ON THE ERGODICITY OF A CLASS OF LEVEL-DEPENDENT QUASI-BIRTH-AND-DEATH PROCESSES

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Abstract

We examine necessary and sufficient conditions for recurrence and positive recurrence of a class of irreducible, level-dependent quasi-birth-and-death (LDQBD) processes with a block tridiagonal structure that exhibits asymptotic convergence in the rows as the level tends to infinity. These conditions are obtained by exploiting a multidimensional Lyapunov drift approach, along with the theory of generalized Markov group inverses. Additionally, we highlight analogies to well-known average drift results for level-independent quasi-birth-and-death (QBD) processes.

Keywords: LDQBD processes; positive recurrence; Lyapunov potential

2010 Mathematics Subject Classification: Primary 60J10

Secondary 37A30

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1. Introduction

In this paper, we examine necessary and sufficient conditions for recurrence and positive recurrence of a class of multidimensional Markov processes, namely level-dependent quasi-birth-and-death processes. These conditions are established by way of Lyapunov (potential) functions and variants of Foster's criterion [17]. Additionally, we demonstrate a correspondence between these conditions and similar results for related Markov processes.

The quasi-birth-and-death (QBD) process is a bivariate, continuous-time Markov process $\{(X_t, Y_t) : t \geq 0\}$ with countable state space $S = \{(i, j) : i \geq 0, j = 1, \ldots, K_i\}$, where i is called the *level* of the process, j is the *phase* and K_i is the number of phases in level i. Generally speaking, K_i can be either a finite, positive integer or infinity for each i. The process is restricted in level jumps only to its nearest neighbors but is unrestricted in the phase dimension. That is, from state $(i, j) \in S$ the process may transition to the states (i, k), $(i - 1, k)$ or $(i + 1, k)$, but not to states of the form $(i \pm n, k)$ where $n \geq 2$. It extends the standard birth-and-death process, whose state space consists only of the level i. If the transition rates of a QBD process are independent of i, it is termed a *homogeneous* or *level-independent* QBD process; if the rates change with i, it is termed a *nonhomogeneous* or *level-dependent* QBD (LDQBD) process. The broad class of LDQBD processes is widely applicable to stochastic models arising in queueing theory, computer and communications systems, reliability theory, inventory theory and many other areas. The level-independent version has been studied extensively in the applied probability literature and is given a comprehensive treatment by Latouche and Ramaswami [24]; the level-dependent version has received comparatively less attention.

For the class of LDQBD models examined here, it is assumed that $K_i = K$ for all i, where K is a finite, positive integer. Since important attributes (e.g., recurrence and transience) of an irreducible continuous-time Markov chain can be ascertained via its embedded discrete-time Markov chain (DTMC) at jump epochs, we analyze this chain and denote it by $\Phi := \{(X_n, Y_n) : n \in \mathbb{Z}_+\}$. This process has state space $S = \{(i, j) : i \geq 0, j = 1, \ldots, K\}$, and its one-step transition probability matrix (P) possesses the classical block tridiagonal form of a QBD process:

$$
P = \begin{pmatrix}
A_1^{(0)} & A_0^{(0)} & 0 & 0 & 0 & \cdots \\
A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & 0 & 0 & \cdots \\
0 & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & 0 & \cdots \\
0 & 0 & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \cdots \\
0 & 0 & 0 & A_2^{(4)} & A_1^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The structure of P indicates that the process is skip-free in the level in both directions. For general LDQBD processes, the blocks along the main diagonal of P ($A_1^{(i)}$ for $i \geq 0$) are square matrices, while those on the secondary diagonals ($A_0^{(i)}$ and $A_2^{(i)}$ for $i \geq 0$) can be rectangular. The model we examine here assumes that all rows of P contain only square matrices of order K. Therefore, for each $i \geq 0$ and $m = 0, 1, 2$, $A_m^{(i)}$ is a nonnegative square matrix of order K, and the row sums of $A^{(i)} := A_2^{(i)} + A_1^{(i)} + A_0^{(i)}$ for $i \geq 1$ are equal to 1, as are those of $A_1^{(0)} + A_0^{(0)}$. When it exists, denote the limiting distribution of Φ by a positive row vector $\boldsymbol{x}$ that uniquely solves the linear system $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}e = 1$, where $\boldsymbol{e}$ is a column vector of ones. The vector $\boldsymbol{x}$ is partitioned by levels into subvectors so that $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$, and $\boldsymbol{x}_i$ contains the limiting probabilities for states in level i, namely those in the set $L_i := \{(i, 1), (i, 2), \ldots, (i, K)\}$. If Φ is irreducible, aperiodic and positive recurrent, then there exist matrices $\{R_i : i \geq 1\}$ such that $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} R_i$, where $\{R_i : i \geq 1\}$ is the minimal nonnegative solution of the set of equations

$$
R_i = A_0^{(i-1)} + R_i\, A_1^{(i)} + R_i\, R_{i+1}\, A_2^{(i+1)}, \quad i \geq 1 \tag{1}
$$

(see Theorem 12.1.1. of [24]). Solving the system of equations (1) is typically relegated to numerical techniques, such as those described in [8]. The LDQBD process is positive recurrent if and only if there exists a positive solution to the system of equations (Theorem 12.1.4, [24])

$$
\boldsymbol{x}_0 = \boldsymbol{x}_0 \left(A_1^{(0)} + A_0^{(0)} G_1 \right)
$$

subject to the normalization condition

$$
\boldsymbol{x}_0 \left(\sum_{n=0}^{\infty} \prod_{i=1}^{n} R_i \right) \boldsymbol{e} = 1,
$$

where G_1 contains the probabilities of reaching level 0 in finite time starting from level 1, and the empty product (when $n = 0$) results in the identity matrix. Although the matrices G_1 and $\{R_i : i \geq 1\}$ can be obtained to any desired accuracy using, for example, the algorithms of Bright and Taylor [8], our primary aim here is to determine analytical necessary and sufficient conditions for a special class of LDQBD processes.

Level-independent QBD processes arise frequently in the context of queueing models with nonstandard arrival and/or service mechanisms (e.g., randomly-varying arrival and/or service rates). One such example is the $M/M/1$ queue in a two-state environment [14, 50] in which the level i corresponds to the number of customers in the system, and the phase j is the state of an exogenous environment process that modulates the arrival and/or service rates. These early models were later extended by Neuts [34, 35, 36] and Purdue [40] to consider environment transitions not restricted to service completion epochs. Neuts [38] formally categorized Markov-modulated queueing systems as homogeneous QBD processes and provided an average drift condition for this class of stochastic processes. Latouche and Taylor [25] provided a general proof for drift conditions of homogeneous QBD processes, which are also applicable to general matrix-analytic models. The emergence of retrial queueing models spawned variants of the homogeneous QBD process, including the LDQBD process in which the block tridiagonal structure varies over an infinite number of levels. Some representative examples of such models include [1, 2, 3, 5, 7, 23, 26, 33, 46]. Bright and Taylor [8] developed an efficient row-truncation algorithm to approximate the limiting distribution of an LDQBD process when it exists; however, the method does not readily yield information about the recurrence characteristics of the LDQBD process. Some models possess special properties to facilitate analysis of stability criterion, as in [4], whereas improvements in numerical algorithms such as ETAQA [9] for computing the stationary distribution of LDQBD processes have made the numerical approach more tenable for queueing applications (see [6, 10, 32]).

For one-dimensional Markov chains with infinitely denumerable state spaces, it is well-known that Lyapunov functions, along with Foster's criterion [17], can be used to establish sufficient conditions for the ergodicity of irreducible chains. As a special case of Foster's criterion, Pakes' lemma [39, Theorem 2] is perhaps most useful in that it generalized to conditions on the subsequential upper bound of drift terms, and

it uses the identity function as the Lyapunov (potential) function. Other extensions considering broader classes of Markov processes, including continuous-time Markov chains (CTMCs), appear in [29, 30, 43, 48, 49]. Nonergodicity was first studied first by Kaplan [19] and subsequently by Sennott, et al. [44, 45]. However, the analysis of LDQBD processes falls more appropriately within the realm of multidimensional Markov processes, which have been studied extensively over the past several decades. Tweedie [49] approached the problem from the vantage point of general state spaces, followed by Szpankowsky [47], Rosberg [42], and Sennott [44]. To the best of our knowledge, the first appearance of a necessary and sufficient Foster-Lyapunov recurrence criterion appears in Mertens, et al. [30]. A corresponding necessary and sufficient criterion for the positive recurrence of Markov chains over countable state spaces is proved in [27]. This result, along with those of Mertens, et al. [30], are incorporated in the monograph by Fayolle, et al. [16] on the constructive theory of countable Markov chains.

