_2(W(v(x)) - Y_2(v(S_2x))) + \mu_1(Y_1(v(x)) - Y_1(v(S_2x)) \\ + \xi \left[Tv(S_1^{-2}x) - Tv(S_2S_1^{-2}x)\right] + p(S_2x) \left[W(v(x)) - W(v(S_2x))\right] \\ \leq 0.\end{aligned}$$

Now, we show that if $v \in \Theta$, then $Bv(S_1x) \leq Bv(S_2x)$ for all x such that $\Psi_0(x) = \{1, 2\}$. After some simplification, by considering (6), (16c), and Proposition 1, we see that

$$\begin{aligned}Bv(S_1x) - Bv(S_2x) &\leq \mu_1 \left[Y_1(v(S_1x)) - W(S_2x)\right] + \mu_2 \left[W(S_1x) - Y_2(v(S_2x))\right] \\ &\quad + \xi \left[Tv(S_1^{-1}x) - v(S_2S_1^{-1}x)\right] + (p(x) - \mu_1 - \mu_2) \left[W(S_1x) - W(S_2x)\right] \\ &\leq \xi \left[\min\{v(S_0S_1^{-1}x), v(S_2S_1^{-1}x)\} - v(S_2S_1^{-1}x)\right] \\ &\quad + \mu_1 \left[Y_1(v(S_1x)) - W(v(S_2x))\right] + \mu_2 \left[W(v(S_1x)) - Y_2(v(S_2x))\right] \qquad (18) \\ &\leq \mu_1 \left[Y_1(v(S_1x)) - W(v(S_2x))\right] + \mu_2 \left[W(v(S_1x)) - Y_2(v(S_2x))\right], \qquad (19)\end{aligned}$$

where (18) is due to Proposition 1, (10), and (16c). When $n(x) = 0$, by (7), (10), and (19)

$$\begin{aligned}Bv(S_1x) - Bv(S_2x) &\leq \mu_1(v(x) - v(S_2x)) + \mu_2(v(S_1x) - v(x)) \\ &= (\mu_1 - \mu_2)(v(x) - v(S_2x)) + \mu_2(v(S_1x) - v(S_2x)) \\ &\leq 0\end{aligned}$$

by (16b) and (16c). When we consider the case $n(x) > 0$, by (7), (10), and (19)

$$\begin{aligned}Bv(S_1x) - Bv(S_2x) &\leq \mu_1 \left[Tv(S_0^{-1}x) - Tv(S_0^{-1}S_2x)\right] + \mu_2 \left[Tv(S_0^{-1}S_1x) - Tv(S_0^{-1}x)\right] \\ &= (\mu_1 - \mu_2) \left[Tv(S_0^{-1}x) - Tv(S_0^{-1}S_2x)\right] + \mu_2 \left[Tv(S_0^{-1}S_1x) - Tv(S_0^{-1}S_2x)\right] \\ &\leq 0\end{aligned}$$

by Proposition 1.

Lastly, we show that if $v \in \Theta$, then $Bv(S_1x) \leq Bv(S_0x)$ for all x such that $1 \in \Psi_0(x)$. After some algebraic manipulation by considering (6) and Proposition 1, we obtain

$$\begin{aligned} Bv(S_1x) - Bv(S_0x) \leq & \sum_{k \in \Psi_1(x)} \mu_k \left[Y_k(v(S_1x)) - Y_k(v(S_0x))\right] + \mu_1 Y_1(v(S_1x)) \\ & + \xi \left[Tv(S_1^{-1}x) - v(S_1^{-1}S_0x)\right] + p(S_1x)W(v(S_1x)) - p(S_0x)W(v(S_0x)). \end{aligned} \tag{20}$$

First, it is easy to see that, when $k \in \Psi_1(x)$, $Y_k(v(S_1x)) \leq Y_k(v(S_0x))$ by (7), (16c), (16d), and Proposition 1. Second, by Lemma 1,

$$\begin{aligned} Tv(S_1^{-1}x) - v(S_1^{-1}S_0x) &= \min\{v(S_0S_1^{-1}x), v(S_iS_1^{-1}x) : i \in \Psi_0(S_1^{-1}x)\} - v(S_1^{-1}S_0x) \\ &\leq 0. \end{aligned} \tag{21}$$

Then, by (9), (10), (20), and (21),

$$Bv(S_1x) - Bv(S_0x) \leq \mu_1(Y_1(v(S_1x)) - Tv(x)) + p(S_1x)(W(v(S_1x)) - Tv(x)). \tag{22}$$

When $n(S_1x) = n(x) = 0$, by (16b), (16c), (16d), and (22)

$$\begin{aligned} Bv(S_1x) - Bv(S_0x) &\leq \mu_1(v(x) - Tv(x)) + p(S_1x)(v(S_1x) - Tv(x)) \\ &= \mu_1(v(x) - v(S_1x)) + p(S_1x)(v(S_1x) - v(S_1x)) \\ &\leq 0. \end{aligned}$$

When $n(S_1x) > 0$, by (22) and Proposition (1),

$$Bv(S_1x) - Bv(S_0x) \leq \mu_1(Tv(S_0^{-1}x) - Tv(x)) + p(S_1x)(Tv(S_0^{-1}S_1x) - Tv(x)) \leq 0.$$

■

The preservation of properties (16a)–(16d) under operator B facilitates use of the value iteration algorithm to establish the optimal policies for servers 1 and 2. The next section characterizes these policies and examines the average cost criterion.

4 Optimal Policies

In this section, we establish the existence and optimality of threshold policies for both servers 1 and 2 to minimize the discounted expected total number of customers in the system. Due to the memoryless property of the service, inter-failure and repair times, and the fact that there are no additional fixed costs associated with a failure event, it is reasonable to conjecture that the optimal policy is to route customers to the faster server (server 1) when it is available. To see why this might be so, consider a scenario in which both servers are initially idle, there is only a single arrival to the system, and the objective is to minimize the mean sojourn time. Let W denote the customer's random time in system. If the customer is routed to the reliable server (server 2), then clearly, $\mathbb{E}(W) = 1/\mu_2$. However, if the customer is sent to server 1, then by the memoryless property of the service and inter-failure times, it is easy to see that $\mathbb{E}(W) = 1/(\mu_1 + \xi)$ with probability $\mu_1/(\mu_1 + \xi)$ and $\mathbb{E}(W) = 1/(\mu_1 + \xi) + 1/\mu_2$ with probability $\xi/(\mu_1 + \xi)$, given that we route the customer to server 2 in the event of an active failure at server 1. So if the customer is initially routed to server 1,

$$\mathbb{E}(W) = \frac{\xi^2 + (\mu_1 + \mu_2)\xi + \mu_1\mu_2}{\mu_2(\mu_1 + \xi)^2} \leq \frac{(\mu_1 + \xi)^2}{\mu_2(\mu_1 + \xi)^2} = \frac{1}{\mu_2}. \tag{23}$$

Hence, it is always advantageous to route the customer to server 1, given that we prefer routing the customer to server 2 when active failures occur. The policy of routing a single customer to server 1 in this case is intuitive; however, it is nontrivial to prove the optimality of this policy for the main model of Section 2 with Poisson arrivals.

Let $a^*(x)$ denote the optimal action in state $x \in E$, which will be determined using value iteration. The optimal policy for the fast server (server 1) is next established via Proposition 2.

4.1 Optimal Policy for the Fast Server

Here, we prove that the optimal policy for the fast server is of threshold type and that its threshold value is zero.

Theorem 1 *For any state $x \in E$ such that $1 \in \Psi_0(x)$, $a^*(x) = 1$, i.e., it is always optimal to route to server 1, if it is available.*

Proof. In Section 2.1, it was shown that the value iteration algorithm converges to the unique value function; hence, we employ the value iteration technique to prove Theorem 1. Let V_k denote the value function approximation at the kth iteration, $k \in \{0\} \cup \mathbb{N}$ and assume that $V_0(x) = 0$ for all $x \in E$. Then it is clear that $V_0 \in \Theta$, i.e., V_0 satisfies inequalities (16a)–(16d). Since $V_1 = BV_0$, $V_1 \in \Theta$ by Proposition 2. Continuing inductively, $V_k \in \Theta$ for all $k \in \{0\} \cup \mathbb{N}$ as $V_k = BV_{k-1}$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} V_k(x) = V(x)$ for all $x \in E$, $V \in \Theta$. This implies that $V(S_1x) \leq V(S_2x)$ for all x such that $\Psi_0(x) = \{1, 2\}$ (i.e., for any state x in which both servers are idle), and $V(S_1x) \leq V(S_0x)$ for all x such that $1 \in \Psi_0(x)$. Hence, at any decision epoch, if server 1 is idle, the optimal action is to route a customer to it. ∎

The policy for server 1 is intriguing in that it mirrors the optimal policy established by Lin and Kumar [11] for a model in which neither server fails. That is, routing to server 1 is always preferred to routing to server 2, irrespective of ξ or α, given that (1) is satisfied (i.e., the system is stable). It is important to note that two features of our model ensure that Proposition 2 holds. The first is that server 1 is subject to both active and idle failures that occur at the same rate (ξ), and the second is that actions may be taken at active failure epochs (i.e., the interrupted customer can be routed either to the queue or to server 2, if it is idle). These features ensure that the terms multiplied by ξ are non-positive in the proof of Proposition 2.

Furthermore, by Proposition 2 and value iteration, it also follows that $V(x) \leq V(S_0x)$ for all $x \in E$, and $V(x) \leq V(S_jx)$ for all x such that $j \in \Psi_0(x)$. However, if actions are forbidden at fictitious transition epochs, the latter result is not necessarily valid when $n(x) > 0$ as we now illustrate by way of a numerical example. Consider a system with $\beta = 0.2$, $\lambda = 1$, $\mu_1 = 15$, $\mu_2 = 2$, $\xi = 3$, and $\alpha = 3$. We solved this problem numerically using the value iteration algorithm (cf. Puterman [14], p. 161) by approximating the original queueing system by one with a finite queue having capacity C. That is, if the queue is full, and neither server is available, arriving customers are rejected. Additionally, if an active failure occurs at server 1 when the queue is at capacity and server 2 is busy, then the customer whose service is interrupted leaves the system. Initially, we chose $C = 1200$, and a stopping criterion (for the value iteration algorithm) of 10^{-9}. Subsequently, we examined systems with $C = 300, 500, 1000$ and observed nearly identical results; therefore, $C = 1200$ was deemed to be acceptable as an approximation to the original infinite-capacity queue. It is worth noting that, for each numerical illustration provided herein, we chose the parameters λ, μ_1, μ_2, α and ξ so that $\lambda \ll \bar{\mu}$, i.e., the system is in light traffic.

For this example, when actions are not allowed at fictitious transition epochs, $V(x) > V(S_1x)$ for all x such that $1 \in \Psi_0(x)$ and $n(x) > 0$ because μ_1 is significantly greater than λ, μ_2, and ξ, i.e., the marginal benefit of keeping server 1 busy is very high. That is why, although starting in state S_1x has the disadvantage of an additional customer in the system as compared to starting in state x, nonetheless $V(x) > V(S_1x)$ because server 1 is initially busy in state S_1x and can quickly complete the service of the current customer and proceed to depleting the queue. On the other hand, if the system starts from state x, server 1 is idle until the next decision epoch (either a customer arrival or a service completion at server 2, if it is busy). This leads to an inefficiency, as the fast server is not being utilized. However, if actions are permitted at fictitious transition epochs, then starting in state x, the system controller will route a customer from the queue to server 1 if the next decision epoch is a fictitious transition epoch by Theorem 1, and this increases the utilization of server 1. Note that, if server 2 is idle (busy) in state x, then the probability that the next decision epoch is a fictitious transition epoch is $1 - (\lambda + \xi)/\nu \approx 0.83$ $(1 - (\lambda + \mu_2 + \xi)/\nu = 0.75)$. This means that, with a high probability, in the next decision epoch the system controller will route a customer from the queue to server 1. That is why, in this case, $V(x) \leq V(S_1x)$ and this can be proved formally by Proposition 2 and value iteration.

Theorem 1 gives the optimal control policy for server 1 and provides some information about the optimal policy of server 2, namely that it is not optimal to route customers to server 2 when server 1 is idle. Theorem 2 of Section 4.2 characterizes the optimal policy for server 2 when server 1 is busy or failed.

