Optimal Replacement Policies Under Environment-Driven Degradation

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Abstract

We examine the problem of optimally maintaining a stochastically degrading system using preventive and reactive replacements. The system's rate of degradation is modulated by an exogenous stochastic environment process, and the system fails when its cumulative degradation level first reaches a fixed deterministic threshold. The objective is to minimize the total expected discounted cost of preventively and reactively replacing such a system over an infinite planning horizon. To this end, we present and analyze a Markov decision process model. It is shown that, for each environment state, there exists an optimal threshold-type replacement policy. Additionally, empirical evidence suggests that, when the environment process is monotone, and the state-dependent degradation rates are totally ordered, the optimal threshold is monotone. Lastly, we derive closed-form bounds on the optimal thresholds.

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1 Introduction

The use of sensors for real-time monitoring and condition assessment of critical components or systems has dramatically increased due to recent advances in sensor technology. These sensors have given rise to condition-based maintenance strategies that seek to preempt unanticipated failures and reduce the cost of maintaining critical systems. The condition-based paradigm stands in stark contrast to the traditional failure-based approach in which failure data might be sparse or even nonexistent (e.g., for highly reliable systems, or those that are prohibitively expensive to run to failure). Consequently, researchers and practitioners have been advocating condition-based techniques as an alternative to failure-based reliability assessment and maintenance optimization in order to exploit available data now attainable through advanced sensors.

Sensors, and the condition-based information they provide, can be especially useful for critical systems that operate in environments that vary randomly over time. That is, changes in the system's environmental conditions can often be related to the rate at which degradation accumulates. For example, the decomposition of a chemical coating can be assessed by considering its exposure to a variety of weather conditions. While some studies in the literature have focused on computing reliability and availability indices for such systems, determining the best strategies for maintaining or replacing such systems have not been fully explored. Optimal maintenance strategies for systems in randomly-varying environments will find wide applicability in a variety of contexts (e.g., wind and solar energy systems, manufacturing equipment, and aircraft engine components, to name a few).

The main objective of this work is to examine optimal replacement strategies for a stochastically degrading system whose linear rate of degradation is governed by an exogenous environment in which it resides and operates. The governing process, called the *random environment*, evolves as an ergodic continuous-time Markov chain (CTMC) on a finite state space. The system degrades over time until its cumulative level of degradation reaches a critical threshold, at which time the system is declared to be failed. Based on observations of the cumulative degradation and the current state of the environment, the system may be preventively replaced in advance of a failure at some fixed cost, or reactively replaced following a failure at a significantly higher cost. Our main objective is to analyze replacement policies for such a system and to explore the effect of a few model parameters on the optimal policy. To this end, we formulate the problem as a continuous-time, infinite-horizon discounted Markov decision process (MDP) model, examine the properties of the value function and analyze the structure of the resulting optimal policies.

Stochastic models that describe the deterioration of repairable and irreparable systems have a long history in the operations research and applied probability literature dating back to the early 1950s. Extensive surveys on the subject can be found in [14, 15, 17, 19, 22, 24]. Most classical models assume that the status of the system evolves as a finite-state Markov chain (in discrete or continuous time), and that the system operates in a static environment (i.e., degradation stems only from normal usage in a benign environment). Researchers who have considered the effects of a time-varying operating environment on single- and multi-unit systems have generally focused on modeling the degradation process itself with the aim of deriving and computing reliability or availability indices (cf. [3, 4, 6, 9, 12, 18, 21]). The vast majority of these studies examine stochastic shock and/or wear models in a random environment. Singpurwalla [20] provides an extensive review of a variety of stochastic deterioration models for systems that evolve in randomly-varying environments. Despite the extensive literature on systems under Markovian deterioration (i.e., those whose deterioration status evolves as a Markov chain), relatively few studies have considered the problem of prescribing optimal maintenance or replacement policies when the system is influenced by a time-varying, stochastic environment. Çekyay and Özekici [2] recently surveyed condition-based maintenance models, particularly those that lead to a structured control-limit policy. Waldmann

[23] appears to be the first to analyze the structure of an optimal replacement policy for a system subjected to stochastic deterioration in a random environment. More specifically, Waldmann [23] considers the effects of uncontrollable internal and external factors on the progression of the system's deterioration status in a continuous-time shock model and derives sufficient conditions to establish the optimality of a control-limit policy with respect to the cumulative damage of the system. Özekici [13] models the deterioration of a single-unit system in an uncontrollable environment by its intrinsic age where the environment evolves as a semi-Markov jump process, and the intrinsic age of the system is determined by the total cumulative degradation. It is shown that, if the system's lifetime distribution has an increasing failure rate in each environment state, then a control-limit policy, with respect to the system's status, is optimal; however, for a given system status, the relationship between the optimal actions in different environment states is not explored. Kurt and Maillart [11] examine the optimal replacement of a system that fails due to random shocks arriving according to a Poisson process whose rate is modulated by a discrete-time Markov chain. They consider both controllable and uncontrollable Markovian environments and analyze the structure of the resulting optimal cost function with respect to the shock arrival rate and the cumulative number of shocks received. Recently, Kurt and Kharoufeh [10] extend the model of [11] by relaxing the fixed cost assumption.

The model we present here differs from existing optimal replacement models in that the cumulative degradation of the system is represented as a continuous random variable, the rate of degradation is modulated by a continuous-time stochastic process, and the modulating environment process is uncontrollable (i.e., it evolves independently of the degradation process even after a replacement has occurred). These model attributes introduce some significant challenges to analyzing the properties of the cost function and the structure of the optimal policy. However, we posit that our assumptions are more realistic and provide a framework within which condition-and environment-based data can be used to provide informed strategies that do not ignore the future profile of the system's environment which can significantly affect future actions. We prove the monotonicity of the cost function with respect to the level of degradation for each environment state, and the existence of a threshold-type replacement policy. Under mild conditions, we also prove the monotonicity of the cost function with respect to the environment state for a fixed degradation level. Under the same conditions, numerical analysis provides evidence that the optimal threshold is monotone. Additionally, we establish closed-form bounds on the optimal thresholds. Several numerical examples are provided to illustrate the structure of the optimal policy and assess the impact of a few model parameters.

The remainder of the paper is organized as follows. In Section 2, we describe the dynamics of the cumulative degradation process before formulating the MDP model and presenting the optimality equations in Section 3. Section 4 presents the main results of the paper, namely the conditions needed to ensure the optimality of a threshold-type policy and the bounds on the optimal thresholds. In Section 5, we illustrate the main results of Section 4 through a few numerical examples, while Section 6 provides a few concluding remarks.