Due to the difficulties associated with analyzing general, multidimensional Markov processes with nonhomogeneous transition matrices, attention has shifted to classes of structured chains that become homogeneous at a finite level of the state space. One important example is the class of LDQBD processes for which there exists a level L above which the process behaves as a homogeneous QBD process. Such models are termed LDQBDs with a large number of boundary states (see [8, 38]) and are relevant to our work here in that their recurrence properties can be determined by examining the behavior of the process beyond level L. Bright and Taylor [8] showed how to construct a stochastically-dominating LDQBD process whose recurrence is sufficient to guarantee the recurrence of the process it dominates. Klimenok and Dudin [21] defined and analyzed the so-called asymptotically quasi-Toeplitz Markov chain (AQTMC). The AQTMC is the nonhomogeneous, asymptotically row-convergent form of the multidimensional QTMC introduced in [12], which was motivated by their previous work on a retrial queue whose input is a batch Markovian arrival process (BMAP) [11, 13, 20]. The QTMC is more widely known as an $M/G/1$-type Markov chain [37], as its structure mirrors that of the system size process of an $M/G/1$ queue embedded at service completion epochs.

It is well-known that the LDQBD process is a subclass of nonhomogeneous $M/G/1$-

type Markov chains. Furthermore, if it is assumed that the blocks of the transition probability matrix converge (as $i \to \infty$), then the LDQBD process can be viewed as a discrete-time AQTMC. While the AQTMC analyzed in [21] is more general, they provide only sufficient conditions for ergodicity and non-ergodicity of these types of chains. Koukoutsidis, et al. [22] devised a nonhomogeneous QBD model for admission control of a WCDMA system. They conjectured an ergodicity condition for the case in which the blocks of the infinitesimal generator matrix converge asymptotically to those of a homogeneous QBD process. For chains that satisfy such convergence properties, the LDQBD process is ergodic if the QBD process to which it converges is ergodic; conversely, if the LDQBD process is nonergodic, then so too is the QBD process. The sketch of their proof uses stochastic dominance concepts.

The primary aim of the present paper is to provide necessary and sufficient conditions for the recurrence and positive recurrence of a class of discrete-time LDQBD processes whose block matrices converge as $i \to \infty$. More specifically, we consider a continuous-time LDQBD process whose embedded jump chain satisfies the convergence criteria

$$\lim_{i \to \infty} A_k^{(i)} = A_k^*, \ k = 0, 1, 2 \quad \text{and} \quad \lim_{i \to \infty} A^{(i)} = A^*,$$

where the above limits hold element-wise. This class of processes encompasses the class of LDQBD processes with a large number of boundary states. It can also be viewed as a special case of the asymptotically quasi-Toeplitz Markov chains; however, our approach and results differ in two important ways. First, we employ a Foster-Lyapunov drift approach to derive positive recurrence (and hence, ergodicity) conditions for this special type of LDQBD without the use of transforms. Second, we show that this condition is not only sufficient, but also necessary, and demonstrate an analogy between our results and the average drift conditions for level-independent QBD processes.

The remainder of the paper is organized as follow. Section 2 first reviews LDQBD processes and some preliminaries related to drift functions and Foster's criterion for one-dimensional Markov chains. In Section 3, we derive the generalized drift function for a specific form of Lyapunov function and employ a modified version of Foster's criterion to provide sufficient conditions for recurrence, and necessary and sufficient conditions for positive recurrence, of the embedded DTMC Φ. Section 4 establishes the

existence of a limiting average drift and characterizes necessary and sufficient conditions for recurrence, null recurrence, positive recurrence and transience using this drift term. Additionally, we highlight analogies between the homogeneous QBD process and the LDQBD process. A few concluding remarks are provided in Section 5.

2. Preliminaries

Although the LDQBD process is generally a continuous-time, discrete-state stochastic process, we analyze its discrete-time counterpart (as the continuous-time version can be analyzed via embedding at transition epochs). Throughout the manuscript, we shall adopt the following notation and conventions: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative reals, $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers and $\mathbb{Z}_+ = \{0, 1, \ldots\}$ the set of nonnegative integers. The vector $\boldsymbol{e} = (1, 1, \ldots, 1)'$ denotes a column vector of ones, and $\mathbf{0}$ shall denote the zero column vector (or zero matrix), as needed. Finally, for any two column vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we write $\boldsymbol{x} \leq \boldsymbol{y}$ to indicate that the inequality holds componentwise.

Let $\Phi := \{(X_n, Y_n) : n \geq 0\}$ be a time-homogeneous, irreducible and aperiodic discrete-time Markov chain (DTMC) on the countable state space

$$S := \big\{ (i, j) : i \in \mathbb{Z}_+, \, j \in \mathscr{P} \big\},$$

where $\mathscr{P} := \{1, \ldots, K\}$ for some $K \in \mathbb{N}$. The components i and j of the state are termed the *level* and *phase* of the process, respectively. The DTMC Φ is a discrete-time quasi-birth-and-death (QBD) process because its one-step transition probability matrix (P) exhibits the block-tridiagonal form

$$P = \begin{pmatrix}
A_1^{(0)} & A_0^{(0)} & 0 & 0 & 0 & \cdots \\
A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & 0 & 0 & \cdots \\
0 & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & 0 & \cdots \\
0 & 0 & A_2^{(3)} & A_1^{(3)} & A_0^{(3)} & \cdots \\
0 & 0 & 0 & A_2^{(4)} & A_1^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{2}$$

The time-homogeneity of Φ implies that P is independent of time n. For each $i \in \mathbb{Z}_+$, the entries $A_0^{(i)}$, $A_1^{(i)}$ and $A_2^{(i)}$ are nonnegative $K \times K$ matrices comprised of transition

probabilities between phase states and levels. As P is a one-step transition probability matrix, we note that

$$
A^{(i)} := \begin{cases} A_1^{(i)} + A_0^{(i)}, & i = 0, \\ A_2^{(i)} + A_1^{(i)} + A_0^{(i)}, & i \geq 1, \end{cases}
$$

is a stochastic matrix. The structure of P shows that the process is skip-free in the level in both directions. More precisely, for $i \in \mathbb{N}$ and any $j, j' \in \mathscr{P}$,

$$
\left[A_0^{(i)} \right]_{jj'} = \mathbb{P}(X_{n+1} = i + 1, Y_{n+1} = j' | X_n = i, Y_n = j),
$$

$$
\left[A_1^{(i)} \right]_{jj'} = \mathbb{P}(X_{n+1} = i, Y_{n+1} = j' | X_n = i, Y_n = j),
$$

$$
\left[A_2^{(i)} \right]_{jj'} = \mathbb{P}(X_{n+1} = i - 1, Y_{n+1} = j' | X_n = i, Y_n = j).
$$

If $A_k^{(i)} = A_k^{(i')}$ for every $i, i' \in \mathbb{Z}_+$ (except possibly for the boundary level $i = 0$) and for each $k \in \{0, 1, 2\}$, then Φ is called a level-independent QBD (or simply QBD) process. However, if $A_k^{(i)} \neq A_k^{(i')}$ for two or more nonzero levels $i \neq i'$, then Φ is called a level-dependent QBD (LDQBD) process.

Our aim is to establish necessary and sufficient conditions for the positive recurrence (and thus, ergodicity) of the LDQBD Φ by analyzing the drift of the process. To that end, we first review drift functions, as well as an extension of Foster's criterion [17] for a one-dimensional irreducible, aperiodic DTMC $\xi := \{X_n : n \geq 0\}$ on a countable state space S. We begin with the following definition.

Definition 1. Let $\nu : S \to (0, \infty)$ be a positive function, and for each $x \in S$ let

$$
\delta_\nu(x) := \mathbb{E}(\nu(X_{n+1}) - \nu(X_n) | X_n = x).
$$

The function ν is called a Lyapunov (or potential) function, and $\delta_\nu(x)$ is the generalized drift of ξ in state x. Furthermore, if $v(X_n) = X_n$ for each n, then $\delta_\nu(x)$ is simply the drift in state x.

Foster [17] provided a sufficient condition for the positive recurrence of a one-dimensional, irreducible DTMC on the state space $\mathbb{Z}_+$. Fayolle et al. [16] extended Foster's result by proving necessity of the drift condition and expanding its applicability to DTMCs with general countable state spaces. For convenience, we restate Theorem 2.2.3 of Fayolle et al. [16] here as Theorem 1.

Theorem 1. (Fayolle et al. [16].) *An irreducible, aperiodic DTMC $\{X_n : n \geq 0\}$ on a countable state space S is positive recurrent if and only if there exist a function $\nu : S \to (0, \infty)$, a number $\epsilon > 0$ and a finite set $B \subset S$ such that*

$$\delta_\nu(x) \leq -\epsilon, \ x \notin B \quad and \quad \mathbb{E}(\nu(X_{n+1})|X_n = x) < \infty, \ x \in B. \tag{3}$$

If we take $S = \mathbb{Z}_+$, condition (3) can be equivalently restated as follows: For some number $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\delta_\nu(i) \leq -\epsilon, \ i \geq N_\epsilon \quad \text{and} \quad \mathbb{E}(\nu(X_{n+1})|X_n = i) < \infty, \ i < N_\epsilon.$$

In Section 3 we derive the generalized drift function for a specific form of the Lyapunov function ν and employ a modified version of Theorem 1 to establish sufficient conditions for recurrence, and necessary and sufficient conditions for positive recurrence, of the LDQBD process Φ.