4.2 Optimal Policy for the Slow Server

In this subsection, we prove that the optimal policy for server 2 is of threshold type, and its threshold depends on whether server 1 is busy or failed. The main result is stated in the following theorem.

Theorem 2 *If* $1 \in \Psi_1(x)$ $(1 \in \Psi_{-1}(x))$, *there exists an* n_1^* $(n_{-1}^*) \in \mathbb{N} \cup \{0\}$ *such that if* $n(x) \geq n_1^*$ $(n(x) \geq n_{-1}^*)$, *then* $a^*(x) = 2$; *otherwise* $a^*(x) = 0$.

The proof of Theorem 2 is provided in the Appendix. This result states that, when the fast server is busy or failed, and the number of customers in the queue is above a threshold level, it is optimal to route a customer to the slow server. Note that the thresholds n_1^* and n_{-1}^* are, in general, not equal (i.e., they depend on the state of the fast server). We illustrate this fact with an example. In the example of Section 4.1, $n_1^* = 3$, but $n_{-1}^* = 1$. The fact that $n_{-1}^* < n_1^*$ is intuitive in this specific case because it is more detrimental for the fast server to be failed than busy; therefore, it makes sense to route customers to the slower server earlier. However, it is not always true that $n_{-1}^* \leq n_1^*$. To see this, consider a system in which $\beta = 0.2$, $\lambda = 1$, $\mu_1 = 15$, $\mu_2 = 2$, $\xi = 1$, and $\alpha = 50$. We solve this numerical example in the same way described in Section 4.1. In this example, $n_1^* = 5$ but $n_{-1}^* = 6$. Due to the low failure rate of the fast server and its high repair rate, its failure events are not significant; therefore, the threshold for routing to the slow server is higher. Moreover, it is conjectured that, for this example, $n_{-1}^* = n_1^* + 1$, because $r(S_1x) = r(S_{-1}x) + 1$ as server 1 is occupied and processing an additional customer. Indeed, the following proposition establishes that the threshold for routing customers to server 2 when server 1 is failed is bounded above by $n_1^* + 1$.

Proposition 3 *The threshold values for server 2 are such that $n^*_{-1} \leq n^*_1 + 1$.*

Proof. The result will be proved by contradiction. Suppose that $n^*_{-1} > n^*_1 + 1$; hence, $n^*_{-1} = n^*_1 + k + 1$ for some $k \in \mathbb{N}$. Moreover, suppose the current state is $(n^*_1 + k, -1, 0)$ and a customer arrives to the system. If the optimal policy is followed, the customer is routed to the queue, and the next state of the system is $(n^*_1 + k + 1, -1, 0)$; however, if the customer is routed to server 2 (a suboptimal action), the new state of the system is $(n^*_1 + k, -1, 1)$. Now consider two processes, say Q_1 and Q_2 defined on the same probability space so that they see the same arrivals, service completions, server failures, and server repairs. Suppose Q_1 and Q_2 are initially in states $(n^*_1 + k + 1, -1, 0)$ and $(n^*_1 + k, -1, 1)$, respectively. Assume that the optimal policy is followed in process Q_1, and that some possibly sub-optimal policy (described in what follows) is executed in process Q_2. Let τ be the first time that either Q_1 or Q_2 encounters a decision epoch. That is, the next decision epoch occurs at either a repair completion at server 1, an arrival to both systems, a service completion at server 2 (which corresponds to a fictitious transition epoch at Q_1), or a fictitious transition epoch in both Q_1 and Q_2. Note that the discounted expected total numbers in system in Q_1 and Q_2 are equal up to time τ. We next examine the value function for each of four distinct cases.

Case 1: Suppose time τ is a repair completion epoch at server 1. Then by Theorem 1, Q_1 will next enter state $(n^*_1 + k, 1, 0)$ since it is optimal to route a customer from the queue to server 1. In Q_2, it is also possible to route a customer to server 1 because $k \geq 1$ so that Q_2 next enters state $(n^*_1 + k - 1, 1, 1)$. From this point forward, the discounted expected total number of customers in Q_1 is $V(n^*_1 + k, 1, 0)$, and if we use the optimal policy in Q_2, then the discounted expected total number of customers in Q_2 is $V(n^*_1 + k - 1, 1, 1)$. But, since $k \geq 1$, $V(n^*_1 + k - 1, 1, 1) \leq V(n^*_1 + k, 1, 0)$ by Theorem 2.

Case 2: Suppose τ is a customer arrival epoch. Then Q_1 will enter state $(n^*_1 + k + 1, -1, 1)$ by Theorem 2, and Q_2 will enter state $(n^*_1 + k + 1, -1, 1)$ since neither server is available. From this point forward, if we use the optimal policy in Q_2, then the discounted expected total number of customers in both systems is $V(n^*_1 + k + 1, -1, 1)$.

Case 3: Suppose τ is a (potential) service completion epoch at server 2. Then, the state of process Q_1 remains $(n^*_1 + k + 1, -1, 0)$ because of the fact that $n^*_{-1} = n^*_1 + k + 1$, (10), and Theorem 2; and in Q_2, we choose to do nothing at this decision epoch so that the state of Q_2 becomes $(n^*_1 + k, -1, 0)$. From this point forward, the discounted expected number of customers in Q_1 is $V(n^*_1 + k + 1, -1, 0)$, and if the optimal policy is used in Q_2, then the discounted expected total number of customers in Q_2 is $V(n^*_1 + k, -1, 0)$. However, $V(n^*_1 + k, -1, 0) \leq V(n^*_1 + k + 1, -1, 0)$ by Proposition 2 and value iteration.

Case 4: Suppose τ is a is a fictitious transition epoch in both Q_1 and Q_2. Then, the state of process Q_1 remains $(n^*_1 + k + 1, -1, 0)$ because of the fact that $n^*_{-1} = n^*_1 + k + 1$, (10), and Theorem 2. Since none of the servers is idle in the process Q_2, the state of this process remains $(n^*_1 + k, -1, 1)$. From this point forward, by Cases 1-4, the discounted expected total number of customers in Q_2 is less than or equal to the number in Q_1.

Considering these four cases, it is seen that the discounted expected total number of customers in Q_2 is less than or equal to the number in Q_1, which is a contradiction since we reach the initial state of Q_2 via a suboptimal action. Since $k \in \mathbb{N}$ is arbitrary, we conclude that $n^*_{-1} \leq n^*_1 + 1$. ∎

Because we allow actions at the fictitious transition epochs, we need to check whether the optimal policy proved in this section remains optimal for the original model in which actions are not permitted at these epochs. Proposition 4 asserts that the system controller does not alter the state of the system at fictitious transition epochs, assuming the optimal policy is followed. Therefore, allowing actions at fictitious transition epochs is inconsequential.

Proposition 4 *Starting from any suitable state, such as* $(0,0,0)$*, if the optimal policy is followed, the controller will not alter the state of the system at fictitious transition epochs.*

Proof. By (5) and (10), at a fictitious transition epoch, the controller can alter the state of the system by routing a customer from the queue to an idle server; in order to alter the state of the system, the queue cannot be empty. However, by Theorem 1, for any state $x \in E$ for which server 1 is idle, we must have $n(x) = 0$. Hence, the controller cannot route a customer from the queue to server 1 at any fictitious transition epoch.

Let τ be the first time at which the controller alters the state of the system at a fictitious time epoch. Then, the controller must route a customer from the queue to server 2 at τ, and server 2 must be idle just before τ. At τ, server 1 can be either failed or busy. We will consider each case separately. Let server 1 be in the failed state at τ. Then, just before τ, the system should be in state $x \in E$ such that $x = (n(x), -1, 0)$. Since the optimal action at τ is to route a customer from the queue to server 2, then $n(x) > n^*_{-1}$ by Theorem 2 and (10). We will prove that if the system starts from any suitable state, say state $(0,0,0)$, it cannot reach a state of the form $(n(x), -1, 0)$ where $n(x) > n^*_{-1}$. Let τ_0 be the time of the last decision epoch before τ. The system can be in three different states just before τ_0.

Case 1: The system can be in state $(n(x), -1, 1)$ where $n(x) > n^*_{-1}$ just before τ_0 and at τ_0 there is a service completion at server 2. However, since $n(x) > n^*_{-1}$, the controller will route a customer from the queue to server 2 at τ_0; hence, the state of the system at τ_0 will be $(n(x) - 1, -1, 1)$.

Case 2: The system can be in state $(n(x) - 1, -1, 0)$ where $n(x) > n^*_{-1}$ just before τ_0 and at τ_0 there is a customer arrival to the system. However, since $n(x) - 1 \geq n^*_{-1}$, the controller will route a customer from queue to server 2 at τ_0 by Theorem 2. Therefore, the state of the system at τ_0 will be $(n(x) - 1, -1, 1)$.

Case 3: The system can be in state $(n(x) - 1, 1, 0)$ where $n(x) > n^*_{-1}$ just before τ_0 and at τ_0 there is an active failure in server 1. However, since $n(x) - 1 \geq n^*_{-1}$, the controller will route the customer from server 1 to server 2 at τ_0 by Theorem 2, and the resulting state at τ_0 will be $(n(x) - 1, -1, 1)$. Therefore, the system cannot reach a state of the form $(n(x), -1, 0)$ where $n(x) > n^*_{-1}$. This implies that server 1 cannot be failed at τ and, therefore, must be busy. Hence, just before τ, the system should be in some state $x \in E$ such that $x = (n(x), 1, 0)$ where $n(x) > n^*_1$ by Theorem 2 and (10).

Let τ_1 be the time of the last decision epoch before τ. The system can be in four different states just before τ_1.

Case 4: The system can be in state $(n(x), 1, 1)$ where $n(x) > n^*_1$ just before τ_1, and at τ_1 there is a service completion at server 2. However, since $n(x) > n^*_1$, the controller will route a customer from the queue to server 2 at τ_1, and the resulting state at τ_1 will be $(n(x) - 1, 1, 1)$.

Case 5: The system can be in state $(n(x) - 1, 1, 0)$ where $n(x) > n^*_1$ just before τ_1, and at τ_1 there is a customer arrival to the system. However, since $n(x) - 1 \geq n^*_1$, the controller will route a customer from the queue to server 2 at τ_1 by Theorem 2. Therefore, the state of the system at τ_1 will be $(n(x)-1,1,1)$.

Case 6: The system can be in state $(n(x)+1,-1,0)$ where $n(x)>n_1^*$ just before τ_1, and at τ_1 there is the repair of server 1. However, $n(x)+1>n_1^*+1\geq n_{-1}^*$ by Proposition 3, and it is known that it is not possible to reach a state of the form $(n(x)+1,-1,0)$ where $n(x)+1>n_{-1}^*$ by Cases 1, 2, and 3.

Case 7: The system can be in state $(n(x)+1,1,0)$ where $n(x)>n_1^*$ just before τ_1, and at τ_1 there is a service completion at server 1. However, by Cases 1–6, state $(n(x)+1,1,0)$ (with $n(x)>n_1^*$) can only be reached from $(n(x)+2,1,0)$, and similarly the state $(n(x)+2,1,0)$ can only be reached from $(n(x)+3,1,0)$. Inductively, state $(n(x)+1,1,0)$ where $n(x)>n_1^*$ can be reached by state $(+\infty,1,0)$, which is a contradiction because any work conserving policy has the maximum throughput rate $\bar{\mu}$ so that $\lambda<\bar{\mu}$, and the mean number of customers in the system is finite. Hence, the mean number of customers in the system will not reach to $+\infty$ in the optimal policy as well.

Therefore, starting from any suitable state, such as $(0,0,0)$, if the optimal policy is used, the controller will not alter the state of the system at the fictitious transition epochs. The term "suitable state" reflects the set of states $x\in E$ such that if the system starts from state x, and if the first decision epoch is a fictitious transition epoch, the controller will not change the state of the system. Note that, in the system that does not allow actions at fictitious transition epochs, it can be shown that $n_{-1}^*\leq n_1^*+1$ using the same argument employed in the proof of Proposition 3. ■

To clarify the notion of a "suitable state," consider a state x in which server 1 is idle and the queue is non-empty (i.e., $1\in\Psi_0(x)$ and $n(x)>0$). Any such state is not a suitable state because starting in this state, if the next decision epoch is a fictitious transition epoch, the system controller will route a customer from the queue to server 1 by Theorem 1. This is why in the numerical example of Section 4.1, if actions are not allowed at fictitious transition epochs, $V(x)>V(S_1x)$; otherwise, $V(x)\leq V(S_1x)$.