2 Degradation Model

In this section, we describe our model of a system that deteriorates due to Markov-modulated degradation. In what follows, all random variables are defined on a common, complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The unit is placed into service at time zero in perfect working order and accumulates degradation until a deterministic, critical threshold value ψ is reached or exceeded, at which time the system is said to be failed. Denote this random first passage time by T_ψ. The degradation accrued over time by the system is attributed to environment-induced wear which is modulated by an external stochastic process (the random environment). The random environment is assumed to be a finite, irreducible continuous-time Markov chain (CTMC), $\mathcal{Z} \equiv \{Z_t : t \geq 0\}$, on a finite state space $S \equiv \{1, 2, \ldots, m\}$. The CTMC has an infinitesimal generator matrix $Q \equiv [q_{ij}]$, $i, j \in S$. Whenever $Z_t = i$, the system accumulates degradation at a linear rate r_i ($r_i > 0$), $i \in S$. For each $t \geq 0$, the cumulative degradation up to time t, denoted by X_t, is random quantity

$$X_t = X_0 + \int_0^t r_{Z_u} \mathrm{d}u. \tag{1}$$

Here we assume $X_0 = 0$ almost surely (a.s.) and

$$\int_0^t |r_{Z_u}| \, \mathrm{d}u < \infty, \quad \text{a.s.}$$

so that X_t is well defined for each $t \geq 0$ a.s. The process, $\mathcal{X} \equiv \{X_t : t \geq 0\}$, is a cumulative stochastic process (i.e., an additive functional of a regenerative process) with continuous state space $[0, \infty)$. The positivity of the degradation rates, $r_1, r_2, \ldots, r_m$, ensures that the sample paths of $\mathcal{X}$ are almost surely piecewise linear and monotone increasing in t; therefore, for each $x \geq 0$, the events $\{X_t \leq x\}$ and $\{T_x > t\}$ are equivalent. The lifetime of the system is the first passage time $T_\psi = \inf\{t > 0 : X_t \geq \psi\}$, or the first time the degradation process $\mathcal{X}$ reaches the failure threshold ψ. Kharoufeh et al. [6, 7, 8, 9] have analyzed the distribution function and moments of T_ψ by way of Laplace-Stieltjes transforms; however, the optimal replacement of a system subject to these dynamics has not been studied. In Section 3, we formulate a sequential decision process to examine the existence and structure of optimal strategies.

3 Markov Decision Process Formulation

In this section, we formulate an MDP model to examine optimal strategies for replacing a system subject to the degradation dynamics described in Section 2. Our objective is to minimize the total expected discounted cost of performing *preventive replacements* (those electively performed in advance of a failure) and *reactive replacements* (those performed in response to a failure).

The state of the MDP is an ordered pair (x, i), a realization of the joint process $(\mathcal{X}, \mathcal{Z})$, where x denotes the current cumulative level of degradation, and i is the current state of the environment. For convenience, define the state space $\Upsilon \equiv [0, \infty) \times S$. Because the environment evolves as a CTMC on a finite state space, we solve for the optimal policy using an embedded discrete-time process via the method of uniformization (see Puterman [16]). Let q_i denote the total rate at which the environment transitions out of state $i \in S$. We select a positive rate q satisfying $q \geq \max\{q_1, q_2, \ldots, q_m\}$ and assume that the system is inspected at exponentially distributed intervals with mean $1/q$. That is, the uniformized process transitions at rate q, which corresponds to opportunities for performing a preventive replacement. (Note, therefore, that the minimum inspection rate that can be chosen is dictated by the environment process.) Define the transition probabilities of the uniformized process by $p_{ij} = q_{ij}/q$, $i, j \in S$ $(j \neq i)$. Given that the environment is currently in state $i \in S$, at the next inspection epoch the environment transitions from state i to state $j \in S$ $(j \neq i)$ with probability p_{ij}, and it remains in state i with probability $p_{ii} = 1 - \sum_{j \neq i} p_{ij}$. Let $P \equiv [p_{ij}]$ be the one-step transition probability matrix containing the values p_{ij}, $i, j \in S$. Denote by Y_i be the total amount of degradation accumulated during a single period when the environment assumes state i. Note that Y_i is an exponential random variable with parameter q/r_i because the duration of a single period is exponentially distributed with parameter q, and degradation is accumulated at a deterministic rate r_i during this period. The cumulative distribution function (c.d.f.) of Y_i is

$$F_i(y) \equiv \mathbb{P}(Y_i \leq y) = 1 - \exp(-qy/r_i), \quad y \geq 0.$$

The set of feasible actions is $\mathcal{A} \equiv \{0, 1\}$ where action 0 corresponds to waiting until the next inspection epoch, and action 1 corresponds to scheduling a preventive replacement to occur at the end of the period.

Inspections are assumed to be costless and performed instantaneously. If an inspection reveals that the system is not failed in state (x, i), then a preventive replacement may be scheduled to occur at the end of the period at a cost c_1 $(0 < c_1 < \infty)$, or the system may be allowed to operate for one more period. Preventive replacements are scheduled at the start of the period and are performed (instantaneously) at the end of the period; therefore, the cumulative degradation level is reset to 0 at the start of the subsequent period. In case a failure occurs during a period in which a preventive replacement has been scheduled, the failure is detected when the preventive replacement is initiated. However, the cost of this replacement is still c_1 as it was already planned. On the other hand, if the system is found to be failed at an inspection epoch, then it must be immediately reactively replaced at a cost c_2 $(c_1 < c_2 < \infty)$; reactive replacements are also assumed to be instantaneous. All costs are incurred at the beginning of each period and discounted at a rate λ $(0 < \lambda < 1)$ with

$$\lambda = \frac{q}{q + \alpha}, \tag{2}$$

where $\alpha > 0$ is the continuous-time discount rate. The objective is to minimize the total expected discounted cost of preventive and reactive replacements over an infinite time horizon. The optimal action in state $(x, i) \in [0, \psi) \times S$ is denoted by $a(x, i)$ so that $a(x, i) = 1$ if the optimal action is to schedule a preventive replacement, and $a(x, i) = 0$ if it is optimal to wait until the next inspection.

We denote by $V(x, i)$ the minimum total expected λ-discounted cost starting in state (x, i). For each $(x, i) \in \Upsilon$, the optimality equation is given by

$$V(x, i) = \begin{cases} \min \begin{cases} c_1 + \lambda \sum_{j=1}^{m} V(0, j) p_{ij} \\ \lambda \sum_{j=1}^{m} \left(\int_0^\infty V(x + y, j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \end{cases} & \text{if } 0 \leq x < \psi, \\ c_2 + V(0, i) & \text{if } x \geq \psi. \end{cases} \tag{3}$$

This formulation assumes that if the cumulative level of degradation crosses the threshold ψ between any two consecutive inspection epochs, the failure is detected at the subsequent inspection epoch, and the system is replaced reactively. In Section 4, we study the properties of the cost function (3) and the structure of the resulting maintenance policy.