3. Component and Average Drift Conditions

For the results that follow, we consider a specific class of Lyapunov functions for the LDQBD process Φ and its associated drift function. For each $(i, j) \in S$, define $\nu : S \to (0, \infty)$ as follows:

$$\nu(i, j) = \begin{cases} k_{0j}, & i = 0, \ j \in \mathscr{P}, \\ i + k_{ij}, & i \geq 1, \ j \in \mathscr{P}, \end{cases} \tag{4}$$

where the constant $k_{0j} > 0$ for any $j \in \mathscr{P}$ and k_{ij} is a nonnegative constant for each $i \in \mathbb{N}$ and $j \in \mathscr{P}$. We denote by $\boldsymbol{k} = [k_{ij}]$ the matrix comprised of these constants, and let $\boldsymbol{k}^{(i)} = (k_{i1}, k_{i2}, \ldots, k_{iK})'$ denote the transpose of the ith row of $\boldsymbol{k}$ with the requirement that $\boldsymbol{k}^{(0)} > \boldsymbol{0}$. Let

$$\delta_\nu(i, j) := \mathbb{E}(\nu(X_{n+1}, Y_{n+1}) - \nu(X_n, Y_n)|(X_n, Y_n) = (i, j)) \tag{5}$$

be the generalized drift in state (i, j) and define the column vector

$$\Delta^{(i)} = (\delta_\nu(i, 1), \ldots, \delta_\nu(i, K))'$$

as the level-i drift vector. The following proposition shows how to obtain $\Delta^{(i)}$ for each $i \in \mathbb{Z}_+$.

Proposition 1. *Let Φ be the discrete-time LDQBD process with state space S and one-step transition probability matrix given by* (2). *For the Lyapunov function of equation* (4), *the generalized level-i drift vector is*

$$\Delta^{(i)} = \begin{cases} A_0^{(0)}\boldsymbol{e} + A_0^{(0)}\boldsymbol{k}^{(1)} + A_1^{(0)}\boldsymbol{k}^{(0)} - \boldsymbol{k}^{(0)}, & i = 0, \\ \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A_0^{(i)}\boldsymbol{k}^{(i+1)} + A_1^{(i)}\boldsymbol{k}^{(i)} + A_2^{(i)}\boldsymbol{k}^{(i-1)}\right) - \boldsymbol{k}^{(i)}, & i \geq 1. \end{cases} \tag{6}$$

Proof. For notational brevity, let $x = (i,j)$ and $x' = (i',j')$ be two states in S, and denote by $p_{xx'}$ the one-step transition probability from x to x'. First we consider the case when $i \geq 1$. Applying equations (4) and (5), the generalized drift in state $x = (i,j)$ is

$$\delta_\nu(x) = \sum_{x' \in S} p_{xx'}\left[\nu(x') - \nu(x)\right] = \sum_{x' \in S} p_{xx'}\left[(i' + k_{i'j'}) - (i + k_{ij})\right]. \tag{7}$$

For each $i \in \mathbb{N}$, consider the ith-level partition $(S_i^-, S_i^\circ, S_i^+)$ of S, where

$$S_i^- = \{(i-1,j) : j \in \mathscr{P}\}, \quad S_i^\circ = \{(i,j) : j \in \mathscr{P}\}, \quad S_i^+ = \{(i+1,j) : j \in \mathscr{P}\},$$

which allows us to decompose the summation in (7) as follows:

$$\delta_\nu(x) = \sum_{x' \in S} p_{xx'}\left[(i' + k_{i'j'}) - (i + k_{ij})\right]$$

$$= \sum_{x' \in S_i^-} p_{xx'}(k_{i-1,j'} - k_{ij} - 1) + \sum_{x' \in S_i^\circ} p_{xx'}(k_{ij'} - k_{ij})$$

$$+ \sum_{x' \in S_i^+} p_{xx'}(k_{i+1,j'} - k_{ij} + 1). \tag{8}$$

Equation (8) can be equivalently expressed as

$$\delta_\nu(x) = \sum_{j' \in \mathscr{P}} \left[A_2^{(i)}\right]_{jj'}(k_{i-1,j'} - k_{ij} - 1) + \sum_{j' \in \mathscr{P}} \left[A_1^{(i)}\right]_{jj'}(k_{ij'} - k_{ij})$$

$$+ \sum_{j' \in \mathscr{P}} \left[A_0^{(i)}\right]_{jj'}(k_{i+1,j'} - k_{ij} + 1)$$

$$= \sum_{j' \in \mathscr{P}} \left(\left[A_0^{(i)}\right]_{jj'} - \left[A_2^{(i)}\right]_{jj'}\right) - k_{ij} \sum_{j' \in \mathscr{P}} \left[A^{(i)}\right]_{jj'}$$

$$+ \sum_{j' \in \mathscr{P}} k_{i-1,j'} \left[A_2^{(i)}\right]_{jj'} + k_{ij'} \left[A_1^{(i)}\right]_{jj'} + k_{i+1,j'} \left[A_0^{(i)}\right]_{jj'},$$

which in matrix-vector form gives

$$\Delta^{(i)} = \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} + \left(A_0^{(i)} \boldsymbol{k}^{(i+1)} + A_1^{(i)} \boldsymbol{k}^{(i)} + A_2^{(i)} \boldsymbol{k}^{(i-1)} \right) - \boldsymbol{k}^{(i)}.$$

Second, for the boundary case $i = 0$, noting that there are no downward jumps from level 0, we can repeat the derivation for the case $i \geq 1$ and exclude all terms containing $A_2^{(i)}$. Doing so yields

$$\Delta^{(0)} = A_0^{(0)} \boldsymbol{e} + A_0^{(0)} \boldsymbol{k}^{(1)} + A_1^{(0)} \boldsymbol{k}^{(0)} - \boldsymbol{k}^{(0)},$$

and the proof is complete. $\qquad\qquad\square$

We now consider the generalized drift vector $\Delta^{(i)}$ restricted to the class of Lyapunov functions (4) for which $\boldsymbol{k}$ has identical rows, i.e., $\boldsymbol{k}^{(i)} = \boldsymbol{c} \geq \boldsymbol{0}$ for all $i \in \mathbb{N}$. In this case, we obtain a simplified (homogeneous) version of the drift vector given by

$$\Delta_h^{(i)} = \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} + A^{(i)} \boldsymbol{c} - \boldsymbol{c}, \quad i \in \mathbb{N}. \tag{9}$$

The relationship between $\Delta_h^{(i)}$ and $\Delta^{(i)}$ is critical to our analysis. We will establish the fact that, under suitable conditions, the existence of solutions $\boldsymbol{c}$ that satisfy $\Delta_h^{(i)} \leq -\boldsymbol{\epsilon}$ imply the existence of solutions $\{\boldsymbol{k}^{(i-1)}, \boldsymbol{k}^{(i)}, \boldsymbol{k}^{(i+1)}\}$ that satisfy $\Delta^{(i)} \leq -\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} = (\epsilon, \epsilon, \ldots, \epsilon)'$ for some $\epsilon > 0$.

Our first result, Theorem 2, follows immediately by applying Foster's criterion to the homogeneous drift vector $\Delta_h^{(i)}$. Before stating this result, we impose a partial ordering $(\leq_s)$ on the state space S. For any two states (i, j) and (i', j') in S, let

$$(i, j) \leq_s (i', j') \quad \text{if and only if} \quad i \leq i', \tag{10}$$

for any $j, j' \in \mathscr{P}$ (i.e., S is ordered according to the level of each state). This ordering allows us to apply results suited for processes whose state space is $\mathbb{Z}_+$.

Theorem 2. *Suppose that Φ is an irreducible, aperiodic discrete-time LDQBD process. Then,*

1. *Φ is recurrent if there exists a constant vector $\hat{\boldsymbol{c}} \in \mathbb{R}_+^K$ such that, for some $N \in \mathbb{N}$,*

$$\left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} \leq \left(I - A^{(i)} \right) \hat{\boldsymbol{c}}, \quad \text{for all } i \geq N.$$

2. Φ *is positive recurrent if there exists a constant vector* $\hat{\boldsymbol{c}} \in \mathbb{R}_+^K$ *such that, for some* $\epsilon > 0$, *there exists an* $N_\epsilon \in \mathbb{N}$ *such that*

$$\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} \le \left(I - A^{(i)}\right)\hat{\boldsymbol{c}} - \boldsymbol{\epsilon}, \quad \text{for all } i \ge N_\epsilon.$$

where $\boldsymbol{\epsilon} = (\epsilon, \epsilon, \ldots, \epsilon)'$.

Proof. For recurrence, we first note that $\nu(i,j) = i + \hat{c}_j \to \infty$ as $i \to \infty$ for all $j \in \mathscr{P}$ due to the partial ordering (10) and nonnegativity of $\hat{c}_j$. By supposition, there exists $\hat{\boldsymbol{c}} \in \mathbb{R}_+^K$ and some $N \in \mathbb{N}$ such that the level-i drift vector

$$\Delta_h^{(i)} = \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + A^{(i)}\hat{\boldsymbol{c}} - \hat{\boldsymbol{c}} \le \boldsymbol{0}, \quad \text{for all } i \ge N,$$

or equivalently,

$$\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} \le \left(I - A^{(i)}\right)\hat{\boldsymbol{c}}, \quad \text{for all } i \ge N.$$

This condition holds for all $(i,j) \notin B := \{0, 1, \ldots, N-1\} \times \mathscr{P}$, where $|B| < \infty$. Therefore, by Theorem 2.1 of [15], the process is recurrent.