Finally, to conclude this section, we summarize the optimal control policy for the system. In state $x\in E$, the optimal action $a^*(x)$ is

$$a^*(x)=\begin{cases}1, & \text{if } 1\in\Psi_0(x),\\ 2, & \text{if } 2\in\Psi_0(x),\ 1\in\Psi_1(x) \text{ and } n(x)\geq n_1^*,\\ 2, & \text{if } 2\in\Psi_0(x),\ 1\in\Psi_{-1}(x) \text{ and } n(x)\geq n_{-1}^*,\\ 0, & \text{otherwise.}\end{cases} \tag{24}$$

4.3 Average Cost Criterion

A common objective in the control of queueing systems is the minimization of the average time customers spend in the system, which is equivalent to minimizing the number of customers in system (via Little's Law). In this section, we examine the average cost criterion and prove that the optimal policy has the same threshold structure as the one proved for the discounted case. The results of Section 4 show that, for any $\beta'\in(0,1)$, the optimal policy for the discounted problem is a threshold policy for each server, and that the optimal threshold for the fast server is zero (by Theorems 1 and 2). The minimum long-run average number of customers in the system, given the initial state $x_0 \in E$, is

$$g^*(x_0) = \inf_{\theta \in \Pi} \left\{ \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_\theta \left[\int_0^T \left(N(t) + \sum_{j=1}^{2} \mathbb{I}_{\{X_j(t)=1\}} \right) \mathrm{d}t \middle| (N(0), X_1(0), X_2(0)) = x_0 \right] \right\},$$

where Π is the set of all admissible policies. The equivalent version of this objective in the uniformized model is

$$g^*(x_0) = \inf_{\theta \in \Pi} \left\{ \limsup_{n\to\infty} \frac{1}{n} \mathbb{E}_\theta \left[\sum_{m=0}^{n-1} L_m \middle| (N(0), X_1(0), X_2(0)) = x_0 \right] \right\}, \tag{25}$$

where L_m denotes the number of customers in the system just after the mth transition. Henceforth, we will consider only the objective in (25). To show the existence of an optimal stationary policy, and that it is a threshold policy under the average cost criterion, we employ the main theorem of Sennott [17], and to do this we need to check whether the main model of Section 4 satisfies Assumptions 1, 2, 3, and 3* of [17].

In order to check Assumptions 1 and 3 of [17], we employ Proposition 5, part (i) of [17]. Consider any stationary work conserving policy, i.e., the servers can be idle only if the queue is empty. Then, the maximum throughput rate of the system is $\bar{\mu}$; therefore, the system is stable by (1) and the average number of customers in the system is finite; hence, condition (i) of Proposition 4 of [17] is satisfied as is Proposition 5, part (i) of [17]. In order to check Assumption 2 of [17], we need the following proposition, which is proved in the Appendix.

Proposition 5 *In the discounted case, $V(0,0,0) \le V(x)$ for all $x \in E$.*

Let $V_\beta(x)$ denote the discounted expected total number of customers in the system starting from state $x \in E$ when the discount parameter is β and let $h_\beta(x) = V_\beta(x) - V_\beta(0,0,0)$. By Proposition 5, $h_\beta(x) \ge 0$ for all $x \in E$, thus Assumption 2 of [17] is satisfied. Since our model satisfies Assumption 3 of [17], there exists a nonnegative M_x, such that $h_\beta(x) \le M_x$ for all $\beta \in (0,1)$ and $x \in E$. Let $q_{xy}(a)$ denotes the probability that the system occupies state $y \in E$ after action $a \in A(x)$ in state $x \in E$. Since there are only finitely many actions at each state $x \in E$, and the system transitions to a single state after each action, then $\sum_y q_{xy}(a) M_y < \infty$ for all $x \in E$ and $a \in A(x)$. Therefore, Assumption 3* of [17] is satisfied. Then, by the lemma stated in [17] (p. 628) and Theorems 1 and 2, we know that for any sequence of discount parameters, $\{\beta_n\}$, such that $\beta_n \to 0$ (respectively $\beta'_n \to 1$) as $n \to \infty$, the policy is stationary and of threshold type at the limit point; therefore, by the main theorem of [17], the optimal policy is stationary and of threshold type. Furthermore, the optimal threshold value for the fast server is zero because, for each β_n, the threshold is zero. However, for server 2, although it is known that the optimal policy is of threshold type, the optimal threshold values may depend not only on the state of server 1, but also the value of β. Lastly, since $n^*_{-1} \le n^*_1 + 1$ for all β_n, this inequality holds at the limit point as well.

5 Variant of the Main Model

In this section, we consider a model with K ($K \ge 2$) servers with $\mu_j = \mu$ for each $j = 1, \dots, K$, and the servers are subject to distinct failure and repair rates. Let ξ_j and α_j respectively denote the failure and repair rates of server j, $j = 1, \dots, K$. The system corresponds to an $M/M/K$

queue with unreliable servers. The state space is $E = \{0, 1, 2, \dots\} \times \{0, 1, -1\}^K$ and the maximum throughput rate of the system is

$$\bar{\mu} = \sum_{j=1}^{K} \left(\frac{\alpha_j}{\alpha_j + \xi_j} \right) \mu,$$

where it is assumed that $\lambda < \bar{\mu}$ for stability. We first consider the objective of minimizing the discounted expected total number of customers in the system and subsequently show that the structure of the optimal policy does not change as the discount parameter approaches zero, i.e. $\beta \to 0$. The uniformization rate is

$$\nu = \lambda + K\mu + \sum_{j=1}^{K} (\xi_j + \alpha_j),$$

and the uniformized discounted expected total number of customers in the system is

$$V(x) = \frac{1}{\nu + \beta} \Bigg\{ r(x) + \lambda TV(x) + \sum_{j \in \Psi_1(x)} \Big(\mu Y_j(V(x)) + \xi_j TV(S_j^{-2} x) \Big) + \sum_{j \in \Psi_{-1}(x)} \alpha_j Z_j(V(x)) + \sum_{j \in \Psi_0(x)} \xi_j V(S_j^{-1} x) + p(x) V(x) \Bigg\} = B' V(x), \quad (26)$$

where the operator T is defined as in (5), B' is defined by (26), and the operator Y_j is defined as in (7). When $j \in \Psi_{-1}(x)$,

$$Z_j(V(x)) = \begin{cases} TV(S_0^{-1} S_j x), & \text{if } n(x) > 0, \\ V(S_j x), & \text{if } n(x) = 0, \end{cases} \quad (27)$$

and the rate $p(x)$ is modified such that

$$p(x) = \nu - \lambda - \sum_{j \in \Psi_1(x)} (\mu + \xi_j) - \sum_{j \in \Psi_{-1}(x)} \alpha_j - \sum_{j \in \Psi_0(x)} \xi_j. \quad (28)$$

Note that, in this model, actions are not permitted at fictitious transition epochs as they are not needed to facilitate the main results. Without loss of generality, we again assume $\nu + \beta = 1$. It can be shown that there exists an optimal stationary policy and the value iteration algorithm converges with the same method used in Subsection 2.1. Given that we define the one-to-one mapping Δ in the same way from the $K + 1$ dimensional state space E to the state space $\mathbb{N}_0$, if we define $w(s) = \lfloor s/3^K \rfloor + K$ for all $s \in \mathbb{N}_0$, $u = 1$, $\kappa = (4 \times 3^K - 1)(K+2)/K$, and for each $\gamma \in [0, 1)$, $\rho = \gamma$, J_0 is in $\mathbb{N}$ such that $\gamma^{J_0 - 1}(4 \times 3^K - 1) J_0 (2J_0 + K)/K \le 1$; then our model satisfies Assumptions 6.10.1 and 6.10.2 of Puterman [14] (p. 232), and by Theorem 6.10.4 of [14] (p. 236), there exists an optimal stationary policy, and the value iteration algorithm converges. In the Appendix, we prove the following proposition from which the optimal control policy of this model follows immediately.

Proposition 6 *Let $v : E \to \mathbb{R}$ and Θ_2 be the set of all v satisfying*

$$v(x) \le v(S_0 x), \quad \forall\, x \in E, \quad (29a)$$

$$v(x) \le v(S_j x), \quad \forall\, x \in E \text{ such that } j \in \Psi_0(x) \text{ and } n(x) = 0, \quad (29b)$$

$$v(S_i x) = v(S_j x), \quad \forall\, x \in E \text{ such that } i, j \in \Psi_0(x), \quad (29c)$$

$$v(S_j x) \le v(S_0 x), \quad \forall\, x \in E \text{ such that } j \in \Psi_0(x). \quad (29d)$$

Then, if $v \in \Theta_2$, $B'v \in \Theta_2$.

Theorem 3 *It is always optimal to route customers to an idle server.*

Proof. As before, we employ the value iteration technique to prove Theorem 3. Let V_k denote the value function approximation at the kth iteration, $k \in \{0\} \cup \mathbb{N}$ and let us assume that $V_0(x) = 0$ for all $x \in E$. Then, it is clear that $V_0 \in \Theta_2$, i.e., V_0 satisfies the inequalities (29a)–(29d). Since $V_1 = B^{'}V_0$, $V_1 \in \Theta_2$ by Proposition 6. Thus, inductively, $V_k \in \Theta_2$ for all $k \in \{0\} \cup \mathbb{N}$ because $V_k = B^{'}V_{k-1}$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} V_k(x) = V(x)$ for all $x \in E$, then $V \in \Theta_2$. This implies that $V(S_i x) = V(S_j x)$ for all $x \in E$ such that $i, j \in \Psi_0(x)$ and $V(S_j x) \leq V(S_0 x)$ for all $x \in E$ such that $j \in \Psi_0(x)$. Hence, whenever a server is idle at a decision epoch, it is optimal to route a customer to the server. ∎

Theorem 3 states that, when the service rates are equal, it is never optimal to delay service, even if the only idle server is highly failure prone (i.e., the policy is insensitive to the failure and repair rates, as long as the system is stable). It is conjectured that this insensitivity property does not hold for general service, inter-failure, and repair times.

Theorem 3 also holds under the average cost criterion. We can prove this result by employing the main theorem of Sennott [17], and to do this we need to check whether the main model of Section 4 satisfies Assumptions 1, 2, 3, and 3* of [17]. We can prove that the model of Section 5 satisfies Assumptions 1, 3, and 3* of [17] in the same way described in Section 4.3. We can check the Assumption 2 of [17] in the following way. First, we need to allow actions at fictitious transition epochs to facilitate the proofs and we can can prove Proposition 6 and Theorem 3 in the same way. Then, we need to show that the system controller does not alter the state of the system at fictitious transition epochs, if the optimal policy is followed and given that the system starts from a suitable state, such as $(0, \ldots, 0)$. The proof is straightforward because according to Theorem 3, a server can be idle only if the queue is empty. Hence, there is not a decision epoch such that the system controller can route a customer from the queue to an idle server. Then, we can prove that $V(x) \leq V(S_i^{-1}x)$ for all $x \in E$ such that $i \in \Psi_0(x)$ and $n(x) = 0$ in the same way that we proved Proposition 5, and this result together with (29a) and (29b) imply that $V(0, \ldots, 0) \leq V(x)$ for all $x \in E$. Then, Assumption 2 of [17] is also satisfied. Therefore, by the lemma stated in [17] (p. 628) and Theorem 3, the optimal policy in the average cost criterion is the same as that of the discounted case.

6 Concluding Remarks

We have considered the control of a two-server queueing system in which the servers are heterogeneous and distinguished by their service rates and/or reliability attributes. For the main model, the servers are differentiated by their service rates and the fact that the fast server is unreliable. Proved was the optimality of a threshold control policy for both servers. For minimizing the long-run average number of customers in the system, the results are consistent with the case of two reliable servers - customers should always be routed to the faster server, irrespective of its failure or repair rates. However, customers should only be routed to the slower (reliable) server if the queue length exceeds a threshold value that depends on the status of the faster server.

Using the framework for analyzing this model, we examined a variant of the main model, in which there are multiple (more than two) servers with equivalent service rates but distinct reliability characteristics. It was shown that it is optimal to keep all servers busy, irrespective of their reliability attributes, i.e., delayed customers should not wait for the server that is less failure prone, but rather should commence their service as early as possible. This result suggests that, at least in the case of exponentially distributed inter-arrival, service, inter-failure, and repair times, reliability attributes may not be important for minimizing congestion in the system.