4 Structural Results

This section analyzes the attributes of the cost function and the optimal policy as a function of the state components (namely the cumulative degradation and the environment state). The first result establishes the existence of an optimal stationary policy and the convergence of the value iteration algorithm.

Lemma 1 *There exists an optimal, non-randomized stationary replacement policy, and the value iteration algorithm converges to the optimal value.*

Proof. Denote by $\mathcal{B}(\mathbb{R}^2)$ the Borel σ-field generated by $\mathbb{R}^2$ and note that $\Upsilon \in \mathcal{B}(\mathbb{R}^2)$ (i.e., Υ is a Borel set). Moreover, the action space, $\mathcal{A} = \{0, 1\}$, is finite, the MDP problem includes strictly positive, bounded immediate costs, and is discounted. Therefore, the problem satisfies the conditions of Corollary 9.17.1 in Bertsekas and Shreve [1], which establishes the existence of a non-randomized optimal policy and the convergence of the value iteration algorithm to the optimal value. ■

The following definitions set the stage for our main results. For notational convenience, let $W(x, i)$ be the cost of waiting in state $(x, i) \in [0, \psi) \times S$, i.e.,

$$W(x, i) = \lambda \sum_{j=1}^{m} \left(\int_0^{\infty} V(x + y, j) \mathrm{d}F_i(y) \right) p_{ij},$$

or equivalently,

$$W(x,i) = \lambda e^{-q(\psi-x)/r_i}\left(c_2 + \sum_{j=1}^{m} V(0,j)p_{ij}\right) + \lambda \sum_{j=1}^{m}\left(\int_0^{\psi-x} V(x+y,j)\frac{q}{r_i}e^{-qy/r_i}\mathrm{d}y\right)p_{ij},$$

because $V(x,i) = c_2 + V(0,i)$ for $x \geq \psi$. Let $R(x,i)$ be the cost of preventive replacement, i.e.,

$$R(x,i) = c_1 + \lambda \sum_{j=1}^{m} V(0,j)p_{ij}.$$

The main results are derived using the notion of monotonicity of a DTMC. This concept is equivalent to the increasing failure rate (IFR) property that is commonly assumed to establish structural results in MDP models.

Definition 1 *Let $P = [p_{ij}]$ be the transition probability matrix of a DTMC with state space $S = \{1, 2, \ldots, m\}$. Then P (or the DTMC) is said to be increasing failure rate (IFR), or monotone, if*

$$\gamma_k(i) \equiv \sum_{j=k}^{m} p_{ij}$$

is non-decreasing in i for all $k = 1, \ldots, m$. Equivalently, P is monotone if $T^{-1}PT \geq \underline{0}$, where $\underline{0}$ is the zero matrix and T is a square matrix with ones on or below the diagonal and zeros above the diagonal.

Now let Q be the infinitesimal generator matrix of a CTMC on S and let $P(t)$, $t \geq 0$, be its transition matrix. As $P(t)$ is stochastic for each $t \geq 0$, one can apply the notion of monotonicity to it.

Definition 2 *A CTMC is said to be monotone if $P(t)$ is monotone for every $t \geq 0$.*

The following result, Theorem 2.1 of Keilson and Kester [5], establishes the necessary and sufficient condition for a CTMC to be monotone.

Theorem 1 *(Keilson and Kester [5]). Let $P(t)$ be the transition matrix of a finite CTMC with infinitesimal generator matrix Q. Then $P(t)$ is monotone for all $t \geq 0$ if and only if*

$$\left[T^{-1}QT\right]_{ij} \geq 0$$

for all $i, j \in S$ such that $j \neq i$, where T is a square matrix with ones on or below the diagonal and zeros above the diagonal.

Monotonicity of the CTMC is especially useful for establishing monotonicity of a uniformized version of the chain. Lemma 2 will be used to establish several results that follow.

Lemma 2 *Let Q be the infinitesimal generator matrix of a monotone CTMC for which $\sup\{q_i : i \in S\} < \infty$. Then there exists some $q \in (0, \infty)$ such that the uniformized chain with transition probability matrix P_q is IFR (monotone).*

Proof. The proof follows directly from the proof of Theorem 2.1 of [5]. ∎

Proposition 1 shows that the optimal total expected discounted cost is monotone non-decreasing in the degradation level x for a given environment state $i \in S$. That is, for a given environment condition, higher degradation leads to a higher optimal cost.

Proposition 1 *For each $i \in S$, $V(x, i)$ is non-decreasing in $x \in [0, \infty)$.*

Proof. Fix the environment state $i \in S$. Let $V_n(x, i)$ be the value function corresponding to the nth iteration of the value iteration algorithm. Similarly, let $W_n(x, i)$ and $R_n(x, i)$ denote the cost of waiting and scheduling a preventive replacement in state (x, i) at stage n, respectively. We prove the proposition by induction on $V_n(x, i)$, where

$$V_{n+1}(x,i) = \begin{cases} \min\begin{cases} c_1 + \lambda \sum_{j=1}^{m} V_n(0,j) p_{ij}, \\ \lambda \sum_{j=1}^{m} \left(\int_0^\infty V_n(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij}, \end{cases} & \text{if } 0 \le x < \psi, \\ c_2 + V_{n+1}(0,i), & \text{if } x \ge \psi. \end{cases} \tag{4}$$

Note that if $x \ge \psi$, V_{n+1} appears on both sides of the equation because reactive replacements are performed immediately and instantaneously. Neither the cost of preventive nor the cost of reactive replacement depends on x; therefore, it is sufficient to show the monotonicity of $W_n(x,i)$, $n \geq 1$ and that

$$c_2 + V_{n+1}(0,i) \geq c_1 + \lambda \sum_{j=1}^{m} V_n(0,j)p_{ij} = R_n(x,i), \tag{5}$$

for each $i \in S$ to prove the monotonicity of $V(x,i)$ in x. The cost of waiting in state (x,i) at stage $n+1$ is

$$W_{n+1}(x,i) = \lambda \sum_{j=1}^{m} \left(\int_0^\infty V_n(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij}.$$

Without loss of generality, assume that

$$V_0(x,j) = \begin{cases} 0, & \text{if } x \in [0,\psi), \\ c_2, & \text{if } x \geq \psi. \end{cases}$$

Then for $n = 0$,

$$W_1(x,i) = \lambda\, e^{-q(\psi - x)/r_i}\, c_2$$

and is monotone non-decreasing in $x \in [0,\psi)$ for each $i \in S$. For the induction hypothesis suppose $V_n(x,j)$ is non-decreasing in $x \in [0,\infty)$ for each $j \in S$. Then $W_{n+1}(x,i)$ is non-decreasing in $x \in [0,\psi)$ by the induction hypothesis.