For positive recurrence, suppose there exists a constant vector $\hat{\boldsymbol{c}} \in \mathbb{R}_+^K$, a number $\epsilon > 0$ and a positive integer N_ϵ such that

$$\Delta_h^{(i)} = \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + A^{(i)}\hat{\boldsymbol{c}} - \hat{\boldsymbol{c}} \le -\boldsymbol{\epsilon}, \quad \text{for all } i \ge N_\epsilon,$$

or equivalently,

$$\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} \le \left(I - A^{(i)}\right)\hat{\boldsymbol{c}} - \boldsymbol{\epsilon}, \quad \text{for all } i \ge N_\epsilon.$$

Thus, we let $B_\epsilon := \{0, 1, \ldots, N_\epsilon - 1\} \times \mathscr{P}$ be a finite subset of S and see that $\delta_\nu(i,j) \le -\epsilon$ for all $(i,j) \notin B_\epsilon$. It remains to show that

$$\mu_{ij} := \mathbb{E}(\nu(X_{n+1}, Y_{n+1})|(X_n, Y_n) = (i,j)) < \infty$$

for all $(i,j) \in B_\epsilon$. In fact, we will show that μ_{ij} is finite for all $(i,j) \in S$, since level jumps are bounded in a LDQBD. Fixing $x = (i,j)$ and letting $x' = (i', j')$, we observe that

$$\mu_{ij} \le |\mu_{ij}| = \left|\sum_{x' \in S} p_{xx'}(i' + k_{i'j'})\right| \le \sum_{x' \in S} |p_{xx'}| \cdot |i' + k_{i'j'}|. \tag{11}$$

To see that the right-most term of inequality (11) is finite, we note that $p_{xx'} = 0$ for any state $x' \in S_{ij}^c$, where for each $(i, j) \in S$,

$$S_{ij} := \begin{cases} \{(i', j') \in S : i' \in \{i, i+1\}, \, j' \in \mathscr{P}\}, & i = 0, \\ \{(i', j') \in S : i' \in \{i-1, i, i+1\}, \, j' \in \mathscr{P}\}, & i \geq 1. \end{cases}$$

Noting that $|S_{ij}| < \infty$, and by inequality (11),

$$\mu_{ij} \leq \sum_{x' \in S} |p_{xx'}| \cdot |i' + k_{i'j'}| \leq \sum_{x' \in S_{ij}} 1 \cdot |i' + k_{i'j'}| = \sum_{x' \in S_{ij}} \nu(i', j') < \infty. \qquad (12)$$

Therefore, by Theorem 2.2.3 of [16], the process is positive recurrent. $\qquad \square$

Next we state our main results, namely necessary and sufficient conditions for recurrence and positive recurrence, respectively, based on an average drift criterion. Before stating the main results, we first state the following assumption, which describes the specific class of LDQBD processes under consideration.

Assumption 1. *For the discrete-time LDQBD process Φ, there exist $K \times K$ sub-stochastic matrices, A_2^*, A_1^*, and A_0^* whose sum*

$$A^* = A_2^* + A_1^* + A_0^*$$

is stochastic and which are related to the process Φ via the limit

$$\lim_{i \to \infty} A_k^{(i)} = A_k^*, \quad k = 0, 1, 2,$$

where the limit holds element-wise.

Assumption 1 asserts that, if the level is sufficiently large, then the LDQBD process evolves in a manner similar to that of a level-independent QBD process. The class of models satisfying this condition is quite large, encompassing many models arising in queueing, reliability, inventory and telecommunications settings.

Theorem 3. *Let Φ be an irreducible, aperiodic discrete-time LDQBD process satisfying Assumption 1 and define the scalar quantities*

$$D^{(i)} = \boldsymbol{\pi}^{(i)} \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e}, \qquad (13)$$

where $\boldsymbol{\pi}^{(i)}$ is the invariant row vector of $A^{(i)} = A_0^{(i)} + A_1^{(i)} + A_2^{(i)}$ for each $i \in \mathbb{Z}_+$. Then,

(i) Φ is recurrent if and only if there exists an $N \in \mathbb{N}$ such that

$$D^{(i)} \leq 0, \quad \text{for all } i \geq N.$$

(ii) Φ is positive recurrent if and only if, for some positive number ϵ, there exists an $N_\epsilon \in \mathbb{N}$ such that

$$D^{(i)} < -\epsilon, \quad \text{for all } i \geq N_\epsilon.$$

Before proving Theorem 3, let us define an operator $\Gamma^{(i)} : \mathbb{R}^K \to \mathbb{R}^K$ by

$$\Gamma^{(i)}(\boldsymbol{c}) = \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} + A^{(i)} \boldsymbol{c} - \boldsymbol{c}$$

such that $\Gamma^{(i)}(\boldsymbol{c}) = \Delta_h^{(i)}$. Additionally, for each $i \in \mathbb{Z}_+$, define the null sets for $\Gamma^{(i)}$ by

$$\mathcal{N}(\Gamma^{(i)}) = \left\{ \boldsymbol{c} \in \mathbb{R}^K : \Gamma^{(i)}(\boldsymbol{c}) = \boldsymbol{0} \right\} \quad \text{and} \quad \mathcal{N}^+(\Gamma^{(i)}) = \mathcal{N}(\Gamma^{(i)}) \cap \mathbb{R}_+^K.$$

The proof of Theorem 3 requires two technical lemmas.

Lemma 1. $D^{(i)} = 0$ if and only if $\mathcal{N}^+(\Gamma^{(i)}) \neq \emptyset$.

Proof. ($\Leftarrow$) Fix the level i and suppose there exists some $\boldsymbol{c} \in \mathcal{N}^+(\Gamma^{(i)})$. Then

$$\Gamma^{(i)}(\boldsymbol{c}) = \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} + A^{(i)} \boldsymbol{c} - \boldsymbol{c} = \boldsymbol{0},$$

which implies

$$\left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} = -A^{(i)} \boldsymbol{c} + \boldsymbol{c}. \tag{14}$$

Left-multiplying both sides of (14) by $\boldsymbol{\pi}^{(i)}$ yields

$$D^{(i)} = \boldsymbol{\pi}^{(i)} \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} = -\boldsymbol{\pi}^{(i)} A^{(i)} \boldsymbol{c} + \boldsymbol{\pi}^{(i)} \boldsymbol{c} = -\boldsymbol{\pi}^{(i)} \boldsymbol{c} + \boldsymbol{\pi}^{(i)} \boldsymbol{c} = 0.$$

($\Rightarrow$) Now suppose $D^{(i)} = 0$. We seek to show the existence of a $\boldsymbol{c} \in \mathbb{R}_+^K$ such that

$$\Gamma^{(i)}(\boldsymbol{c}) = \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} + \left(A^{(i)} - I \right) \boldsymbol{c} = \boldsymbol{0}. \tag{15}$$

To this end, we employ two related Theorems of Alternative – one due to Farkas and the other due to Gordan – both of which are summarized in [28] and restated without proof in the Appendix. Using (15), we can write the linear system of equalities

$$\left(A^{(i)} - I \right) \boldsymbol{c} = - \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e}. \tag{16}$$

Setting $\boldsymbol{c} = \boldsymbol{e}$, we note that $\left(A^{(i)} - I\right)\boldsymbol{e} = \boldsymbol{0}$; hence, $\boldsymbol{e}$ is a nonnegative, nonzero solution to the linear system of equations

$$\left(A^{(i)} - I\right)\boldsymbol{c} = \boldsymbol{0}.$$

Therefore, Alternative (ii) of Gordan's theorem [18] cannot be satisfied; hence,

$$\boldsymbol{y}'\left(A^{(i)} - I\right) \ngtr \boldsymbol{0}, \text{ for all } \boldsymbol{y} \in \mathbb{R}^K.$$

This implies that the system in Alternative (ii) of Farkas's lemma, namely $\boldsymbol{y}'\left(A^{(i)} - I\right) \geq \boldsymbol{0}$, can only be satisfied in equality. Because $A^{(i)}$ is finite and irreducible, the unique solution at equality is $\boldsymbol{y}' = \boldsymbol{\pi}^{(i)}$ (up to a multiplicative constant), since $\boldsymbol{\pi}^{(i)}\left(A^{(i)} - I\right) = \boldsymbol{0}$. However,

$$-\boldsymbol{y}'\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} = -\boldsymbol{\pi}^{(i)}\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} = -D^{(i)} = 0.$$

Hence, there does not exist $\boldsymbol{y} \in \mathbb{R}^K$ satisfying Alternative (ii) of Farkas's lemma, which implies that Alternative (i) of Farkas holds. That is, there exists a $\boldsymbol{c} \geq \boldsymbol{0}$ that solves (16), from which we conclude that $\mathcal{N}^+(\Gamma^{(i)}) \neq \emptyset$. $\qquad\square$