We first considered the objective of minimizing the discounted expected total number of customers in the system and subsequently proved the optimality of the same threshold-type policy under the average cost criterion. When the objective is to minimize the congestion in the system, the long-run average cost criterion is more suitable than the discounted cost model. On the other hand, if the objective is minimizing a monetary value, such as the cost of holding customers in the system, then it is more intuitive to consider the discounted cost model. The results presented herein can, in fact, be extended to consider a cost-based objective function. For example, if h denotes the holding cost per customer per unit time in the system, then the term $r(x)$ in (6) is replaced by $h\,r(x)$. In this case, minimizing the discounted expected total holding cost is a more logical objective than the average cost criterion, and it is easy to see that the optimal policy has the same structure as the one proved in Section 4. On the other hand, the optimality of a threshold-type policy is not apparent if the servers and the queue have different holding cost rates, or if there exists a server switching cost for customers that are routed from a failed server to an idle server. It may be possible to extend the slow-server problem with failures and different cost structures, but further analysis is needed to determine the optimal control policy in each case.

Ultimately, it will be instructive to examine queueing systems with more than two servers having distinct service, failure, and repair rates. However, it is not clear if the optimal policies will be of threshold type when the controller can route customers to one of multiple idle servers, all of which are unreliable. It may also be difficult to establish the optimal policies for systems in which the active and idle failure rates differ. These extensions present formidable challenges, as does a relaxation of the exponential inter-arrival, service, inter-failure, and repair times in the two-server model.

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Appendix

Proof of Theorem 2

Here, using value iteration, we prove that when server 1 is busy or failed, the optimal control policy for server 2 is of threshold type. Our proof is similar to that of Koole [8]; however, ours is more involved in that we cannot collapse the state space to two dimensions due to the unreliability of server 1 (i.e., we need to track whether or not server 1 is failed, and this leads to additional cases that must be considered). To establish the theorem, we will first state and prove several lemmas. Let $v : E \to \mathbb{R}$ be a real-valued function, and let Θ_1 be the set of all v such that $v \in \Theta$ and

$$v(S_0x) - v(S_2x) - v(S_0^2x) + v(S_0S_2x) \le 0, \ \forall\, x \text{ s.t. } 1 \notin \Psi_0(x) \text{ and } 2 \in \Psi_0(x), \tag{30a}$$

$$v(S_0x) - v(x) - v(S_0S_2x) + v(S_2x) \le 0, \ \forall\, x \text{ s.t. } 1 \notin \Psi_0(x) \text{ and } 2 \in \Psi_0(x), \tag{30b}$$

$$v(0,1,0) - v(0,0,1) - v(1,1,0) + v(0,1,1) \le 0, \tag{30c}$$

$$v(0,1,0) - v(0,0,0) - v(0,1,1) + v(0,0,1) \le 0. \tag{30d}$$

Note that, inequality (30a) is the key result which establishes the threshold policy for server 2, inequality (30b) is supermodularity, and (30c) and (30d) are boundary conditions encountered when $n(x) = 0$. Summing (30a) and (30b), we obtain the following inequality:

$$2v(S_0x) - v(x) - v(S_0^2x) \le 0 \quad \forall\, x \text{ such that } 1 \notin \Psi_0(x) \text{ and } 2 \in \Psi_0(x). \tag{31}$$

Next, summing (30a) and (30b) while replacing x by S_0x in (30b), we obtain

$$2v(S_0S_2x) - v(S_2x) - v(S_0^2S_2x) \le 0 \quad \forall\, x \text{ such that } 1 \notin \Psi_0(x) \text{ and } 2 \in \Psi_0(x). \tag{32}$$

We will prove that the operator B preserves the properties of the functions belonging to Θ_1. To this end, we state and prove the following lemma.

Lemma 2 *If $v \in \Theta_1$, then for all $x \in E$ such that $1 \notin \Psi_0(x)$ and $2 \in \Psi_0(x)$*

$$W(v(S_0x)) - Y_2(v(S_2x)) - W(v(S_0^2x)) + Y_2(v(S_0S_2x)) \le 0. \tag{33}$$

Proof. First, by (7) and (10),

$$\begin{aligned} W(v(S_0x)) - Y_2(v(S_2x)) - W(v(S_0^2x)) + Y_2(v(S_0S_2x)) \\ &= 2Tv(x) - Y_2(v(S_2x)) - Tv(S_0x) \\ &=: \Gamma_0. \end{aligned}$$

When $n(x) = 0$, we have two cases to consider. First, when $v(S_0^2 x) \le v(S_0 S_2 x)$, by (7),

$$\begin{aligned}\Gamma_0 &= 2Tv(x) - v(x) - Tv(S_0 x)\\ &= 2\min\{v(S_0 x), v(S_2 x)\} - v(x) - \min\{v(S_0^2 x), v(S_0 S_2 x)\}\\ &= 2v(S_0 x) - v(x) - v(S_0^2 x)\\ &\le 0\end{aligned}$$

by (30a) and (31). Second, when $v(S_0 S_2 x) \le v(S_0^2 x)$,

$$\begin{aligned}\Gamma_0 &= 2\min\{v(S_0 x), v(S_2 x)\} - v(x) - v(S_0 S_2 x)\\ &\le v(S_0 x) + v(S_2 x) - v(x) - v(S_0 S_2 x)\\ &\le 0\end{aligned}$$

by (30b). When $n(x) > 0$, we have three cases to consider due to (30a). The first one is $v(x) \le v(S_0^{-1} S_2 x)$ and $v(S_0^2 x) \le v(S_0 S_2 x)$. In this case, by (7),

$$\begin{aligned}\Gamma_0 &= 2Tv(x) - Tv(S_0^{-1} x) - Tv(S_0 x)\\ &= 2\min\{v(S_0 x), v(S_2 x)\} - \min\{v(x), v(S_0^{-1} S_2 x)\} - \min\{v(S_0^2 x), v(S_0 S_2 x)\}\\ &= 2\min\{v(S_0 x), v(S_2 x)\} - v(x) - v(S_0^2 x)\\ &\le 2v(S_0 x) - v(x) - v(S_0^2 x)\\ &\le 0\end{aligned}$$

by (31). The second case is $v(x) \le v(S_0^{-1} S_2 x)$ and $v(S_0 S_2 x) \le v(S_0^2 x)$. In this case,

$$\begin{aligned}\Gamma_0 &= 2\min\{v(S_0 x), v(S_2 x)\} - v(x) - v(S_0 S_2 x)\\ &\le v(S_0 x) + v(S_2 x) - v(x) - v(S_0 S_2 x)\\ &\le 0\end{aligned}$$

by (30b). The last case is $v(S_0^{-1} S_2 x) \le v(x)$ and $v(S_0 S_2 x) \le v(S_0^2 x)$. Here,

$$\begin{aligned}\Gamma_0 &= 2\min\{v(S_0 x), v(S_2 x)\} - v(S_0^{-1} S_2 x) - v(S_0 S_2 x)\\ &\le 2v(S_2 x) - v(S_0^{-1} S_2 x) - v(S_0 S_2 x)\\ &\le 0\end{aligned}$$

by the fact that $n(x) > 0$ and (32). Note that, the case $v(S_0^{-1} S_2 x) \le v(x)$ and $v(S_0^2 x) \le v(S_0 S_2 x)$ is not possible due to (30a). ■

Lemma 3 *If $v \in \Theta_1$, then for all $x \in E$ such that $1 \in \Psi_1(x)$ and $2 \in \Psi_0(x)$,*

$$Bv(S_0 x) - Bv(S_2 x) - Bv(S_0^2 x) + Bv(S_0 S_2 x) \le 0. \tag{34}$$

Proof. By (6) and (9),

$$\begin{aligned}Bv(S_0 x) = r(S_0 x) + \lambda Tv(S_0 x) + \mu_1 Y_1(v(S_0 x)) + \xi Tv(S_1^{-2} S_0 x)\\ + p(S_2 x) W(v(S_0 x)) + \mu_2 W(v(S_0 x)),\end{aligned} \tag{35}$$

$$Bv(S_2x) = r(S_2x) + \lambda Tv(S_2x) + \mu_1 Y_1(v(S_2x)) + \mu_2 Y_2(v(S_2x)) \\ + \xi Tv(S_1^{-2} S_2 x) + p(S_2x) W(v(S_2x)), \quad (36)$$

$$Bv(S_0^2x) = r(S_0^2x) + \lambda Tv(S_0^2x) + \mu_1 Y_1(v(S_0^2x)) + \xi Tv(S_1^{-2} S_0^2 x) \\ + p(S_2x) W(v(S_0^2x)) + \mu_2 W(v(S_0^2x)) \ , \quad (37)$$

$$Bv(S_0S_2x) = r(S_0S_2x) + \lambda Tv(S_0S_2x) + \mu_1 Y_1(v(S_0S_2x)) \\ + \mu_2 Y_2(v(S_0S_2x)) + \xi Tv(S_1^{-2} S_0 S_2 x) + p(S_2x) W(v(S_0S_2x)). \quad (38)$$

We will consider the terms with λ, μ_1, μ_2, ξ, and $p(S_2x)$ in (35), (36), (37), and (38) separately. It is clear that

$$r(S_0x) - r(S_2x) - r(S_0^2x) + r(S_0S_2x) = 0. \quad (39)$$

Second,

$$\lambda \left(Tv(S_0x) - Tv(S_2x) - Tv(S_0^2x) + Tv(S_0S_2x)\right) = \lambda\Big(\min\{v(S_0^2x), v(S_0S_2x)\} \\ - v(S_0S_2x) - \min\{v(S_0^3x), v(S_2S_0^2x)\} + v(S_0^2S_2x)\Big). \quad (40)$$

If $v(S_0^3x) \le v(S_2S_0^2x)$, (40) becomes

$$\begin{aligned}\lambda \left(\min\{v(S_0^2x), v(S_0S_2x)\} - v(S_0S_2x) - \min\{v(S_0^3x), v(S_2S_0^2x)\} + v(S_0^2S_2x)\right) \\ \le \lambda \left(v(S_0^2x) - v(S_0S_2x) - v(S_0^3x) + v(S_0^2S_2x)\right) \\ \le 0\end{aligned}$$

by (30a). If $v(S_2S_0^2x) \le v(S_0^3x)$, (40) becomes

$$\begin{aligned}\lambda \left(\min\{v(S_0^2x), v(S_0S_2x)\} - v(S_0S_2x) - \min\{v(S_0^3x), v(S_2S_0^2x)\} + v(S_0^2S_2x)\right) \\ \le \lambda \left(v(S_0S_2x) - v(S_0S_2x) - v(S_2S_0^2x) + v(S_0^2S_2x)\right) \\ = 0.\end{aligned}$$

Hence,

$$\lambda \left(Tv(S_0x) - Tv(S_2x) - Tv(S_0^2x) + Tv(S_0S_2x)\right) \le 0. \quad (41)$$

Next, we consider the terms with μ_1. If $n(x) = 0$, then

$$\begin{aligned}\mu_1 \left(Y_1(v(S_0x)) - Y_1(v(S_2x)) - Y_1(v(S_0^2x)) + Y_1(v(S_0S_2x))\right) \\ = \mu_1 \left(Tv(S_1^{-1}x) - v(S_1^{-1}S_2x) - Tv(S_0S_1^{-1}x) + Tv(S_1^{-1}S_2x)\right) \\ = \mu_1 \left(v(x) - v(S_1^{-1}S_2x) - v(S_0x) + v(S_2x)\right) \\ = \mu_1 \left(v(0,1,0) - v(0,0,1) - v(1,1,0) + v(0,1,1)\right) \\ \le 0\end{aligned}$$

by (30c). If $n(x) > 0$, then

$$\begin{aligned}
\mu_1 & \left(Y_1(v(S_0x)) - Y_1(v(S_2x)) - Y_1(v(S_0^2x)) + Y_1(v(S_0S_2x))\right) \\
&= \mu_1 \left(Tv(S_1^{-1}x) - Tv(S_0^{-1}S_1^{-1}S_2x) - Tv(S_0S_1^{-1}x) + Tv(S_1^{-1}S_2x)\right) \\
&= \mu_1 \left(v(x) - v(S_0^{-1}S_2x) - v(S_0x) + v(S_2x)\right) \\
&\leq 0
\end{aligned}$$

by (30a). Hence,

$$\mu_1 \left(Y_1(v(S_0x)) - Y_1(v(S_2x)) - Y_1(v(S_0^2x)) + Y_1(v(S_0S_2x))\right) \leq 0. \tag{42}$$