To show (5) we consider two cases. If $V_{n+1}(0,i)$ is equal to the first component of the minimum in (4), then (5) follows directly. If $V_{n+1}(0,i)$ is equal to the second component of the minimum in (4), then

$$\begin{aligned} c_2 + V_{n+1}(0,i) &= c_2 + \lambda \sum_{j=1}^{m} \left(\int_0^\infty V_n(y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \\ &\geq c_2 + \lambda \sum_{j=1}^{m} \left(\int_0^\infty V_n(0,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \qquad (6) \\ &= c_2 + \lambda \sum_{j=1}^{m} V_n(0,j) p_{ij} \\ &\geq c_1 + \lambda \sum_{j=1}^{m} V_n(0,j) p_{ij}, \qquad (7) \end{aligned}$$

where (6) follows from the induction hypothesis and inequality (7) holds because $c_1 < c_2$. Therefore, $V_{n+1}(x,i)$ is monotone non-decreasing in x for each $i \in S$, which implies the same for $V(x,i)$ by Lemma 1. ∎

The monotonicity of $V(x,i)$ gives rise to the existence of optimal degradation thresholds above which it is optimal to perform a preventive replacement. This result is formalized in Theorem 2.

Theorem 2 *For each $i \in S$, there exists an optimal degradation threshold $\ell(i)$, such that*

$$a(x,i) = \begin{cases} 1, & \text{if } x \geq \ell(i), \\ 0, & \text{if } x < \ell(i). \end{cases}$$

Proof. The result follows directly from Proposition 1 because the cost of a preventive replacement in state (x,i),

$$R(x,i) = c_1 + \lambda \sum_{j=1}^{m} V(0,j)\, p_{ij},$$

does not depend on the cumulative degradation level x. ∎

Proposition 2 elucidates the influence of the degradation rates on the value function. More specifically, it establishes that, for a fixed cumulative degradation level x, the value function is monotone in i whenever the CTMC environment process is monotone and the degradation rates are monotone in i. Before presenting Proposition 2, we first provide a useful lemma, similar to Lemma 4.7.2 of Puterman [16].

Lemma 3 *Let P be IFR and suppose $f(i+1,j) \geq (\leq) f(i,j)$, and $f(i,j+1) \geq (\leq) f(i,j)$, for $i,j = 0,1,\ldots,m$. Then*

$$\sum_{j=0}^{m} f(i+1,j) p_{i+1j} \geq (\leq) \sum_{j=0}^{m} f(i,j) p_{ij}$$

for each $i = 0,1,2,\ldots,m$.

Proof. Define $f(i,-1)=0$ for each i, then for the non-decreasing case,

$$\begin{aligned}
\sum_{j=0}^{m} f(i+1,j)p_{i+1j} &\geq \sum_{j=0}^{m} f(i,j)p_{i+1j} \qquad (8)\\
&= \sum_{j=0}^{m} p_{i+1j} \sum_{k=0}^{j} [f(i,k)-f(i,k-1)]\\
&= \sum_{j=0}^{m} [f(i,j)-f(i,j-1)] \sum_{k=j}^{m} p_{i+1k}\\
&= \sum_{j=1}^{m} [f(i,j)-f(i,j-1)] \sum_{k=j}^{m} p_{i+1k} + f(i,0)\sum_{k=0}^{m} p_{i+1k}\\
&\geq \sum_{j=1}^{m} [f(i,j)-f(i,j-1)] \sum_{k=j}^{m} p_{ik} + f(i,0)\sum_{k=0}^{m} p_{ik} \qquad (9)\\
&= \sum_{j=0}^{m} f(i,j)P_{ij}
\end{aligned}$$

where (8) follows from $f(i+1,j) \geq f(i,j)$, and (9) follows from the IFR property. For the non-increasing case, i.e., $f(i+1,j) \leq f(i,j)$, and $f(i,j+1) \leq f(i,j)$, for $i,j = 0,1,\ldots,m$, the result is obtained by multiplying $f(i,j)$ by -1 for each (i,j) and applying the result for the non-decreasing case. ∎

Proposition 2 *If the CTMC environment process is monotone, and $\{r_i : i \in S\}$ is monotone non-decreasing (non-increasing) in i, then $V(x,i)$ is non-decreasing (non-increasing) in i for each $x \in [0,\infty)$.*

Proof. We prove the result by induction on the iterates of the value iteration algorithm given by (4). Without loss of generality, assume that $V_0(x,i) = 0$ for all $(x,i) \in \Upsilon$. For the induction hypothesis, suppose that $V_n(x,j)$ is non-decreasing in $j \in S$ for all $x \in [0,\infty)$. By Lemma 2, as $\mathcal{Z}$ is monotone, there exists a finite q such that the uniformized chain P is IFR, and we assume that we select such a q. By Lemma 4.7.2 of Puterman [16], for $n \geq 1$,

$$c_1 + \lambda \sum_{j=1}^{m} V_n(0,j)\, p_{ij}$$

is non-decreasing in $i \in S$. For the non-increasing case, the reverse inequality is obtained by multiplying $V(0,j)$ by -1 for all j and applying Lemma 4.7.2 again. Now, since $\{r_i : i \in S\}$ is monotone non-decreasing (non-increasing) we have

$$\int_t^\infty \frac{q}{r_{i+1}} e^{-qy/r_{i+1}} \mathrm{d}y = e^{-qt/r_{i+1}} \geq (\leq) e^{-qt/r_i} = \int_t^\infty \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y,$$

for all $t \geq 0$, which corresponds to first order stochastic dominance. Therefore,

$$\int_0^\infty V_n(x+y,j) \frac{q}{r_{i+1}} e^{-qy/r_{i+1}} \mathrm{d}y \geq (\leq) \int_0^\infty V_n(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y,$$

since $V_n(x+y,j)$ is a non-decreasing function of y for all x and j. Additionally, by the induction hypothesis, $V_n(x+y,j)$ is non-decreasing (non-increasing) in $j \in S$, so the integral expression

$$\int_0^\infty V_n(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y,$$

is non-decreasing (non-increasing) in both $i \in S$ and $j \in S$. By Lemma 3

$$\lambda \sum_{j=1}^{m} \left(\int_0^\infty V_n(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij}$$

is non-decreasing (non-increasing) in $i \in S$, which shows that $V_{n+1}(x,i)$ is non-decreasing (non-increasing) in $i \in S$. The result follows by induction and Lemma 1. ■

Proposition 3 provides simple conditions that ensure the optimality of the preventive replacement action. That is, if the degradation rates are non-decreasing and condition (10) holds for state $(x,i) \in [0,\psi) \times S$, then it is optimal to preventively replace for all states $(y,j) \in [0,\psi) \times S$ such that $y \geq x$ and $j \geq i$.