We next demonstrate the relationship of the average drift $D^{(i)}$ to the solution $\boldsymbol{c}$ of the system of inequalities $\Gamma^{(i)}(\boldsymbol{c}) \leq -\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} = (\epsilon, \epsilon, \ldots \epsilon) > \boldsymbol{0}$. For each $i \in \mathbb{Z}_+$, define the cones

$$\mathcal{C}_\epsilon(\Gamma^{(i)}) = \left\{\boldsymbol{c} \in \mathbb{R}^K : \Gamma^{(i)}(\boldsymbol{c}) \leq -\boldsymbol{\epsilon}\right\} \quad \text{and} \quad \mathcal{C}_\epsilon^+(\Gamma^{(i)}) := \mathcal{C}_\epsilon(\Gamma^{(i)}) \cap \mathbb{R}_+^K.$$

Lemma 2. *For some positive number ϵ, $D^{(i)} \leq -\epsilon$ if and only if $\mathcal{C}_\epsilon^+(\Gamma^{(i)}) \neq \emptyset$.*

Proof. ($\Leftarrow$) Suppose there exists some $\boldsymbol{c} \in \mathcal{C}_\epsilon^+(\Gamma^{(i)})$. That is, for such a solution,

$$\Gamma^{(i)}(\boldsymbol{c}) = \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A^{(i)} - I\right)\boldsymbol{c} \leq -\boldsymbol{\epsilon}.$$

Left-multiplying by the stochastic vector $\boldsymbol{\pi}^{(i)}$ yields

$$\boldsymbol{\pi}^{(i)}\Gamma^{(i)}(\boldsymbol{c}) = \boldsymbol{\pi}^{(i)}\left[\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A^{(i)} - I\right)\boldsymbol{c}\right] \leq -\boldsymbol{\pi}^{(i)}\boldsymbol{\epsilon} = -\epsilon.$$

However, note that $\boldsymbol{\pi}^{(i)}\Gamma^{(i)}(\boldsymbol{c}) = D^{(i)} + 0 = D^{(i)}$; therefore, we conclude that $D^{(i)} \leq -\epsilon$.

($\Rightarrow$) Conversely, suppose that $D^{(i)} \leq -\epsilon$ for some $\epsilon > 0$, which implies that $D^{(i)}\boldsymbol{e} \leq -\boldsymbol{\epsilon}$. Our aim is to show the existence of a nonnegative vector $\hat{\boldsymbol{c}}$ such that

$$\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A^{(i)} - I\right)\hat{\boldsymbol{c}} = D^{(i)}\boldsymbol{e},$$

or equivalently,

$$\left(A^{(i)} - I\right)\hat{\boldsymbol{c}} = D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}. \tag{17}$$

To simplify notation, let $M := A^{(i)} - I$ and $\boldsymbol{b} := D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}$. As shown in the proof of Lemma 1, the system $\boldsymbol{y}'M \geq \boldsymbol{0}$ can only be satisfied in equality, and the unique solution is $\boldsymbol{y}' = \boldsymbol{\pi}^{(i)}$ up to a multiplicative constant. Therefore, it remains to show that $\boldsymbol{y}'\boldsymbol{b} \not< 0$. We see that

$$\boldsymbol{y}'\boldsymbol{b} = \boldsymbol{\pi}^{(i)}\boldsymbol{b} = \boldsymbol{\pi}^{(i)}\left[D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}\right] = \boldsymbol{\pi}^{(i)}D^{(i)}\boldsymbol{e} - \boldsymbol{\pi}^{(i)}\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}$$

$$= D^{(i)} - D^{(i)} = 0.$$

Hence, there does not exist a $\boldsymbol{y} \geq \boldsymbol{0}$ satisfying $\boldsymbol{y}'\boldsymbol{b} < 0$. Therefore, by Farkas' lemma, there is a nonnegative solution $\hat{\boldsymbol{c}}$ to the system (17). Noting that $D^{(i)}\boldsymbol{e} \leq -\boldsymbol{\epsilon}$, we see that $\mathcal{C}_{\epsilon}^{+}(\Gamma^{(i)}) \neq \emptyset$. $\square$

Equipped with Lemmas 1 and 2, we are now prepared to prove the main result, Theorem 3.

Proof of Theorem 3. Part (i): Recurrence. To prove the first result, we will establish that the conditions required by Theorem 2.2.1 of [16] are met, i.e., $\nu(i,j) \to \infty$ as $i \to \infty$ and $\Delta^{(i)} \leq \boldsymbol{0}$ for i sufficiently large. Under the partial ordering (10), it is clear that, for each $j \in \mathscr{P}$, $\nu(i,j) = i + k_{ij} \to \infty$ as $i \to \infty$ due to the nonnegativity of k_{ij}. Hence, it remains to show that, for i sufficiently large, $D^{(i)} \leq 0$ if and only if there exists a sequence of constant vectors $\{\boldsymbol{k}^{(i)} : i \in \mathbb{Z}_+\}$ such that $\Delta^{(i)} \leq \boldsymbol{0}$. To this end, for each $i \in \mathbb{Z}_+$, define the operator $\Lambda^{(i)}$ such that

$$\Lambda^{(i)}(\boldsymbol{k}^{(i)}) := \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A_0^{(i)}\boldsymbol{k}^{(i+1)} + A_1^{(i)}\boldsymbol{k}^{(i)} + A_2^{(i)}\boldsymbol{k}^{(i-1)}\right) - \boldsymbol{k}^{(i)} = \Delta^{(i)}.$$

$(\Rightarrow)$ Suppose there is some $N \in \mathbb{N}$ such that $D^{(i)} \leq 0$ for each $i \geq N$. Our aim is to show that there exists a number $N^* \geq N$ and a sequence of vectors

$$\mathscr{K} = \left\{\boldsymbol{k}^{(i)} \in \mathbb{R}_+^K : i \in \mathbb{Z}_+\right\}$$

such that $\Lambda^{(i)}(\boldsymbol{k}^{(i)}) \leq \boldsymbol{0}$ whenever $i \geq N^*$. To this end, we first show that the component-wise limits

$$\boldsymbol{\ell}_g = \lim_{i \to \infty} \Gamma^{(i)}(\boldsymbol{k}^{(i)}) \text{ and } \boldsymbol{\ell}_l = \lim_{i \to \infty} \Lambda^{(i)}(\boldsymbol{k}^{(i)})$$

exist for a certain choice of $\mathscr{K}$. Next, we will show that $\boldsymbol{\ell}_g$ and $\boldsymbol{\ell}_l$ are equal, and when it has been shown that $\boldsymbol{\ell}_g \leq \mathbf{0}$, we will have demonstrated the sufficiency of part (i) of Theorem 3.

To establish the existence of $\mathscr{K}$, we first obtain a solution to the inequality

$$\Gamma^{(i)}(\boldsymbol{k}^{(i)}) = \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + \left(A^{(i)} - I\right)\boldsymbol{k}^{(i)} \leq \mathbf{0}. \tag{18}$$

By supposition, we have that $D^{(i)} \leq 0$ for each $i \geq N$; hence, we can satisfy (18) by solving the system of linear equalities $\Gamma^{(i)}(\boldsymbol{k}^{(i)}) = D^{(i)}\boldsymbol{e}$. That is,

$$\Gamma^{(i)}(\boldsymbol{k}^{(i)}) = \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e} + (A^{(i)} - I)\boldsymbol{k}^{(i)} = D^{(i)}\boldsymbol{e}.$$

By rearranging the terms in the above expression, we obtain

$$\left(A^{(i)} - I\right)\boldsymbol{k}^{(i)} = D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}. \tag{19}$$

Since $A^{(i)} - I$ is singular, an explicit solution of (19) can alternatively be obtained by employing the theory of group inverses. By Theorem 2.1 of [31], the existence and uniqueness of the group inverse

$$\left(T^{(i)}\right)^{\#} = \left(I - A^{(i)}\right)^{\#}$$

of the matrix $T^{(i)} = I - A^{(i)}$ is guaranteed, as each $A^{(i)}$ is the transition matrix of an irreducible, time-homogeneous Markov chain. Theorem 2.2 of [41] shows that every possible solution $\boldsymbol{k}^{(i)}$ of (19) is of the form

$$\boldsymbol{k}^{(i)} = -\left(T^{(i)}\right)^{\#}\left(D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}\right) + \left(I - \left(T^{(i)}\right)^{\#}T^{(i)}\right)\boldsymbol{z}_i, \tag{20}$$

where $\boldsymbol{z}_i \in \mathbb{R}^K$ is arbitrary for each $i \geq 1$. Equation (20) can be further refined by noting the equivalence

$$\Pi^{(i)} = I - \left(T^{(i)}\right)^{\#}T^{(i)}, \tag{21}$$

where $\Pi^{(i)} \in \mathbb{R}^{K \times K}$ is the matrix whose rows are all the stationary probability vector $\boldsymbol{\pi}^{(i)}$ (see Theorem 2.3 of [31]), from which we obtain

$$\boldsymbol{k}^{(i)} = -\left(T^{(i)}\right)^{\#}\left(D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}\right) + \Pi^{(i)}\boldsymbol{z}_i. \tag{22}$$

For some nonnegative scalar m_i, let $\boldsymbol{z}_i := m_i\,\boldsymbol{e} \in \mathbb{R}^K$, the column vector consisting of nonnegative scalar entries m_i. Equation (22) then reduces to

$$\boldsymbol{k}^{(i)} = -\left(T^{(i)}\right)^{\#}\left(D^{(i)}\boldsymbol{e} - \left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}\right) + m_i\boldsymbol{e}. \tag{23}$$