Next, let us consider the terms involving μ_2. By Lemma 2,

$$\mu_2 \left(W(v(S_0x)) - Y_2(v(S_2x)) - W(v(S_0^2x)) + Y_2(v(S_0S_2x))\right) \leq 0. \tag{43}$$

Considering terms with ξ, we obtain,

$$\begin{aligned}
\xi & \left(Tv(S_1^{-2}S_0x) - Tv(S_1^{-2}S_2x) - Tv(S_1^{-2}S_0^2x) + Tv(S_1^{-2}S_0S_2x)\right) \\
&= \xi \left[\min\{v(S_1^{-2}S_0^2x), v(S_1^{-2}S_0S_2x)\} - v(S_1^{-2}S_0S_2x) \right. \\
&\qquad \left. - \min\{v(S_1^{-2}S_0^3x), v(S_1^{-2}S_0^2S_2x)\} + v(S_1^{-2}S_0^2S_2x)\right] \\
&=: \Gamma_1.
\end{aligned}$$

There are two cases to consider, the first of which is $v(S_1^{-2}S_0^3x) \leq v(S_1^{-2}S_0^2S_2x)$. In this case, by (30a),

$$\Gamma_1 \leq \xi \left(v(S_1^{-2}S_0^2x) - v(S_1^{-2}S_0S_2x) - v(S_1^{-2}S_0^3x) + v(S_1^{-2}S_0^2S_2x)\right) \leq 0. \tag{44}$$

For the second case, if $v(S_1^{-2}S_0^2S_2x) \leq v(S_1^{-2}S_0^3x)$, then

$$\Gamma_1 \leq \xi \left(v(S_1^{-2}S_0S_2x) - v(S_1^{-2}S_0S_2x) - v(S_1^{-2}S_0^2S_2x) + v(S_1^{-2}S_0^2S_2x)\right) = 0. \tag{45}$$

Lastly, we consider terms involving $p(S_2x)$. In this case, by (10) and the fact that $1 \in \Psi_1(x)$ and $2 \in \Psi_0(x)$,

$$\begin{aligned}
& p(S_2x) \left[W(v(S_0x)) - W(v(S_2x)) - W(v(S_0^2x)) + W(v(S_0S_2x))\right] \\
&= p(S_2x) \left[Tv(x) - v(S_2x) - Tv(S_0x) + v(S_0S_2x)\right] \\
&= p(S_2x) \left[\min\{v(S_0x), v(S_2x)\} - v(S_2x) - \min\{v(S_0^2x), v(S_0S_2x)\} + v(S_0S_2x)\right] \\
&=: \Gamma_2.
\end{aligned}$$

There are two cases to consider, the first of which is $v(S_0^2x) \leq v(S_0S_2x)$. In this case, by (30a),

$$\Gamma_2 \leq p(S_2x) \left(v(S_0x) - v(S_2x) - v(S_0^2x) + v(S_0S_2x)\right) \leq 0. \tag{46}$$

Next, if $v(S_0S_2x) \leq v(S_0^2x)$, it is seen that

$$\Gamma_2 \leq p(S_2x) \left(v(S_2x) - v(S_2x) - v(S_0S_2x) + v(S_0S_2x)\right) = 0. \tag{47}$$

Hence, by (39), (41), (42), (43), (44), (45), (46), and (47), inequality (34) holds and Lemma 3 is proved. ∎

Lemma 4 *If $v \in \Theta_1$, then for all $x \in E$ such that $1 \in \Psi_{-1}(x)$ and $2 \in \Psi_0(x)$,*

$$Bv(S_0x) - Bv(S_2x) - Bv(S_0^2x) + Bv(S_0S_2x) \le 0. \tag{48}$$

Proof. By (6) and (9),

$$Bv(S_0x) = r(S_0x) + \lambda Tv(S_0x) + \alpha Z(v(S_0x)) + p(S_2x)W(v(S_0x)) + \mu_2 W(v(S_0x)), \tag{49}$$

$$Bv(S_2x) = r(S_2x) + \lambda Tv(S_2x) + \mu_2 Y_2(v(S_2x)) + \alpha Z(v(S_2x)) + p(S_2x)W(v(S_2x)), \tag{50}$$

$$Bv(S_0^2x) = r(S_0^2x) + \lambda Tv(S_0^2x) + \alpha Z(v(S_0^2x)) + p(S_2x)W(v(S_0^2x)) + \mu_2 W(v(S_0^2x)), \tag{51}$$

$$\begin{aligned} Bv(S_0S_2x) = r(S_0S_2x) + \lambda Tv(S_0S_2x) + \mu_2 Y_2(v(S_0S_2x)) \\ + \alpha Z(v(S_0S_2x)) + p(S_2x)W(v(S_0S_2x)). \end{aligned} \tag{52}$$

We will consider the terms with λ, μ_2, α, and $p(S_2x)$ in (49), (50), (51), and (52) separately. First, it is clear that

$$r(S_0x) - r(S_2x) - r(S_0^2x) + r(S_0S_2x) = 0, \tag{53}$$

Second,

$$\lambda \left(Tv(S_0x) - Tv(S_2x) - Tv(S_0^2x) + Tv(S_0S_2x)\right) \le 0. \tag{54}$$

Note that (54) is obtained in exactly the same way as (41) because if $1 \in \Psi_{-1}(x)$, the derivation of (41) is not altered. Next, let us consider the terms with μ_2. By Lemma 2,

$$\mu_2 \left(W(v(S_0x)) - Y_2(v(S_2x)) - W(v(S_0^2x)) + Y_2(v(S_0S_2x))\right) \le 0. \tag{55}$$

Now for terms involving α, if $n(x) = 0$, then by (8), (16c) and (16d),

$$\begin{aligned} &\alpha \left(Z(v(S_0x)) - Z(v(S_2x)) - Z(v(S_0^2x)) + Z(v(S_0S_2x))\right) \\ &\qquad = \alpha \left(Tv(S_1x) - v(S_1S_2x) - Tv(S_1S_0x) + Tv(S_1S_2x)\right) \\ &\qquad = \alpha \left(v(S_1^2x) - v(S_1S_2x) - v(S_1^2S_0x) + v(S_1^2S_2x)\right) \\ &\qquad = \alpha \left(v(0,1,0) - v(0,0,1) - v(1,1,0) + v(0,1,1)\right) \\ &\qquad \le 0 \end{aligned}$$

by (30c). And if $n(x) > 0$, then by (8), (16c) and (16d),

$$\begin{aligned} &\alpha \left(Z(v(S_0x)) - Z(v(S_2x)) - Z(v(S_0^2x)) + Z(v(S_0S_2x))\right) \\ &\qquad = \alpha \left(Tv(S_1x) - Tv(S_0^{-1}S_1S_2x) - Tv(S_1S_0x) + Tv(S_1S_2x)\right) \\ &\qquad = \alpha \left(v(S_1^2x) - v(S_0^{-1}S_1^2S_2x) - v(S_1^2S_0x) + v(S_1^2S_2x)\right) \\ &\qquad \le 0 \end{aligned}$$

by (30a). Hence,

$$\alpha \left(Z(v(S_0x)) - Z(v(S_2x)) - Z(v(S_0^2x)) + Z(v(S_0S_2x))\right) \le 0. \tag{56}$$

Lastly, we consider the terms with $p(S_2x)$. In this case,

$$p(S_2x)\left(W(v(S_0x)) - W(v(S_2x)) - W(v(S_0^2x)) + W(v(S_0S_2x))\right) \leq 0. \tag{57}$$

Because $1 \in \Psi_{-1}(x)$ does not alter the derivation of (46) and (47), we can derive (57) in the same way that we derived (46) and (47). Hence, by (53), (54), (55), (56), and (57), inequality (48) holds and the lemma is proved. ■

Lemma 5 *If $v \in \Theta_1$, then for all $x \in E$ such that $1 \in \Psi_1(x)$ and $2 \in \Psi_0(x)$,*

$$Bv(S_0x) - Bv(x) - Bv(S_0S_2x) + Bv(S_2x) \leq 0. \tag{58}$$

Proof. By (6) and (9),

$$\begin{aligned} Bv(S_0x) = r(S_0x) + \lambda Tv(S_0x) + \mu_1 Y_1(v(S_0x)) + \xi Tv(S_1^{-2}S_0x) \\ +p(S_2x)W(v(S_0x)) + \mu_2 W(v(S_0x)), \end{aligned} \tag{59}$$

$$Bv(x) = r(x) + \lambda Tv(x) + \mu_1 Y_1(v(x)) + \xi Tv(S_1^{-2}x) + p(S_2x)W(v(x)) + \mu_2 W(v(x)), \tag{60}$$

$$\begin{aligned} Bv(S_0S_2x) = r(S_0S_2x) + \lambda Tv(S_0S_2x) + \mu_1 Y_1(v(S_0S_2x)) + \mu_2 Y_2(v(S_0S_2x)) \\ +\xi Tv(S_1^{-2}S_0S_2x) + p(S_2x)W(v(S_0S_2x)), \end{aligned} \tag{61}$$

$$\begin{aligned} Bv(S_2x) = r(S_2x) + \lambda Tv(S_2x) + \mu_1 Y_1(v(S_2x)) + \mu_2 Y_2(v(S_2x)) \\ +\xi Tv(S_1^{-2}S_2x) + p(S_2x)W(v(S_2x)). \end{aligned} \tag{62}$$

We will consider the terms with λ, μ_1, μ_2, ξ, and $p(S_2x)$ in (59), (60), (61), and (62) separately. First, it is clear that

$$r(S_0x) - r(x) - r(S_0S_2x) + r(S_2x) = 0. \tag{63}$$

Second,

$$\begin{aligned} \lambda\left(Tv(S_0x) - Tv(x) - Tv(S_0S_2x) + Tv(S_2x)\right) = \lambda\big(\min\{v(S_0^2x), v(S_0S_2x)\} \\ - \min\{v(S_0x), v(S_2x)\} - v(S_2S_0^2x) + v(S_0S_2x)\big). \end{aligned} \tag{64}$$

In case $v(S_0x) \leq v(S_2x)$, (64) becomes

$$\begin{aligned} &\lambda\left(\min\{v(S_0^2x), v(S_0S_2x)\} - \min\{v(S_0x), v(S_2x)\} - v(S_2S_0^2x) + v(S_0S_2x)\right) \\ &\leq \lambda\left(v(S_0^2x) - v(S_0x) - v(S_2S_0^2x) + v(S_0S_2x)\right) \\ &\leq 0 \end{aligned}$$

by (30b). In case $v(S_2x) \leq v(S_0x)$, (64) becomes

$$\begin{aligned} &\lambda\left(\min\{v(S_0^2x), v(S_0S_2x)\} - \min\{v(S_0x), v(S_2x)\} - v(S_2S_0^2x) + v(S_0S_2x)\right) \\ &\leq \lambda\left(2v(S_0S_2x) - v(S_2x) - v(S_2S_0^2x)\right) \\ &\leq 0 \end{aligned}$$

by (32). Hence,

$$\lambda \left(Tv(S_0x) - Tv(x) - Tv(S_0S_2x) + Tv(S_2x)\right) \leq 0. \tag{65}$$

Next, we consider the terms with μ_1. If $n(x) = 0$, then by (7),

$$\begin{aligned}
\mu_1 &\left(Y_1(v(S_0x)) - Y_1(v(x)) - Y_1(v(S_0S_2x)) + Y_1(v(S_2x))\right) \\
&= \mu_1 \left(Tv(S_1^{-1}x) - v(S_1^{-1}x) - Tv(S_1^{-1}S_2x) + v(S_1^{-1}S_2x)\right) \\
&= \mu_1 \left(v(x) - v(S_1^{-1}x) - v(S_2x) + v(S_1^{-1}S_2x)\right) \\
&= \mu_1 \left(v(0,1,0) - v(0,0,0) - v(0,1,1) + v(0,0,1)\right) \\
&\leq 0
\end{aligned}$$

by (30d). If $n(x) > 0$, then by (7),

$$\begin{aligned}
\mu_1 &\left(Y_1(v(S_0x)) - Y_1(v(x)) - Y_1(v(S_0S_2x)) + Y_1(v(S_2x))\right) \\
&= \mu_1 \left(Tv(S_1^{-1}x) - Tv(S_0^{-1}S_1^{-1}x) - Tv(S_1^{-1}S_2x) + Tv(S_0^{-1}S_1^{-1}S_2x)\right) \\
&= \mu_1 \left(v(x) - v(S_0^{-1}x) - v(S_2x) + v(S_0^{-1}S_2x)\right) \\
&\leq 0
\end{aligned}$$

by (30b). Hence,

$$\mu_1 \left(Y_1(v(S_0x)) - Y_1(v(x)) - Y_1(v(S_0S_2x)) + Y_1(v(S_2x))\right) \leq 0. \tag{66}$$