Proposition 3 *Suppose $\{r_i : i \in S\}$ is monotone non-decreasing in i and that for some state (x,i),*

$$\frac{c_1}{c_2} \leq \lambda \exp\left[-\frac{q(\psi - x)}{r_i} \right]. \tag{10}$$

Then $a(y,j) = 1$ for any $(y,j) \in [0,\psi) \times S$ such that $y \geq x$ and $j \geq i$.

Proof. To prove the result, we examine the cost of waiting in state (x, i),

$$\begin{aligned} W(x,i) &= \lambda e^{-q(\psi-x)/r_i} c_2 + \lambda \sum_{j=1}^{m} \left(\int_{\psi-x}^{\infty} V(0,j) \mathrm{d}y \right) p_{ij} + \lambda \sum_{j=1}^{m} \left(\int_{0}^{\psi-x} V(x+y,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \\ &\geq \lambda e^{-q(\psi-x)/r_i} c_2 + \lambda \sum_{j=1}^{m} \left(\int_{\psi-x}^{\infty} V(0,j) \mathrm{d}y \right) p_{ij} + \lambda \sum_{j=1}^{m} \left(\int_{0}^{\psi-x} V(0,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \quad (11) \\ &= \lambda e^{-q(\psi-x)/r_i} c_2 + \lambda \sum_{j=1}^{m} \left(\int_{0}^{\infty} V(0,j) \frac{q}{r_i} e^{-qy/r_i} \mathrm{d}y \right) p_{ij} \\ &= \lambda e^{-q(\psi-x)/r_i} c_2 + \lambda \sum_{j=1}^{m} V(0,j) p_{ij}, \end{aligned}$$

where inequality (11) follows from Proposition 1. Then by (10),

$$W(x,i) \geq \lambda e^{-q(\psi-x)/r_i} c_2 + \lambda \sum_{j=1}^{m} V(0,j) p_{ij} \geq c_1 + \lambda \sum_{j=1}^{m} V(0,j) p_{ij} = R(x,i).$$

Thus, the optimal action is to perform a preventive replacement in state (x, i). Because $e^{-q(\psi-x)/r_i} c_2$ is non-decreasing in both x and i, the result is proved. ■

Similarly, if $\{r_i : i \in S\}$ is monotone non-increasing in i, and (10) holds, then $a(y, j) = 1$ for any $(y, j) \in [0, \psi) \times S$ such that $y \geq x$ and $j \leq i$.

Now using condition (10), it is possible to compute an upper bound, $\mu(i)$, on the optimal degradation threshold $\ell(i)$ for each $i \in S$. Specifically,

$$\mu(i) = \psi + \ln \left(\frac{c_1}{c_2} \cdot \frac{1}{\lambda} \right) \frac{r_i}{q}, \quad i \in S, \tag{12}$$

where it is optimal to replace for $x \geq \mu(i)$. It is worth noting that if $c_1 < \lambda c_2$, the sequence of thresholds $\{\mu(i) : i \in S\}$ is non-increasing. Therefore, for some subset of the state space Υ, the optimal actions may be determined without solving the MDP problem. In general, to solve continuous-state MDP problems, the continuous component of the state space can be discretized. However, this discretization can lead to a potentially enormous state space and increased computational burden. By exploiting the degradation threshold bounds of (12), the computational burden can be reduced.

It is tempting to presume that if $\mathcal{Z}$ is monotone, and $\{r_i : i \in S\}$ is non-decreasing, then even if condition (10) does not hold for a fixed cumulative degradation level x, if $a(x, i) = 1$, then $a(x, i+1) = 1$, i.e., the thresholds are monotone in i. However, it is very difficult to establish this result in full generality, hence, we state it as a conjecture.

Conjecture 1 *If the CTMC $\mathcal{Z}$ is monotone and $\{r_i : i \in S\}$ is monotone non-decreasing (non-increasing), then the optimal sequence of thresholds $\{\ell(i) : i \in S\}$ is monotone non-increasing (non-decreasing).*

5 Numerical Examples

In this section, we illustrate the main results through a few numerical examples. For each of the examples we observe that, if the degradation rates are monotone in $i \in S$, and the CTMC environment process $\mathcal{Z}$ is monotone, then the optimal policy exhibits a monotone structure (Conjecture 1). We provide an example which shows that the thresholds may be non-monotone when $\mathcal{Z}$ is not monotone. Furthermore, we numerically analyze the effect of the inspection rate q on the thresholds and the value functions.

To construct these examples, let Π be the original, continuous-state MDP, and Π_n be the MDP obtained by discretizing the continuous state component x so that the degradation assumes one of n values equally spaced on the interval $[0, \psi]$. In (3), we replace integration and the exponential density function with summation and a discrete probability mass function (p.m.f.), respectively. To compute the probability mass functions, we first approximate the area under the exponential density function on the interval $[0, \psi]$ by summing the areas of n equal-width rectangles. Then, we normalize the areas of the rectangles to ensure a proper p.m.f., i.e., the total sum of the rectangle areas plus $\bar{F}(\psi)$ is unity.

In our solution algorithm, we increase n until the observed "change" in the value function is below a pre-specified $\epsilon > 0$. It is important to note that we only need to consider cumulative degradation values in the interval $[0, \psi]$ since the optimal action is to reactively replace for any state (x, i) such that $x > \psi$. We denote the discretized state space by $\Upsilon_n \equiv \Gamma_n \times S$ where $\Gamma_n = \{0 = x_1, x_2, \ldots, x_n = \psi\}$ for some $n \in \mathbb{N}$, and the value function for problem Π_n by $V^*_{\Pi_n}$. The solution procedure is as follows:

Step 0: Set $i \leftarrow 1$. Fix the integer n_0 to obtain Υ_{n_0}. Solve problem Π_{n_0} using value iteration to obtain the optimal values $V^*_{\Pi_{n_0}}(s)$, $s \in \Upsilon_{n_0}$;

Step 1: Set $n_{i+1} \leftarrow (i+1)n_0$ to obtain $\Upsilon_{n_{i+1}}$;

Step 2: Solve $\Pi_{n_{i+1}}$ using value iteration and obtain optimal values of the discretized MDP, $V^*_{\Pi_{n_{i+1}}}(s)$ for all $s \in \Upsilon_{n_{i+1}}$;

Step 3: If $\max_{s \in \Upsilon_{n_0}} \left| V^*_{\Pi_{n_{i+1}}}(s) - V^*_{\Pi_{n_i}}(s) \right| < \epsilon$, stop, otherwise $i \leftarrow i + 1$ and return to Step 1.