Next, we show that there exists a positive sequence $\{m_i : i \in \mathbb{N}\}$ that induces the solutions $\boldsymbol{k}^{(i)}$ to be nonnegative in (23). To this end, it is necessary to show that

$$\left(T^{(i)}\right)^{\#} = \left(I - A^{(i)}\right)^{\#} \quad \text{and} \quad D^{(i)} = \boldsymbol{\pi}^{(i)}\left(A_0^{(i)} - A_2^{(i)}\right)\boldsymbol{e}$$

converge as $i \to \infty$. By Assumption 1, the convergence of sequences $\left\{A_j^{(i)}\right\}$ for $j = 0, 1, 2$ ensures convergence of $I - A^{(i)}$. Next, we use this convergence, along with (21), to show that if the generalized inverse $\left\{\left(T^{(i)}\right)^{\#}\right\}$ converges, then the sequence of stationary vectors $\left\{\boldsymbol{\pi}^{(i)}\right\}$ likewise converges. Lemma 3 of the Appendix establishes that

$$\lim_{i\to\infty} \left(T^{(i)}\right)^{\#} = (T^*)^{\#},$$

ensuring that all terms on the right-hand side of (23) (with the exception of m_i) form convergent sequences. We will now choose an appropriate sequence of numbers $\{m_i\}$. Define the limits

$$D^* := \lim_{i\to\infty} D^{(i)} \qquad T^* := \lim_{i\to\infty} T^{(i)} \qquad \boldsymbol{\pi}^* := \lim_{i\to\infty} \boldsymbol{\pi}^{(i)}$$

and the function $\kappa : \mathbb{R} \to \mathbb{R}^K$ by

$$\kappa(x) := -(T^*)^{\#}\left[D^*\boldsymbol{e} - (A_0^* - A_2^*)\boldsymbol{e}\right] + x\boldsymbol{e}. \tag{24}$$

As the first summand on the right-hand side of (24) is a real constant vector, we may choose a sufficiently large number $M > 0$ such that $\kappa(M) \geq \boldsymbol{0}$ componentwise. The convergence of $\left\{\boldsymbol{k}^{(i)}\right\}$ in (23) when $m_i := M$ ensures there is an integer $J \in \mathbb{N}$ such that $\boldsymbol{k}^{(i)} \geq \boldsymbol{0}$ for every $i \geq J$. We may thus set

$$\mathscr{K} := \left\{\boldsymbol{k}^{(i)} : i = 0\ldots J - 1\right\} \cup \left\{\boldsymbol{k}^{(i)} : i \geq J, m_i = M\right\} \subset \mathbb{R}_+^K,$$

where vectors $\left\{\boldsymbol{k}^{(i)} : i = 0\ldots J - 1\right\} \subset \mathbb{R}_+^K$ may be chosen arbitrarily. In this way, we obtain a sequence $\mathscr{K}$ of feasible solutions to $\Gamma^{(i)}(\boldsymbol{k}^{(i)}) \leq \boldsymbol{0}$ that converges to some vector $\boldsymbol{k}^* \in \mathbb{R}_+^K$. Now, the existence of $\boldsymbol{k}^* = \lim_{i\to\infty} \boldsymbol{k}^{(i)}$ ensures convergence of the sequences $\left\{A_j^{(i)}\boldsymbol{k}^{(i)} : i \in \mathbb{Z}_+\right\}$ for $j = 0, 1, 2$, and hence of $\left\{A^{(i)}\boldsymbol{k}^{(i)} : i \in \mathbb{Z}_+\right\}$. Consequently, we obtain the convergence of $\left\{\Gamma^{(i)}(\boldsymbol{k}^{(i)}) : i \in \mathbb{Z}_+\right\}$ and $\left\{\Lambda^{(i)}(\boldsymbol{k}^{(i)}) : i \in \mathbb{Z}_+\right\}$ with the additional property that, as $i \to \infty$,

$$\Gamma^{(i)}(\boldsymbol{k}^{(i)}) \to (A_0^* - A_2^*)\boldsymbol{e} + (A^* - I)\boldsymbol{k}^* \leq \boldsymbol{0}.$$

We now show that $\left\{\Gamma^{(i)}(\boldsymbol{k}^{(i)}) : i \in \mathbb{Z}_+\right\}$ and $\left\{\Lambda^{(i)}(\boldsymbol{k}^{(i)}) : i \in \mathbb{Z}_+\right\}$ converge to the same limit. To this end, let $\|B\|$ be any matrix norm of a square matrix of K dimensions with real entries and let $\|B\|_{op}$ be the operator norm of a bounded linear transformation $\mathbb{R}^K \to \mathbb{R}^K$ whose operator matrix is B. Observe that for any $\epsilon > 0$ and i sufficiently large,

$$
\begin{aligned}
\left\|\Lambda^{(i)}(\boldsymbol{k}^{(i)}) - \Gamma^{(i)}(\boldsymbol{k}^{(i)})\right\| & \\
&= \left\|\left(A_0^{(i)}\boldsymbol{k}^{(i+1)} - A_0^{(i)}\boldsymbol{k}^{(i)}\right) + \left(A_1^{(i)}\boldsymbol{k}^{(i)} - A_1^{(i)}\boldsymbol{k}^{(i)}\right) + \left(A_2^{(i)}\boldsymbol{k}^{(i-1)} - A_2^{(i)}\boldsymbol{k}^{(i)}\right)\right\| \\
&\leq \left\|A_0^{(i)}\boldsymbol{k}^{(i+1)} - A_0^{(i)}\boldsymbol{k}^{(i)}\right\| + \left\|A_2^{(i)}\boldsymbol{k}^{(i-1)} - A_2^{(i)}\boldsymbol{k}^{(i)}\right\| \qquad (25) \\
&\leq \left\|A_0^{(i)}\right\|_{op}\left\|\boldsymbol{k}^{(i+1)} - \boldsymbol{k}^{(i)}\right\| + \left\|A_2^{(i)}\right\|_{op}\left\|\boldsymbol{k}^{(i-1)} - \boldsymbol{k}^{(i)}\right\| \\
&\leq \eta_0\left\|\boldsymbol{k}^{(i+1)} - \boldsymbol{k}^{(i)}\right\| + \eta_2\left\|\boldsymbol{k}^{(i-1)} - \boldsymbol{k}^{(i)}\right\| \\
&\leq \eta(\epsilon/2) + \eta(\epsilon/2) = \eta\epsilon,
\end{aligned}
$$

where $\eta = \max\{\eta_0, \eta_2\}$, since convergent sequences are Cauchy convergent in finite-dimensional Euclidean space. Thus, there exists an $N^* > N$ such that $\Lambda^{(i)}(\boldsymbol{k}^{(i)}) \leq \boldsymbol{0}$ for each $i \geq N^*$; hence, the process Φ is recurrent.

($\Leftarrow$) Conversely, suppose that Φ is recurrent. By [15, Thm. 2.2.1], there exists some $N^* > 0$ and some sequence $\mathscr{K} = \left\{\boldsymbol{k}^{(i)} : i \in \mathbb{Z}_+\right\} \subset \mathbb{R}_+^K$ of vectors such that

$$
\Lambda^{(i)}(\boldsymbol{k}^{(i)}) \leq \boldsymbol{0}, \quad i \geq N^*. \qquad (26)
$$

It is possible to construct a convergent $\mathscr{K}'$ from the elements of $\mathscr{K}$ that satisfies (26). Indeed, due to Assumption 1, each $A_k^{(i)}$ for $k = 0, 1, 2$ converges element-wise, thereby ensuring the existence of some $N' \geq N^*$ such that (26) holds for the convergent sequence

$$
\mathscr{K}' = \left\{\boldsymbol{k}^{(i)} : i \in \mathbb{Z}_+, \boldsymbol{k}^{(i)} = \boldsymbol{k}^{(N')} \text{ for } i \geq N'\right\}.
$$

Then, from Assumption 1 and the convergence of sequence $\mathscr{K}'$, we obtain the inequality (25) over the sequence $\mathscr{K}'$ whenever $i \geq N''$, for some $N'' > N'$. This, together with the fact that (26) holds over $\mathscr{K}'$ shows that $\Gamma^{(i)}(\boldsymbol{k}^{(i)}) \leq \boldsymbol{0}$ must likewise hold over $\mathscr{K}'$ for i sufficiently large. By Lemmas 1 and 2, it must be the case that $D^{(i)} \leq 0$ for all such i. This completes the proof of part (i).

Part (ii): Positive Recurrence. The assertion of part (ii) follows directly from the proof of part (i) by invoking Theorem 2.2.3 of [16] in place of Theorem 2.2.1 of the same reference. $\qquad\square$

4. Connections to Level-Independent QBD Processes

In this section, we draw parallels between established drift conditions for level-independent QBD processes and conditions derived in Section 3. In Proposition 2 we establish the existence of the limiting drift D^* and show that it can be computed in a manner that is analogous to computing the average drift of its level-independent counterpart.