Now, we consider the terms with μ_2. First, by (17),

$$\mu_2 \left(W(v(S_0x)) - W(v(x)) - Y_2(v(S_0S_2x)) + Y_2(v(S_2x))\right) = 0. \tag{67}$$

Considering the terms with ξ, we obtain

$$\xi \left(Tv(S_1^{-2}S_0x) - Tv(S_1^{-2}x) - Tv(S_1^{-2}S_0S_2x) + Tv(S_1^{-2}S_2x)\right) \leq 0. \tag{68}$$

We can easily derive (68) as we derived (65). These two differ in that $1 \in \Psi_1(x)$ in the former and $1 \in \Psi_{-1}(x)$ in the latter, which is inconsequential because a customer cannot be routed to server 1 in either case. Finally, we consider terms involving $p(S_2x)$. When $n(x) = 0$, by (10),

$$\begin{aligned}
p(S_2x) &\left[W(v(S_0x)) - W(v(x)) - W(v(S_0S_2x)) + W(v(S_2x))\right] \\
&= p(S_2x) \left[Tv(x) - v(x) - Tv(S_2x) + v(S_2x)\right] \\
&= p(S_2x) \left[\min\{v(S_0x), v(S_2x)\} - v(x) - v(S_0S_2x) + v(S_2x)\right] \\
&\leq p(S_2x) \left[v(S_0x) - v(x) - v(S_0S_2x) + v(S_2x)\right] \\
&\leq 0
\end{aligned} \tag{69}$$

by (30b). When $n(x) > 0$, by (10),

$$\begin{aligned}
p(S_2x) &\left[W(v(S_0x)) - W(v(x)) - W(v(S_0S_2x)) + W(v(S_2x))\right] \\
&= p(S_2x) \left[Tv(x) - Tv(S_0^{-1}x) - Tv(S_2x) + Tv(S_0^{-1}S_2x)\right] \\
&\leq 0
\end{aligned} \tag{70}$$

where (70) can be derived as (65) was derived. Then, by (69) and (70),

$$p(S_2x)\left[W(v(S_0x)) - W(v(x)) - W(v(S_0S_2x)) + W(v(S_2x))\right] \le 0. \tag{71}$$

Hence, by (63), (65), (66), (67), (68), and (71), inequality (58) holds, and Lemma 5 is proved. ■

Lemma 6 *If $v \in \Theta_1$, then for all $x \in E$ such that $1 \in \Psi_{-1}(x)$ and $2 \in \Psi_0(x)$,*

$$Bv(S_0x) - Bv(x) - Bv(S_0S_2x) + Bv(S_2x) \le 0. \tag{72}$$

Proof. By (6) and (9),

$$Bv(S_0x) = r(S_0x) + \lambda Tv(S_0x) + \alpha Z(v(S_0x)) + p(S_2x)W(v(S_0x)) + \mu_2 W(v(S_0x)), \tag{73}$$

$$Bv(x) = r(x) + \lambda Tv(x) + \alpha Z(v(x)) + p(S_2x)W(v(x)) + \mu_2 W(v(x)), \tag{74}$$

$$\begin{aligned} Bv(S_0S_2x) = r(S_0S_2x) + \lambda Tv(S_0S_2x) + \mu_2 Y_2(v(S_0S_2x)) \\ +\alpha Z(v(S_0S_2x)) + p(S_2x)W(v(S_0S_2x)), \end{aligned} \tag{75}$$

$$Bv(S_2x) = r(S_2x) + \lambda Tv(S_2x) + \mu_2 Y_2(v(S_2x)) + \alpha Z(v(S_2x)) + p(S_2x)W(v(S_2x)). \tag{76}$$

We will consider the terms with λ, μ_2, α, and $p(S_2x)$ in (73), (74), (75), and (76) separately. First, it is clear that

$$r(S_0x) - r(x) - r(S_0S_2x) + r(S_2x) = 0. \tag{77}$$

Second,

$$\lambda\left(Tv(S_0x) - Tv(x) - Tv(S_0S_2x) + Tv(S_2x)\right) \le 0. \tag{78}$$

Inequality (78) is obtained in the same way as (65). When we consider the terms with μ_2, by (17),

$$\mu_2\left(W(v(S_0x)) - W(v(x)) - Y_2(v(S_0S_2x)) + Y_2(v(S_2x))\right) = 0. \tag{79}$$

Next, consider terms involving α. If $n(x) = 0$, then by (8),

$$\begin{aligned} &\alpha\left(Z(v(S_0x)) - Z(v(x)) - Z(v(S_0S_2x)) + Z(v(S_2x))\right) \\ &\qquad = \alpha\left(Tv(S_1x) - v(S_1x) - Tv(S_1S_2x) + v(S_1S_2x)\right) \\ &\qquad = \alpha\left(v(S_1^2x) - v(S_1x) - v(S_1^2S_2x) + v(S_1S_2x)\right) \\ &\qquad = \alpha\left(v(0,1,0) - v(0,0,0) - v(0,1,1) + v(0,0,1)\right) \\ &\qquad \le 0 \end{aligned}$$

by (30d). If $n(x) > 0$, then by (8),

$$\begin{aligned} &\alpha\left(Z(v(S_0x)) - Z(v(x)) - Z(v(S_0S_2x)) + Z(v(S_2x))\right) \\ &\qquad = \alpha\left(Tv(S_1x) - Tv(S_0^{-1}S_1x) - Tv(S_1S_2x) + Tv(S_0^{-1}S_1S_2x)\right) \\ &\qquad = \alpha\left(v(S_1^2x) - v(S_0^{-1}S_1^2x) - v(S_1^2S_2x) + v(S_0^{-1}S_1^2S_2x)\right) \\ &\qquad \le 0 \end{aligned}$$

by (30b). Hence,

$$\alpha\left(Z(v(S_0x)) - Z(v(x)) - Z(v(S_0S_2x)) + Z(v(S_2x))\right) \le 0. \tag{80}$$

Lastly, we consider the terms with $p(S_2x)$. In this case, we can derive the following inequality as we derived (71), so

$$p(S_2x)\left[W(v(S_0x)) - W(v(x)) - W(v(S_0S_2x)) + W(v(S_2x))\right] \le 0. \tag{81}$$

Hence, by (77), (78), (79), (80), and (81), inequality (72) holds, and Lemma 6 is proved. ■

Lemma 7 *If $v \in \Theta_1$, then*

$$Bv(0,1,0) - Bv(0,0,1) - Bv(1,1,0) + Bv(0,1,1) \le 0. \tag{82}$$

Proof. By (6) and (9),

$$\begin{aligned} Bv(0,1,0) = {} & r(0,1,0) + \lambda Tv(0,1,0) + \mu_1 Y_1(v(0,1,0)) + \xi Tv(0,-1,0) \\ & + p(0,1,1)W(v(0,1,0)) + \mu_2 W(v(0,1,0)), \end{aligned} \tag{83}$$

$$\begin{aligned} Bv(0,0,1) = {} & r(0,0,1) + \lambda Tv(0,0,1) + \mu_2 Y_2(v(0,0,1)) + \xi v(0,-1,1) \\ & + p(0,1,1)W(v(0,0,1)) + \mu_1 W(v(0,0,1)), \end{aligned} \tag{84}$$

$$\begin{aligned} Bv(1,1,0) = {} & r(1,1,0) + \lambda Tv(1,1,0) + \mu_1 Y_1(v(1,1,0)) + \xi Tv(1,-1,0) \\ & + p(0,1,1)W(v(1,1,0)) + \mu_2 W(v(1,1,0)), \end{aligned} \tag{85}$$

$$\begin{aligned} Bv(0,1,1) = {} & r(0,1,1) + \lambda Tv(0,1,1) + \mu_1 Y_1(v(0,1,1)) + \mu_2 Y_2(v(0,1,1)) \\ & + \xi Tv(0,-1,1) + p(0,1,1)W(v(0,1,1)). \end{aligned} \tag{86}$$

We will consider the terms with λ, μ_1, μ_2, ξ, and $p(0,1,1)$ in (83), (84), (85), and (86) separately. First, it is clear that

$$r(0,1,0) - r(0,0,1) - r(1,1,0) + r(0,1,1) = 0. \tag{87}$$

Next, let us consider the terms with λ. By (16d),

$$\begin{aligned} \lambda\left(Tv(0,1,0) - Tv(0,0,1) - Tv(1,1,0) + Tv(0,1,1)\right) = {} & \lambda\big[\min\{v(0,1,1), v(1,1,0)\} \\ & - v(0,1,1) - \min\{v(1,1,1), v(2,1,0)\} + v(1,1,1)\big]. \end{aligned} \tag{88}$$

According to (88), we have two cases to consider. First, in case $v(1,1,1) \le v(2,1,0)$, then

$$\begin{aligned} & \lambda\left(Tv(0,1,0) - Tv(0,0,1) - Tv(1,1,0) + Tv(0,1,1)\right) \\ & \quad \le \lambda\left[\min\{v(0,1,1), v(1,1,0)\} - v(0,1,1) - v(1,1,1) + v(1,1,1)\right] \\ & \quad \le \lambda\left(v(0,1,1) - v(0,1,1) - v(1,1,1) + v(1,1,1)\right) \\ & \quad = 0. \end{aligned}$$

Second, consider the case in which $v(2,1,0) \le v(1,1,1)$. Then,

$$\begin{aligned}
&\lambda\left(Tv(0,1,0) - Tv(0,0,1) - Tv(1,1,0) + Tv(0,1,1)\right)\\
&\qquad\le \lambda\left[\min\{v(0,1,1), v(1,1,0)\} - v(0,1,1) - v(2,1,0) + v(1,1,1)\right]\\
&\qquad\le \lambda\left(v(1,1,0) - v(0,1,1) - v(2,1,0) + v(1,1,1)\right)\\
&\qquad\le 0
\end{aligned}$$

by (30a). Hence,

$$\lambda\left(Tv(0,1,0) - Tv(0,0,1) - Tv(1,1,0) + Tv(0,1,1)\right) \le 0. \tag{89}$$

Next, let us consider terms involving μ_1. By (7), (10), (16c), and (16d),

$$\begin{aligned}
&\mu_1(Y_1(v(0,1,0)) - W(v(0,0,1)) - Y_1(v(1,1,0)) + Y_1(v(0,1,1)))\\
&\qquad= \mu_1(v(0,0,0) - v(0,0,1) - v(0,1,0) + v(0,0,1))\\
&\qquad= \mu_1(v(0,0,0) - v(0,1,0)).
\end{aligned} \tag{90}$$

When we consider the terms with μ_2 we obtain, by (7) and (10),

$$\begin{aligned}
&\mu_2(W(v(0,1,0)) - Y_2(v(0,0,1)) - W(v(1,1,0)) + Y_2(v(0,1,1)))\\
&\qquad= \mu_2(v(0,1,0) - v(0,0,0) - Tv(0,1,0) + v(0,1,0))\\
&\qquad= \mu_2(2v(0,1,0) - v(0,0,0) - \min\{v(1,1,0), v(0,1,1)\}).
\end{aligned} \tag{91}$$

Summing (90) and (91), we obtain the following result by (16a) and (16b):

$$(\mu_1 - \mu_2)\left[v(0,0,0) - v(0,1,0)\right] + \mu_2\left[v(0,1,0) - \min\{v(1,1,0), v(0,1,1)\}\right] \le 0. \tag{92}$$

Therefore, (92) shows that the sum of terms with μ_1 and μ_2 cannot exceed zero. Next, when we consider terms involving ξ,

$$\begin{aligned}
&\xi\left(Tv(0,-1,0) - v(0,-1,1) - Tv(1,-1,0) + Tv(0,-1,1)\right)\\
&\qquad= \xi[\min\{v(0,-1,1), v(1,-1,0)\} - v(0,-1,1)\\
&\qquad\qquad - \min\{v(1,-1,1), v(2,-1,0)\} + v(1,-1,1)]\\
&\qquad=: \Gamma_3.
\end{aligned}$$