Our first example demonstrates the structure described in Theorem 2, i.e., if it is optimal to replace in state (x, i), then it is optimal to replace in state $(x + y, i)$ for any $y \geq 0$ for each $i \in S$.

Example 1: The discount rate is $\lambda = 0.99$, the preventive replacement cost is $c_1 = 3$, the reactive replacement cost is $c_2 = 10$, the critical degradation level is $\psi = 1$, and the inspection rate is $q = 10$. The environment $\mathcal{Z}$ has state space $S = \{1, 2, \ldots, 8\}$ and infinitesimal generator matrix

$$Q = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & -5 & 2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & -5 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & -5 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & -5 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & -5 & 2.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.5 & -5 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 \end{bmatrix}.$$

Note that the CTMC is monotone and the uniformized chain P is IFR (monotone) when $q = 10$. The increasing degradation rates are as follows: $r_1 = 0.5$, $r_2 = 1$, $r_3 = 1.5$, $r_4 = 2$, $r_5 = 2.5$, $r_6 = 3$, $r_7 = 3.5$, $r_8 = 4$. For $\epsilon = 0.0001$, the solution procedure terminated with the state space partitioned into $n = 8300$ intervals. Figure 1 illustrates the change in the optimal cost for states (0,1) and (0.1,2) throughout the solution procedure. From Figure 1, it is clear that the marginal benefit of discretization is diminishing as the level of refinement increases.

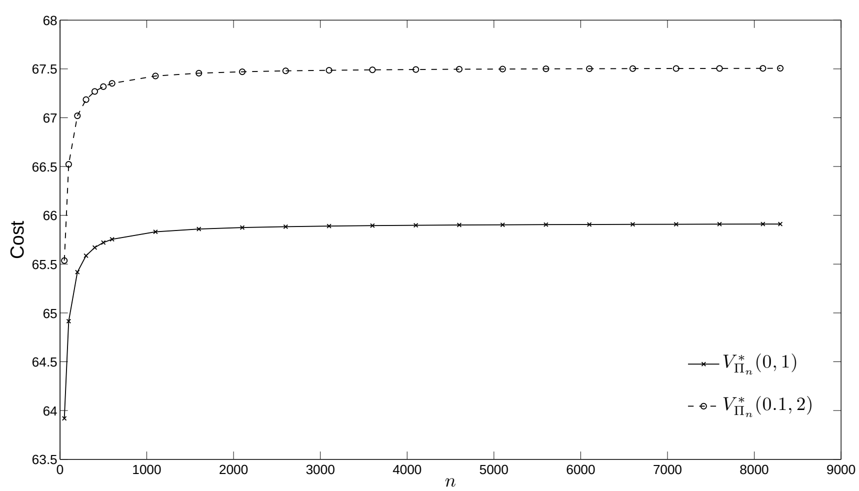

Figure 1: The change in $V^*_{\Pi_n}$ as discretization becomes finer.

Figure 2 depicts the resulting optimal policy and bounds on the degradation thresholds obtained by equation (12) while illustrating the control-limit structure with respect to the cumulative degradation level (as per Theorem 2). Additionally, the optimal threshold is monotone in i (as per Conjecture 1). Example 2 illustrates the case when the degradation rates are non-increasing.

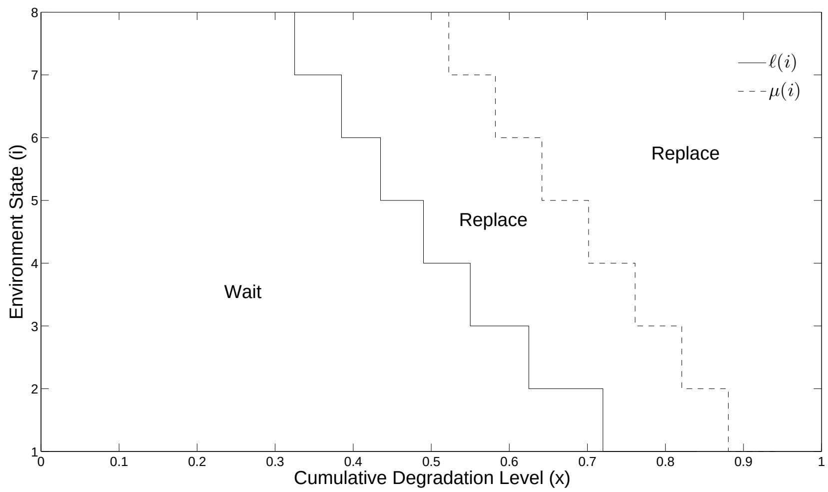

Figure 2: Example 1 optimal threshold (solid line) and its upper bound (dotted line).

Example 2: All the parameters are the same as in Example 1 except that the degradation rates are in decreasing order, i.e., $r_1 = 4$, $r_2 = 3.5$, $r_3 = 3$, $r_4 = 2.5$, $r_5 = 2$, $r_6 = 1.53$, $r_7 = 1$, $r_8 = 0.5$. Figure 3 depicts the resulting optimal policy and bounds on the degradation thresholds. We observe both the control-limit structure and the monotonicity property of the thresholds stated in Theorem 2 and Conjecture 1, respectively.

It is noteworthy that, after extensive empirical testing, we were unable to identify a counterexample to the monotonicity of the threshold in the environment state i when $\mathcal{Z}$ is monotone. However, the monotone structure of the thresholds may not be preserved when $\mathcal{Z}$ is not monotone. Our third example illustrates such a case.

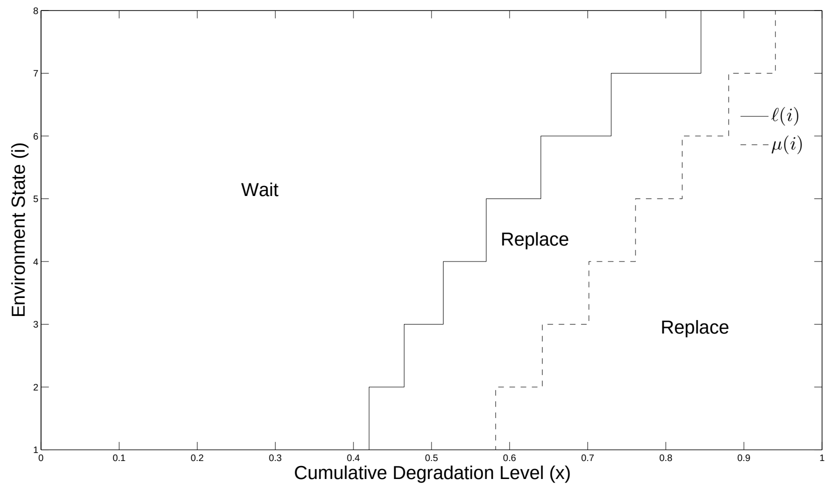

Figure 3: Example 2 optimal threshold (solid line) and its upper bound (dotted line).