Proposition 2. *Suppose that the assumptions of Theorem 3 hold for the discrete-time process Φ. Then the limiting drift D^* exists and is given by*

$$D^* := \lim_{i \to \infty} D^{(i)} = \boldsymbol{\pi}^* \left(A_0^* - A_2^* \right) \boldsymbol{e},$$

where $\boldsymbol{\pi}^*$ is the unique solution to the system of equations

$$\boldsymbol{x} A^* = \boldsymbol{x} \quad and \quad \boldsymbol{x} \boldsymbol{e} = 1, \tag{27}$$

$A^* = A_2^* + A_1^* + A_0^*$, *and the matrices A_j^*, $j = 0, 1, 2$ are defined in Assumption 1.*

Proof. First we establish that the K-dimensional vector $\boldsymbol{\pi}^* := \lim \boldsymbol{\pi}^{(i)}$ is a solution to (27). For each $i \in \mathbb{N}$, $\boldsymbol{\pi}^{(i)} A^{(i)} = \boldsymbol{\pi}^{(i)}$ and $\boldsymbol{\pi}^{(i)} \boldsymbol{e} = 1$; therefore,

$$\lim_{i \to \infty} \boldsymbol{\pi}^{(i)} A^{(i)} = \lim_{i \to \infty} \boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}^* \quad and \quad 1 = \lim_{i \to \infty} \boldsymbol{\pi}^{(i)} \boldsymbol{e} = \boldsymbol{\pi}^* \boldsymbol{e}. \tag{28}$$

Next we show that $\boldsymbol{\pi}^{(i)} A^{(i)} \to \boldsymbol{\pi}^* A^*$ as $i \to \infty$. For each phase $j \in \mathscr{P}$,

$$\lim_{i \to \infty} \left[\boldsymbol{\pi}^{(i)} A^{(i)} \right]_j = \lim_{i \to \infty} \sum_{m \in \mathscr{P}} \boldsymbol{\pi}_m^{(i)} A_{m,j}^{(i)} = \sum_{m \in \mathscr{P}} \lim_{i \to \infty} \boldsymbol{\pi}_m^{(i)} A_{m,j}^{(i)} = \sum_{m \in \mathscr{P}} \left(\boldsymbol{\pi}_m^* \right) \left(A_{m,j}^* \right)$$
$$= \left[\boldsymbol{\pi}^* A^* \right]_j.$$

As the left- and right-hand sides are equal for every j, it follows that

$$\lim_{i \to \infty} \boldsymbol{\pi}^{(i)} A^{(i)} = \boldsymbol{\pi}^* A^*. \tag{29}$$

By equations (28) and (29), $\boldsymbol{\pi}^* A^* = \boldsymbol{\pi}^*$ and $\boldsymbol{\pi}^* \boldsymbol{e} = 1$. To see that $\boldsymbol{\pi}^*$ is the unique solution to (27), note that

$$\boldsymbol{e} = \lim_{i \to \infty} A^{(i)} \boldsymbol{e} = \left(\lim_{i \to \infty} A^{(i)} \right) \boldsymbol{e} = A^* \boldsymbol{e},$$

demonstrating that A^* is stochastic. The irreducibility and finite dimensionality of A^*, along with (28) and (29), prove that $\boldsymbol{\pi}^*$ uniquely solves (27). Finally, the existence of $\boldsymbol{\pi}^*$, along with Assumption 1, establish the existence of

$$D^* = \lim_{i \to \infty} \boldsymbol{\pi}^{(i)} \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} = \left(\lim_{i \to \infty} \boldsymbol{\pi}^{(i)} \right) \left(\lim_{i \to \infty} \left(A_0^{(i)} - A_2^{(i)} \right) \boldsymbol{e} \right)$$
$$= \boldsymbol{\pi}^* \left(A_0^* - A_2^* \right) \boldsymbol{e},$$

and the proof is complete. $\qquad \square$

For the embedded DTMC of irreducible, level-independent QBD processes in which A_0, A_1 and A_2 are constant matrices over the levels i, it is well-known that the average drift of the process is

$$D = \boldsymbol{\pi}(A_0 - A_2)\boldsymbol{e},$$

where $\boldsymbol{\pi}$ is the unique solution of the system of equations $\boldsymbol{\pi}(A_0 + A_1 + A_2) = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$. Furthermore, it can be shown that the process is positive recurrent if and only if $D < 0$, null recurrent if and only if $D = 0$, and transient if $D > 0$ (see [24, 38]). Proposition 3 asserts that similar conditions can be stated for the level-dependent process Φ considered herein.

Proposition 3. *Suppose the assumptions of Theorem 3 hold. Then the discrete-time LDQBD process Φ is*

  (i) *recurrent if and only if $D^* \leq 0$ and $D^{(i)} > 0$ for at most finitely many i,*

  (ii) *positive recurrent if and only if $D^* < 0$,*

  (iii) *null recurrent if and only if $D^* = 0$ and $D^{(i)} > 0$ for at most finitely many i,*

  (iv) *transient if and only if $D^* \geq 0$ and $D^{(i)} > 0$ for infinitely many i.*

Proof. For part (i), suppose first that $D^* \leq 0$ and $D^{(i)} > 0$ for at most a finite number of terms i. Then there must be some $N > 0$ such that the terms $D^{(i)} \leq$

0 for every $i \geq N$, by existence of $D^* = \lim_{i \to \infty} D^{(i)}$. Thus, there is a finite number $\epsilon = \inf_{i \geq N} \left(-D^{(i)} \right) \geq 0$. So by Lemmas 1 and 2, there is a sequence $\mathscr{K} = \left\{ \boldsymbol{k}^{(i)} \in \mathbb{R}_+^K : i \in \mathbb{Z}_+ \right\}$ such that

$$\Gamma^{(i)}(\boldsymbol{k}^{(i)}) \leq -\epsilon \quad \text{for } i \geq N. \tag{30}$$

It was shown in the proof of Theorem 3 that $\mathscr{K}$ is convergent and, furthermore, that there exists an $N^* > N$ such that

$$\Lambda^{(i)}(\boldsymbol{k}^{(i)}) \leq -\epsilon, \quad i \geq N^*,$$

thus establishing the recurrence of Φ. Conversely, if Φ is recurrent, then by Theorem 3, there exists an $\epsilon \geq 0$ and $N > 0$ such that $D^{(i)} \leq -\epsilon$ for all $i \geq N$. As $D^{(i)} \to D^*$, we must have that the limit point $D^* \leq -\epsilon$, which completes the proof of part (i).

For part (ii), assuming the strict inequality $D^* < 0$ implies the existence some $N > 0$ such that the terms $D^{(i)} < 0$ for every $i > N$. This further implies the existence of a finite number $\epsilon = \inf_{i \geq N} \left(-D^{(i)} \right) > 0$. By Lemma 2, we then obtain a sequence $\mathscr{K} = \left\{ \boldsymbol{k}^{(i)} \in \mathbb{R}_+^K : i \in \mathbb{Z}_+ \right\}$ such that

$$\Gamma^{(i)}(\boldsymbol{k}^{(i)}) \leq -\epsilon < 0 \text{ for } i \geq N.$$

As in the proof of part (i), we may then infer the existence of an $N^* > N$ such that

$$\Lambda^{(i)}(\boldsymbol{k}^{(i)}) \leq -\epsilon < 0, \quad i \geq N^*,$$

which establishes the positive recurrence of Φ. On the other hand, if Φ is positive recurrent, then Theorem 3 gives the existence of an $\epsilon > 0$ and $N > 0$ such that $D^{(i)} \leq -\epsilon < 0$ for all $i \geq N$. As $D^{(i)} \to D^*$, we must have that the limit point $D^* \leq -\epsilon < 0$, which completes the proof of part (ii).

Part (iii) follows immediately from parts (i) and (ii), since the region of null recurrence indicated by D^* must be the complement of the set $\{D^* \in \mathbb{R} : D^* < 0\}$ within the indicated region of recurrence.

Finally, part (iv) follows from part (i) due to the fact that the regions of transience and recurrence must be complementary. □

5. Conclusion

In this paper, we have provided both necessary and sufficient conditions for the recurrence and positive recurrence of a class of LDQBD processes that exhibit *convergence in the rows* by employing a Foster-Lyapunov approach. For such processes, a limiting drift term D^* was shown to exist, and this term provides an analytical means by which to characterize the recurrence properties of the process. The parallels to the level-independent QBD process are apparent, namely that a negative average drift is necessary and sufficient to ensure positive recurrence. The analysis contained herein provides a simpler means by which to perform stability analysis for broad classes of queueing models with nonstandard input and output processes.