Now we have two cases two consider. First, if $v(1,-1,1) \le v(2,-1,0)$,

$$\Gamma_3 \le \xi\left(v(0,-1,1) - v(0,-1,1) - v(1,-1,1) + v(1,-1,1)\right) = 0. \tag{93}$$

Second, if $v(2,-1,0) \le v(1,-1,1)$, by (30a),

$$\Gamma_3 \le \xi\left(v(1,-1,0) - v(0,-1,1) - v(2,-1,0) + v(1,-1,1)\right) \le 0. \tag{94}$$

Lastly, we consider the terms with $p(0,1,1)$. By (10),

$$\begin{aligned}
&p(0,1,1)\left(W(v(0,1,0)) - W(v(0,0,1)) - W(v(1,1,0)) + W(v(0,1,1))\right)\\
&\qquad= p(0,1,1)\left[v(0,1,0) - v(0,0,1) - \min\{v(1,1,0), v(0,1,1)\} + v(0,1,1)\right]\\
&\qquad=: \Gamma_4.
\end{aligned}$$

First consider the case of $v(1,1,0) \leq v(0,1,1)$. By (30c),

$$\Gamma_4 = p(0,1,1)\left(v(0,1,0) - v(0,0,1) - v(1,1,0) + v(0,1,1)\right) \leq 0. \tag{95}$$

Second, if $v(0,1,1) \leq v(1,1,0)$, then by (16c),

$$\Gamma_4 = p(0,1,1)(v(0,1,0) - v(0,0,1)) \leq 0. \tag{96}$$

Hence, by (87), (89), (92), (93), (94), (95), and (96), inequality (82) holds, so Lemma 7 is proved. ■

Lemma 8 *If $v \in \Theta_1$, then*

$$Bv(0,1,0) - Bv(0,0,0) - Bv(0,1,1) + Bv(0,0,1) \leq 0. \tag{97}$$

Proof. By (6) and (9),

$$\begin{aligned} Bv(0,1,0) = r(0,1,0) + \lambda Tv(0,1,0) + \mu_1 Y_1(v(0,1,0)) + \xi Tv(0,-1,0) \\ +p(0,1,1)W(v(0,1,0)) + \mu_2 W(v(0,1,0)), \end{aligned} \tag{98}$$

$$\begin{aligned} Bv(0,0,0) = r(0,0,0) + \lambda Tv(0,0,0) + \xi v(0,-1,0) + p(0,1,1)W(v(0,0,0)) \\ +\mu_1 W(v(0,0,0)) + \mu_2 W(v(0,0,0)), \end{aligned} \tag{99}$$

$$\begin{aligned} Bv(0,1,1) = r(0,1,1) + \lambda Tv(0,1,1) + \mu_1 Y_1(v(0,1,1)) + \mu_2 Y_2(v(0,1,1)) \\ +\xi Tv(0,-1,1) + p(0,1,1)W(v(0,1,1)), \end{aligned} \tag{100}$$

$$\begin{aligned} Bv(0,0,1) = r(0,0,1) + \lambda Tv(0,0,1) + \mu_2 Y_2(v(0,0,1)) + \xi v(0,-1,1) \\ +p(0,1,1)W(v(0,0,1)) + \mu_1 W(v(0,0,1)). \end{aligned} \tag{101}$$

We will consider the terms with λ, μ_1, μ_2, ξ, and $p(0,1,1)$ in (98), (99), (100), and (101) separately. First, it is clear that

$$r(0,1,0) - r(0,0,0) - r(0,1,1) + r(0,0,1) = 0. \tag{102}$$

Next, let us consider the terms with λ. By (16c) and (16d),

$$\begin{aligned} &\lambda\left(Tv(0,1,0) - Tv(0,0,0) - Tv(0,1,1) + Tv(0,0,1)\right) \\ &\quad = \lambda\left[\min\{v(0,1,1), v(1,1,0)\} - v(0,1,0) - v(1,1,1) + v(0,1,1)\right] \\ &\quad \leq \lambda\left(v(1,1,0) - v(0,1,0) - v(1,1,1) + v(0,1,1)\right) \\ &\quad \leq 0, \end{aligned} \tag{103}$$

where (103) is due to (30b). Next, let us consider the terms with μ_1. By (7) and (10),

$$\begin{aligned} &\mu_1(Y_1(v(0,1,0)) - W(v(0,0,0)) - Y_1(v(0,1,1)) + W(v(0,0,1))) \\ &\quad = \mu_1(v(0,0,0) - v(0,0,0) - v(0,0,1) + v(0,0,1)) \\ &\quad = 0. \end{aligned} \tag{104}$$

When we consider the terms with μ_2, by (7) and (10),

$$\begin{aligned}\mu_2(W(v(0,1,0)) - W(v(0,0,0)) - Y_2(v(0,1,1)) + Y_2(v(0,0,1)))&\\ = \mu_2(v(0,1,0) - v(0,0,0) - v(0,1,0) + v(0,0,0))&\\ = 0.& \qquad (105)\end{aligned}$$

Next, let us consider the terms with ξ.

$$\begin{aligned}&\xi\,(Tv(0,-1,0) - v(0,-1,0) - Tv(0,-1,1) + v(0,-1,1))\\ &\quad = \xi\,[\min\{v(0,-1,1), v(1,-1,0)\} - v(0,-1,0) - v(1,-1,1) + v(0,-1,1)]\\ &\quad \le \xi\,(v(1,-1,0) - v(0,-1,0) - v(1,-1,1) + v(0,-1,1))\\ &\quad \le 0, \qquad (106)\end{aligned}$$

where (106) is due to (30b). Lastly, we consider the terms with $p(0,1,1)$. In this case, by (10),

$$\begin{aligned}&p(0,1,1)\,(W(v(0,1,0)) - W(v(0,0,0)) - W(v(0,1,1)) + W(v(0,0,1)))\\ &\quad = p(0,1,1)\,(v(0,1,0) - v(0,0,0) - v(0,1,1) + v(0,0,1))\\ &\quad \le 0, \qquad (107)\end{aligned}$$

where (107) is due to (30d). Hence, by (102), (103), (104), (105), (106), and (107), inequality (97) holds, so Lemma 8 is proved. ∎

Lemma 9 *If $v \in \Theta_1$, then $Bv \in \Theta_1$.*

Proof. The proof follows directly from Proposition 2 and Lemmas 3, 4, 5, 6, 7, and 8. ∎

In Section 2.1, it was shown that the value iteration algorithm converges to the unique value function; hence, we use this technique to prove Theorem 2. Let V_k denote the value function approximation at the kth iteration, $k \in \{0\} \cup \mathbb{N}$, and assume that $V_0(x) = 0$ for all $x \in E$. Then it is clear that $V_0 \in \Theta_1$, i.e., V_0 satisfies inequalities (16a)–(16d) and (30a)–(30d). Since $V_1 = BV_0$, $V_1 \in \Theta_1$ by Lemma 9. Thus, inductively, $V_k \in \Theta_1$ for all $k \in \{0\} \cup \mathbb{N}$ because $V_k = BV_{k-1}$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} V_k(x) = V(x)$ for all $x \in E$, then $V \in \Theta_1$. This implies that for all $x \in E$ such that $1 \notin \Psi_0(x)$ and $2 \in \Psi_0(x)$,

$$V(S_0x) - V(S_2x) \le V(S_0^2x) - V(S_0S_2x). \qquad (108)$$

Formally, if it is optimal to route a customer to the queue in state $S_0x \in E$ such that $1 \notin \Psi_0(S_0x)$ and $2 \in \Psi_0(S_0x)$, then it is also optimal to route a customer to the queue in state $x \in E$ such that $1 \notin \Psi_0(x)$ and $2 \in \Psi_0(x)$ by (108). Hence, a threshold-type policy is optimal for the slow server.

Proof of Proposition 6

Here we prove Proposition 6. To this end, we first state and prove the following proposition, from which Proposition 6 follows directly.

Proposition 7 *If $v \in \Theta_2$, then $Tv \in \Theta_2$ where the operator T is defined by* (5).

Proof. Let us first consider preservation of (29a). Considering the fact that $\Psi_0(x) = \Psi_0(S_0x)$,

$$\begin{aligned} Tv(x) - Tv(S_0x) &= \min_{a\in A(x)} v(S_ax) - \min_{a\in A(S_0x)} v(S_aS_0x) \\ &= \min\{v(S_0x), v(S_ix) : i \in \Psi_0(x)\} - \min\{v(S_0^2x), v(S_iS_0x) : i \in \Psi_0(x)\} \\ &\leq 0 \end{aligned}$$

by (29a) and Lemma 1. Next, we consider the preservation of (29b). Note that,

$$\begin{aligned} &Tv(x) - Tv(S_jx) \\ &\quad = \min\{v(S_0x), v(S_ix) : i \in \Psi_0(x)\} - \min\{v(S_0S_jx), v(S_iS_jx) : i \in \Psi_0(S_jx)\} \\ &\quad \leq 0 \end{aligned}$$

by (29a), (29b), and Lemma 1. Next, by (29c) and (29d),

$$\begin{aligned} Tv(S_ix) - Tv(S_jx) &= \min\{v(S_0S_ix), v(S_kS_ix) : k \in \Psi_0(S_ix)\} \\ &\qquad - \min\{v(S_0S_jx), v(S_kS_jx) : k \in \Psi_0(S_jx)\} \\ &= v(S_jS_ix) - v(S_iS_jx) \\ &= 0. \end{aligned}$$

Finally, using (29c), (29d) and Lemma 1,

$$\begin{aligned} Tv(S_jx) - Tv(S_0x) = \min\{v(S_0S_jx), v(S_iS_jx) : i \in \Psi_0(S_jx)\} \\ - \min\{v(S_0^2x), v(S_iS_0x) : i \in \Psi_0(S_0x)\} \leq 0 \end{aligned}$$

∎

Now, to prove Proposition 6, we we will show that the operator B' preserves the properties of functions satisfying (29a), (29b), (29c), and (29d), respectively. We first show that if $v \in \Theta_2$, then $B'v(x) \leq B'v(S_0x)$ for all $x \in E$. It is clear that $r(x) - r(S_0x) = -1$ and $p(x) = p(S_0x)$ by (28). Moreover, by (7), (27), and Proposition 7, it is clear that, if $v \in \Theta_2$ and $n(x) > 0$, $Y_j(v(x)) \leq Y_j(v(S_0x))$ for all $j \in \Psi_1(x)$ and $Z_j(v(x)) \leq Z_j(v(S_0x))$ for all $j \in \Psi_{-1}(x)$. When $n(x) = 0$, for all $j \in \Psi_1(x)$,

$$\begin{aligned} Y_j(v(x)) - Y_j(v(S_0x)) &= v(S_j^{-1}x) - Tv(S_j^{-1}x) \\ &= v(S_j^{-1}x) - \min\{v(S_0S_j^{-1}x), v(S_iS_j^{-1}x) : i \in \Psi_0(S_j^{-1}x)\} \\ &\leq 0 \end{aligned}$$

by (29a), (29b), and Lemma 1. In a similar way, when $n(x) = 0$ and if $j \in \Psi_{-1}(x)$, $Z_j(v(x)) \leq Z_j(v(S_0x))$ by (29a), (29b), and Lemma 1. Then, by the fact that $\Psi_i(x) = \Psi_i(S_0x)$ for all $i \in$

$\{-1,0,1\}$, (29a), (26), and Proposition 7,

$$\begin{aligned}
&B'v(x) - B'v(S_0x) \\
&\quad = r(x) - r(S_0x) + \lambda(Tv(x) - Tv(S_0x)) + \sum_{j\in\Psi_1(x)} \mu(Y_j(v(x)) - Y_j(v(S_0x))) \\
&\quad + \sum_{j\in\Psi_1(x)} \xi_j(Tv(S_j^{-2}x) - Tv(S_j^{-2}S_0x)) + \sum_{j\in\Psi_{-1}(x)} \alpha_j(Z_j(v(x)) - Z_j(v(S_0x))) \\
&\quad + \sum_{j\in\Psi_0(x)} \xi_j(v(S_j^{-1}x) - v(S_0S_j^{-1}x)) + p(x)(v(x) - v(S_0x)) \\
&\quad \le 0.
\end{aligned}$$