Example 3: The discount rate is $\lambda = 0.90$, the preventive replacement cost is $c_1 = 5$, the reactive replacement cost is $c_2 = 20$, the critical degradation level is $\psi = 1$, and the inspection rate is $q = 10$. The environment $\mathcal{Z}$ has state space $S = \{1, 2, \ldots, 8\}$ and infinitesimal generator matrix

$$Q = \begin{bmatrix} -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -6 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -6 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -9.9 & 9.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9.9 & 9.9 \\ 9.9 & 0 & 0 & 0 & 0 & 0 & 0 & -9.9 \end{bmatrix}.$$

Note that the environment process is not monotone in this case; therefore, the uniformized chain P is not IFR for any $q > 0$. The increasing degradation rates are as follows: $r_1 = 0.1$, $r_2 = 0.2$,

$r_3 = 0.3$, $r_4 = 0.4$, $r_5 = 0.5$, $r_6 = 0.6$, $r_7 = 0.7$, $r_8 = 0.8$. For $\epsilon = 0.0001$, the solution procedure terminated with the state space partitioned into $n = 4000$ intervals. Figure 4 depicts the resulting optimal policy. As shown in Figure 4, the optimal policy exhibits a control limit structure only

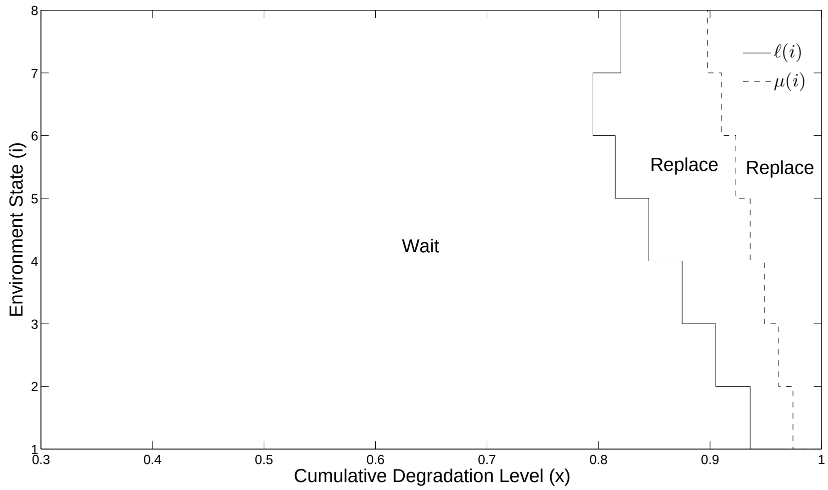

Figure 4: Example 3 optimal threshold (solid line) and its upper bound (dotted line).

with respect to the cumulative degradation level; the monotone structure of the optimal threshold is not preserved. Recall that Theorem 2 does not place any conditions on the CTMC environment process, but Conjecture 1 assumes that $\mathcal{Z}$ is monotone.

Finally, we provide a fourth example to illustrate the effect of the selected inspection rate q on the optimal value functions and the optimal thresholds. For each $i \in S$, let $\ell_q(i)$ be the optimal threshold in state i, given that the system is inspected at rate q.

Example 4: The preventive replacement cost is $c_1 = 3$, the reactive replacement cost is $c_2 = 10$, the critical degradation level is $\psi = 1$, and the degradation rates are $r_1 = 1$, $r_2 = 1.5$, $r_3 = 2$. The generator matrix of the governing environment process is

$$Q = \begin{bmatrix} -6 & 1 & 5 \\ 3 & -7 & 4 \\ 2 & 6 & -8 \end{bmatrix}.$$

Our aim here is to examine the behavior of the optimal thresholds $\ell_q(i)$, $i \in S$, as a function of q. Therefore, to ensure an equivalent continuous-time discount rate α, λ is computed as a function of q via (2). Initially, we choose $\lambda = 0.95$ with $q = 10$, and these values yield $\alpha = 0.5/0.95$, which is fixed throughout. Figure 5 depicts the behavior of the optimal thresholds, $\ell_q(i)$, $i = 1, 2, 3$.

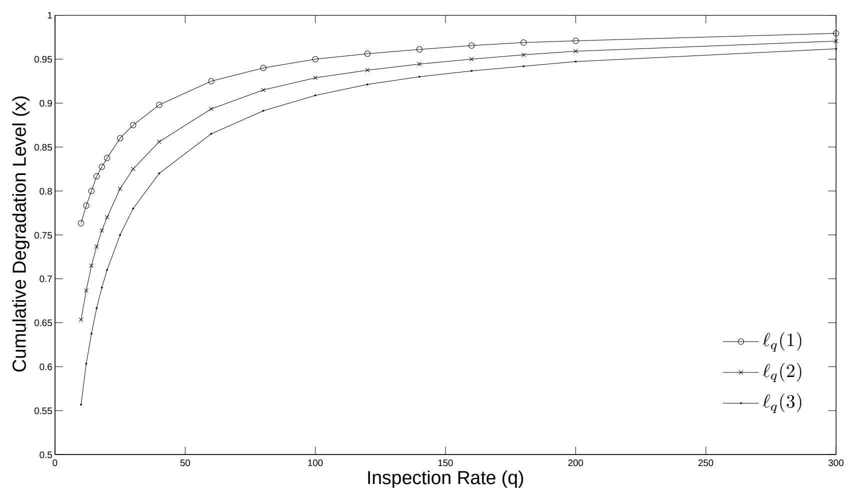

Figure 5: Optimal degradation thresholds as a function of q.

Figure 5 indicates that observing the system more frequently yields higher degradation thresholds. This behavior is consistent with intuition as the likelihood of catching the system before failure increases with the inspection frequency. Moreover, as the inspection rate approaches infinity, the problem coincides with a continuous review problem, which leads to an optimal policy of replacing preventively just before failure to minimize the total discounted cost.

We also consider the effect of the inspection rates on the optimal value functions. Let $V_q^*(x,i)$ be the optimal cost starting in state (x,i) when the inspection rate is q. Figure 6 depicts the behavior of the optimal cost starting in states (0,1), (0.25,2), and (0.5,3) as the inspection rate q increases. By observation of Figure 6, the optimal costs appear to be monotone decreasing in q.

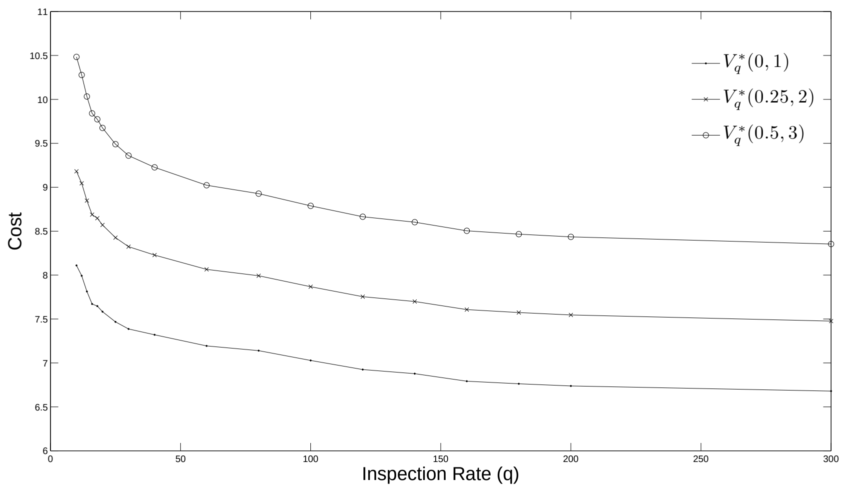

Figure 6: Optimal cost as a function of q.

This result is not surprising because we are less likely to encounter reactive replacements (which are more costly) as the inspection frequency increases. We provide some concluding remarks in Section 6.

6 Conclusions

In this paper we have presented a framework for optimally replacing a system that degrades due to the influence of its random environment, where the objective is to minimize the total expected discounted cost over an infinite horizon. Although optimal replacement problems have been extensively studied, there is limited research on systems degrading under the effect of a random environment. We formulated a continuous-time, continuous-state space MDP model and analyzed the structure of the optimal policy. In particular, we showed that the value function is monotone in both the degradation level and environment state under certain conditions, and the structure of the optimal policy can be characterized as a threshold-type policy. Furthermore, we established some simple conditions that guarantee the optimal action is to preventively replace the system and provided some numerical examples to illustrate the optimal policies and show the effect of the inspection rate on the thresholds and value functions.

The results presented here are very useful for determining optimal policies for systems operating in complex environments, once the system parameters have been properly estimated. Our technique requires observations of the cumulative level of degradation, the environment's current state, its transition times, and observation of its subsequent state. In addition, the estimation of the degradation rates, r_i, $i \in S$, that describe the evolution of degradation as a function of the environment is not a trivial task. Obviously, for this purpose, real degradation data is required.

Proving the monotonicity of the optimal thresholds with respect to the environment state is a key future research question, and it is also important to consider different cost structures. For example, in real settings, the preventive and replacement costs might depend on the current degradation level and the environment state rather than being independent of the state. Moreover, in many applications, inspections are very costly and/or time consuming. When inspections are neither free nor instantaneous, the model presents some very challenging features that are worthy of investigation. Finally, the model presented herein assumes that the environment evolves as a CTMC on a finite state space. Realistically (depending on the application), the environment might evolve on a continuous state space. Proving the existence of an optimal threshold in this case is challenging but will lead to results that may be applicable in a number of engineering applications (e.g., the optimization of wind turbine maintenance strategies).

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References

[1] D. Bertsekas and S. Shreve. *Stochastic Optimal Control: The Discrete Time Case*. Academic Press, New York, NY, 1978.

[2] B. Çekyay and S. Özekici. Condition-based maintenance under Markovian deterioration. In J. Cochran, A. Cox, P. Keskinocak, J. P. Kharoufeh, and J. C. Smith, editors, *Wiley Encyclopedia of Operations Research and Management Science*, Hoboken, NJ, 2011. John Wiley & Sons, Inc.

[3] G. Gottlieb. Failure distributions of shock models. *Journal of Applied Probability*, 17:745–752, 1980.

[4] I. Igaki, U. Sumita, and M. Kowada. Analysis of Markov renewal shock models. *Journal of Applied Probability*, 32(3):821–831, 1995.

[5] J. Keilson and A. Kester. Monotone matrices and monotone Markov processes. *Stochastic Processes and Their Applications*, 5:231–241, 1977.

[6] J. Kharoufeh. Explicit results for wear processess in a Markovian environment. *Operations Research Letters*, 31(3):237–244, 2003.

[7] J. Kharoufeh and S. Cox. Stochastic models for degradation-based reliability. *IIE Transactions*, 37(6):533–542, 2005.

[8] J. Kharoufeh, D. Finkelstein, and D. Mixon. Availability of periodically inspected systems subject to Markovian wear and shocks. *Journal of Applied Probability*, 43(2):303–317, 2006.

[9] J. Kharoufeh, C. Solo, and M. Ulukus. Semi-Markov models for degradation-based reliability. *IIE Transactions*, 42(8):599–612, 2010.

[10] M. Kurt and J. Kharoufeh. Monotone optimal replacement policies for a Markovian deteriorating system in a controllable environment. *Operations Research Letters*, 38(4):273–279, 2010.

[11] M. Kurt and L. Maillart. Structured replacement policies for systems receiving shocks with a Markov-modulated Poisson arrival rate. *Operations Research Letters*, 37(4):280–284, 2009.

[12] G. Li and J. Luo. Shock model in Markovian environment. *Naval Research Logistics*, 52:253–260, 2005.

[13] S. Özekici. Optimal maintenance policies in random environments. *European Journal of Operational Research*, 2(3):283–294, 1995.

[14] H. Pham and H. Wang. Imperfect maintenance. *European Journal of Operational Research*, 94(3):425–438, 1996.

[15] W. Pierskalla and J. Voelker. A survey of maintenance models: The control and surveillance of deteriorating systems. *Naval Research Logistics Quarterly*, 23(3):353–388, 1976.

[16] M. Puterman. *Markov Decision Processes*. John Wiley and Sons Inc., New York, NY, 1994.

[17] P. A. Scarf. On the application of mathematical models in maintenance. *European Journal of Operational Research*, 99(3):493–506, 1997.

[18] J. Shanthikumar and U. Sumita. General shock models associated with correlated renewal sequences. *Journal of Applied Probability*, 20(3):600–614, 1983.

[19] Y. Sherif and M. Smith. Optimal maintenance models for systems subject to failure. *Naval Research Logistics Quarterly*, 28(1):47–74, 1971.

[20] N. Singpurwalla. Survival in dynamic environments. *Statistical Science*, 10:86–103, 1995.

[21] U. Sumita and J. Shanthikumar. A class of correlated cumulative shock models. *Advances in Applied Probability*, 17(2):347–366, 1985.

[22] C. Valdez-Flores and R. Feldman. A survey of preventive maintenance models for stochasically deteriorating single-unit systems. *Naval Research Logistics*, 36(4):419–446, 1989.

[23] K. Waldmann. Optimal replacement under additive damage in randomly varying environments. *Naval Research Logistics*, 30(3):377–386, 1983.

[24] H. Wang. A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research*, 139(3):469–489, 2002.