Next, we show that if $v \in \Theta_2$, then $B'v(x) \le B'v(S_jx)$ for all $x \in E$ such that $j \in \Psi_0(x)$ and $n(x) = 0$. After some algebraic manipulation and considering (26), (28), and Proposition 7,

$$\begin{aligned}
&B'v(x) - B'v(S_jx) \\
&\qquad \le \sum_{k\in\Psi_1(x)} \left[\mu(Y_k(v(x)) - Y_k(v(S_jx)) + \xi_k(Tv(S_k^{-2}x) - Tv(S_k^{-2}S_jx))\right] \\
&\qquad - \mu Y_j(v(S_jx)) - \xi_j Tv(S_j^{-1}x) + \sum_{k\in\Psi_{-1}(x)} \alpha_k(Z_k(v(x)) - Z_k(v(S_jx))) \\
&\qquad + \sum_{k\in\Psi_0(S_jx)} \xi_k(v(S_k^{-1}x) - v(S_k^{-1}S_jx)) + \xi_j v(S_j^{-1}x) \\
&\qquad + p(S_jx)(v(x) - v(S_jx)) + \mu v(x).
\end{aligned} \tag{109}$$

Note that, by (7), it is easy to see that when $n(x) = 0$, $Y_k(v(x)) \le Y_k(v(S_jx)$ for all $k \in \Psi_1(x)$. Similarly when $n(x) = 0$, $Z_k(v(x)) \le Z_k(v(S_jx)$ for all $k \in \Psi_{-1}(x)$ by (27). Moreover,

$$\begin{aligned}
v(S_j^{-1}x) - Tv(S_j^{-1}x) &= v(S_j^{-1}x) - \min\{v(S_0S_j^{-1}x), v(S_iS_j^{-1}x) : i \in \Psi_0(S_j^{-1}x)\} \\
&\le 0
\end{aligned} \tag{110}$$

by (29a), (29b), and Lemma 1. Then by (7), (29b), (109), (110), Proposition 7, and the fact that $n(x) = 0$,

$$B'v(x) - B'v(S_jx) \le \mu[v(x) - Y_j(v(S_jx))] = \mu\,[v(x) - v(x)] = 0.$$

Next, we show that if $v \in \Theta_2$, then $B'v(S_ix) = B'v(S_jx)$ for all $x \in E$ such that $i, j \in \Psi_0(x)$. By (7), (26), (28), (29c), Proposition 7, and the fact that $p(S_ix) = p(S_jx)$ when $i, j \in \Psi_0(x)$; we get the following result,

$$\begin{aligned}
B'v(S_ix) - B'v(S_jx) &= r(S_ix) - r(S_jx) + \lambda(Tv(S_ix) - Tv(S_jx)) + \mu Y_i(v(S_ix)) \\
&+ \xi_i Tv(S_i^{-1}x) + \xi_j v(S_j^{-1}S_ix) - \mu Y_j(v(S_jx)) - \xi_j Tv(S_j^{-1}x) - \xi_i v(S_i^{-1}S_jx) \\
&+ \sum_{k\in\Psi_1(x)} \left[\mu(Y_k(v(S_ix)) - Y_k(v(S_jx))) + \xi_k(Tv(S_k^{-2}S_ix) - Tv(S_k^{-2}S_jx))\right] \\
&+ \sum_{k\in\Psi_0(S_iS_jx)} \xi_k(v(S_k^{-1}S_ix) - v(S_k^{-1}S_jx)) + p(S_ix)v(S_ix) - p(S_jx)v(S_jx) \\
&= \mu(Y_i(v(S_ix)) - Y_j(v(S_jx))) + \xi_i(Tv(S_i^{-1}x) - v(S_i^{-1}S_jx)) \\
&\qquad + \xi_j(v(S_j^{-1}S_ix) - Tv(S_j^{-1}x)).
\end{aligned} \tag{111}$$

Note that, by (7),

$$Y_i(v(S_ix)) = Y_j(v(S_jx)). \tag{112}$$

Moreover,

$$\begin{aligned}
Tv(S_i^{-1}x) - v(S_i^{-1}S_jx) &= \min\left\{v(S_0S_i^{-1}x), v(S_kS_i^{-1}x) : k \in \Psi_0(S_i^{-1}x)\right\} - v(S_i^{-1}S_jx) \\
&= v(S_jS_i^{-1}x) - v(S_i^{-1}S_jx) \\
&= 0,
\end{aligned} \tag{113}$$

by (29c) and (29d). Next,

$$\begin{aligned}
v(S_j^{-1}S_ix) - Tv(S_j^{-1}x) &= v(S_j^{-1}S_ix) - \min\left\{v(S_0S_j^{-1}x), v(S_kS_j^{-1}x) : k \in \Psi_0(S_j^{-1}x)\right\} \\
&= v(S_j^{-1}S_ix) - v(S_iS_j^{-1}x) \\
&= 0,
\end{aligned} \tag{114}$$

by (29c) and (29d). Hence, by (111), (112), (113), and (114),

$$B'v(S_ix) - B'v(S_jx) = 0.$$

Finally, we show that if $v \in \Theta_2$, then $B'v(S_jx) \leq B'v(S_0x)$ for all $x \in E$ such that $j \in \Psi_0(x)$. After some algebraic manipulation and using (26), (28), and Proposition 7, we obtain

$$\begin{aligned}
B'v(S_jx) - B'v(S_0x) &\leq \mu Y_j(v(S_jx)) + \xi_j(Tv(S_j^{-1}x) - v(S_j^{-1}S_0x)) \\
&\quad + \sum_{k\in\Psi_1(x)} \left[\mu(Y_k(v(S_jx)) - Y_k(v(S_0x))) + \xi_k(Tv(S_k^{-2}S_jx) - Tv(S_k^{-2}S_0x))\right] \\
&\quad + \sum_{k\in\Psi_0(S_jx)} \left[\xi_k(v(S_k^{-1}S_jx) - v(S_k^{-1}S_0x))\right] + \sum_{k\in\Psi_{-1}(x)} \left[\alpha_k(Z_k(v(S_jx)) - Z_k(v(S_0x)))\right] \\
&\quad + p(S_jx)(v(S_jx) - v(S_0x)) - \mu v(S_0x).
\end{aligned} \tag{115}$$

First, by Lemma 1,

$$\begin{aligned}
Tv(S_j^{-1}x) - v(S_j^{-1}S_0x) &= \min\{v(S_0S_j^{-1}x), v(S_kS_j^{-1}x) : k \in \Psi_0(S_j^{-1}x)\} - v(S_j^{-1}S_0x) \\
&\leq 0.
\end{aligned} \tag{116}$$

Moreover, when $k \in \Psi_1(x)$ and $n(x) = 0$, by (7),

$$\begin{aligned}
Y_k(v(S_jx)) - Y_k(v(S_0x)) &= v(S_k^{-1}S_jx) - Tv(S_k^{-1}x) \\
&= v(S_k^{-1}S_jx) - \min\left\{v(S_0S_k^{-1}x), v(S_lS_k^{-1}x) : l \in \Psi_0(S_k^{-1}x)\right\} \\
&= v(S_k^{-1}S_jx) - v(S_jS_k^{-1}x) \\
&= 0
\end{aligned} \tag{117}$$

by (29c) and (29d). Second, when $k \in \Psi_1(x)$ and $n(x) > 0$, by (7) and Proposition 7

$$Y_k(v(S_jx)) - Y_k(v(S_0x)) = Tv(S_0^{-1}S_k^{-1}S_jx) - Tv(S_k^{-1}x) \leq 0. \tag{118}$$

Similarly, when $k \in \Psi_{-1}(x)$ and $n(x) = 0$, by (27),

$$\begin{aligned} Z_k(v(S_jx)) - Z_k(v(S_0x)) &= v(S_kS_jx) - Tv(S_kx) \\ &= v(S_kS_jx) - \min\{v(S_0S_kx), v(S_lS_kx) : l \in \Psi_0(S_kx)\} \\ &= v(S_kS_jx) - v(S_jS_kx) \\ &= 0 \end{aligned} \tag{119}$$

by (29c) and (29d). Second, when $k \in \Psi_{-1}(x)$ and $n(x) > 0$, by (27) and Proposition 7

$$Z_k(v(S_jx)) - Z_k(v(S_0x)) = Tv(S_0^{-1}S_kS_jx) - Tv(S_kx) \leq 0. \tag{120}$$

Then, by (29d), (115), (116), (117), (118), (119), (120), and Proposition 7;

$$\begin{aligned} B'v(S_jx) - B'v(S_0x) &\leq \mu(Y_j(v(S_jx)) - v(S_0x)) \\ &=: \Gamma_5. \end{aligned}$$

Now, when $n(x) = 0$, by (7) and (29a),

$$\Gamma_5 = \mu(v(x) - v(S_0x)) \leq 0.$$

When $n(x) > 0$, by (7),

$$\begin{aligned} \Gamma_5 &= \mu(Tv(S_0^{-1}x) - v(S_0x)) \\ &= \mu(\min\{v(x), v(S_iS_0^{-1}x) : i \in \Psi_0(x)\} - v(S_0x)) \\ &\leq \mu(v(x) - v(S_0x)) \\ &\leq 0 \end{aligned}$$

by (29a). Hence, $B'v(S_jx) \leq B'v(S_0x)$ for all $x \in E$ such that $j \in \Psi_0(x)$.

Proof of Proposition 5

Consider the main model described in Section 2. By (16a), (16b), Proposition 2 and value iteration, we can see that $\min\{V(0,0,0), V(0,-1,0)\} \leq V(x)$ for all $x \in E$. So it is enough to prove that $V(0,0,0) \leq V(0,-1,0)$. Let Q_1 and Q_2 be two different processes defined on the same probability space so that they see the same arrivals, service completions, server failures, and server repairs. Suppose Q_1 and Q_2 are initially in states $(0,0,0)$ and $(0,-1,0)$, respectively. Let τ be the first time that server 1 has a failure in the process Q_1, which corresponds to a fictitious transition epoch at Q_2, or server 1 is repaired in the process Q_2, which corresponds to a fictitious transition epoch at Q_1. Assume that, until τ, the optimal policy is followed in process Q_2, and the same policy is executed in process Q_1. Then, just before τ, Q_1 and Q_2 should be in states $(n,0,d_2)$ and $(n,-1,d_2)$ for some $n \in \mathbb{N} \cup \{0\}$ and $d_2 \in \{0,1\}$, respectively. Note that the discounted expected total numbers in Q_1 and Q_2 are equal up to time τ. There are two cases to consider.

Case 1: Suppose time τ is the repair time of server 1 in the process Q_2. Then if $n = 0$, then Q_2 will enter the state $(0,0,d_2)$ and Q_1 will remain in the state $(0,0,d_2)$ by (10). If $n > 0$, then Q_2 will enter the state $(n-1,1,d_2)$ by Theorem 1, and since τ is a fictitious transition epoch for the process Q_1, we route a customer from the queue to server 1 at τ (which may be a sub-optimal action), thus the process Q_1 enters the state $(n-1, 1, d_2)$. From this point forward, if we follow the optimal policy in both Q_1 and Q_2, then the discounted expected total number of customers in Q_1 and Q_2 are equal to each other.

Case 2: Suppose time τ is the failure time of server 1 in the process Q_1. Then, Q_1 will enter the state $(n, -1, d_2)$. Since τ is a fictitious transition epoch in the process Q_2, if $d_2 = 1$, then the process Q_2 will remain in the state $(n, -1, d_2)$ by (10). If $d_2 = 0$, since we follow the optimal policy in the process Q_2 so far, then the optimal action at τ is to not to alter the state of the system by Proposition 4. Hence, the process Q_2 will remain in the state $(n, -1, d_2)$. From this point forward, if we follow the optimal policy in both Q_1 and Q_2, then the discounted expected total number of customers in Q_1 and Q_2 are equal to each other.

Considering these two cases, although we follow a possibly sub-optimal policy in the process Q_1 and the optimal policy in the process Q_2, the discounted expected total number of customers in Q_1 and Q_2 are equal to each other. Hence, $V(0,0,0) \leq V(0,-1,0)$ in the system in which we allow actions at the fictitious transition epochs. However, we know that the system controller will not change the state of the system starting from both the states $(0,0,0)$ and $(0,-1,0)$ by Proposition 4. Hence, $V(0,0,0) \leq V(0,-1,0)$ in the original system in which we do not allow actions at the fictitious transition epochs.