Appendix

Theorem 4. (Farkas's Lemma.) *Let $M \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$ a column vector of dimension m. Then exactly one of the statements (i) or (ii) is true:*

  (i) *There exists some $\boldsymbol{x} \in \mathbb{R}^n$ with $\boldsymbol{x} \geq \boldsymbol{0}$ such that $M\boldsymbol{x} = \boldsymbol{b}$, or*

  (ii) *There exists some $\boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{y}'M \geq \boldsymbol{0}$ and $\boldsymbol{y}'\boldsymbol{b} < 0$.*

Theorem 5. (Gordan's Theorem.) *Let $M \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix. Then exactly one of the statements (i) or (ii) is true:*

  (i) *There exists some $\boldsymbol{x} \in \mathbb{R}^n$ with $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \neq \boldsymbol{0}$ such that $M\boldsymbol{x} = \boldsymbol{0}$, or*

  (ii) *There exists some $\boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{y}'M > \boldsymbol{0}$.*

Proposition 4. *For $n \in \mathbb{N}$, let $\{M_i\}$ be a sequence of elements in $\mathbb{R}^{n \times n}$ that converges elementwise to an invertible limit M_*. Then*

$$\lim_{i \to \infty} M_i^{-1} = M_*^{-1}.$$

Proof. From rudimentary linear algebra, it is known that diagonalizable invertible matrices M_i may be expressed in terms of the decompositions $M_i = P_i D_i P_i^{-1}$, where D_i is a diagonal matrix whose nonzero entries are the (possibly repeated) real and complex eigenvalues of M_i, and is unique up to ordering of the eigenvalues. Otherwise,

if M_i is not diagonalizable, then one may still decompose the matrix as $M_i = P_i J_i P_i^{-1}$, where J_i is the corresponding Jordan canonical form (JCF) of M_i. Likewise, one may also write $M_* = P_* D_* P_*^{-1}$ or $M_* = P_* J_* P_*^{-1}$, where D_* is a diagonal matrix and J_* is a JCF, depending upon whether or not M_* is diagonalizable.

For the special case in which each $M_i = P_i D_i P_i^{-1}$ is a diagonalizable matrix, the fact that $M_i \to M_*$, together with the uniqueness of the diagonal decomposition and the invertibility of P_i, guarantees the convergence $P_i \to P_*$, $P_i^{-1} \to P_*^{-1}$, and hence $D_i \to D_*$. This shows that M_* must likewise be diagonalizable. Setting

$$D_i := \mathrm{diag}\,(\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})\,, \quad D_* = \mathrm{diag}\,(\lambda_{*1}, \lambda_{*2}, \ldots \lambda_{*n})\,,$$

we see that $\lambda_{ik} \to \lambda_{*k}$ must hold for each $k = 1 \ldots n$. Consequently, we must also have $1/\lambda_{ik} \to 1/\lambda_{*k}$ for each k, which proves that $D_i^{-1} \to D_*^{-1}$, and thus $M_i^{-1} \to M_*^{-1}$.

For the case in which each M_i is non-diagonalizable, we may express each J_i for large enough i as $J_i = \mathrm{diag}\,(J_{i1}, J_{i2}, \ldots, J_{ir})$ and $J_* = \mathrm{diag}\,(J_{*1}, J_{*2}, \ldots, J_{*r})$ for some $r \geq 1$. For a fixed $k \in \mathbb{Z}_+$ and complex eigenvalues λ_{ik} and λ_{*k}, the corresponding block entries J_{ik} and J_{*k}, both of which are of the dimension $n_k \times n_k$, appear in the form

$$J_{ik} = \begin{bmatrix} \lambda_{ik} & 1 & & \\ & \lambda_{ik} & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_{ik} & 1 \end{bmatrix}, \quad J_{*k} = \begin{bmatrix} \lambda_{*k} & 1 & & \\ & \lambda_{*k} & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_{*k} & 1 \end{bmatrix}.$$

The dimensions of these matrices must obey the relation $n = \sum_{k=1}^{r} n_k$. By the uniqueness of the JCF decomposition (up to ordering of the eigenvalues in the JCF matrix), together with the invertibility of P_i, the convergence $M_i \to M_*$ implies that $P_i \to P_*$, $P_i^{-1} \to P_*^{-1}$, and thus $J_i \to J_*$. In particular, $J_{ik} \to J_{*k}$ implies that $\lambda_{ik} \to \lambda_{*k}$ for each $k = 1 \ldots r$.

We will now show that $M_i^{-1} \to M_*^{-1}$, where

$$M_i^{-1} = P_i^{-1} J_i^{-1} P_i \quad \text{and} \quad M_*^{-1} = P_*^{-1} J_*^{-1} P_*.$$

Based on the properties of JCF decompositions, it is sufficient to demonstrate the convergence $J_i^{-1} \to J_*^{-1}$. In order to do this, we will consider J_i for large i. Since

$J_i^{-1} = \text{diag}\left(J_{i1}^{-1}, J_{i2}^{-1}, \ldots, J_{ir}^{-1}\right)$, we will choose an arbitrary fixed k and note that

$$
J_{ik}^{-1} = \begin{bmatrix}
\mu_{ik} & \mu_{ik}^2 & \mu_{ik}^3 & \cdots & \mu_{ik}^{n_k} \\
 & \mu_{ik} & \mu_{ik}^2 & \cdots & \mu_{ik}^{n_k-1} \\
 & & \mu_{ik} & \cdots & \mu_{ik}^{n_k-2} \\
 & & & \ddots & \vdots \\
 & & & & \mu_{ik}
\end{bmatrix}, \qquad \mu_{ik} = 1/\lambda_{ik}.
$$

Due to the triangular shape of the matrix, it is possible to find some nilpotent matrix N_{ik} such that $N_{ik}^{n_k+1} = \mathbf{0}$ for some $n_k \geq 1$, and for which

$$
J_{ik}^{-1} = I - \mu_{ik} N_{ik} + \mu_{ik}^2 N_{ik}^2 - \cdots \pm \mu_{ik}^{n_k} N_{ik}^{n_k} = \mu_{ik}\left(I + \mu_{ik} N_{ik}\right)^{-1}. \qquad (31)
$$

Applying the convergence $\lambda_{ik} \to \lambda_{*k}$ to Equation (31) then gives $\mu_{ik} \to \mu_{*k} = 1/\lambda_{*k}$, which in turn produces $J_{ik}^{-1} \to J_{*k}^{-1}$ for each $k = 1, \ldots, r$. Consequently, $J_i^{-1} \to J_*^{-1}$ and

$$
M_*^{-1} = P_*^{-1} J_*^{-1} P_* = \lim_{i \to \infty} P_i^{-1} J_i^{-1} P_i,
$$

and the proof is complete. $\qquad \square$

Lemma 3. *Suppose that Assumption 1 holds. Then*

$$
\lim_{i \to \infty} \left(T^{(i)}\right)^{\#} = \left(T^*\right)^{\#},
$$

where $T^{(i)} = I - A^{(i)}$, $T^* = \lim_{i \to \infty} T^{(i)}$ and $B^{\#}$ denotes the generalized inverse of a matrix $B \in \mathbb{R}^{K \times K}$ defined in [31].

Proof. To prove the assertion, we will appeal to the Jordan canonical forms (JCF) of the operator $T^{(i)}$ and its group inverse $\left(T^{(i)}\right)^{\#}$. The irreducibility of $A^{(i)}$ for every i allows us to invoke Corollary 2.1 of [31], which states that

$$
T^{(i)} = \left(S^{(i)}\right)^{-1} \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & I - U^{(i)} \end{array}\right) S^{(i)}, \quad \left(T^{(i)}\right)^{\#} = \left(S^{(i)}\right)^{-1} \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & \left(I - U^{(i)}\right)^{-1} \end{array}\right) S^{(i)},
$$

where $S^{(i)} \in \mathbb{R}^{K \times K}$ and $U^{(i)} \in \mathbb{R}^{(K-1) \times (K-1)}$ are invertible matrices. By Assumption 1, we know that $T^* = I - A^* = \lim_{i \to \infty} I - A^{(i)}$ exists. Thus, from the same result, we obtain the JCF decompositions

$$
T^* = \left(S^*\right)^{-1} \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & I - U^* \end{array}\right) S^*, \quad \left(T^*\right)^{\#} = \left(S^*\right)^{-1} \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & \left(I - U^*\right)^{-1} \end{array}\right) S^*, \qquad (32)
$$

where $S^* \in \mathbb{R}^{K \times K}$ and $U^* \in \mathbb{R}^{(K-1) \times (K-1)}$ are invertible matrices. It is well-known that the JCF decomposition is unique up to ordering of the distinct eigenvalues, as explained in the proof of Proposition 4. We will assume that the eigenvalues in each $I - U^{(i)}$ are ordered in such a way that the element-wise convergence $\left(I - U^{(i)}\right) \to (I - U^*)$ makes sense, should it occur.

For notational brevity, we will label the following JCFs as

$$
J^{(i)} := \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & I - U^{(i)} \end{array} \right) \quad \text{and} \quad J^* := \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & I - U^* \end{array} \right).
$$

The known convergence of $T^{(i)} = S^{(i)} J^{(i)} \left(S^{(i)}\right)^{-1} \to T^*$, together with the invertibility of $S^{(i)}$ and the uniqueness of the JCF decomposition, allows us to assert the convergence of $S^{(i)} \to S^*$, $U^{(i)} \to U^*$, $J^{(i)} \to J^*$, and $\left(S^{(i)}\right)^{-1} \to (S^*)^{-1}$. The convergence $U^{(i)} \to U^*$ further implies that $J^{(i)} \to J^*$. In summary, we obtain the convergence of $\left\{T^{(i)}\right\}$ to the limiting JCF decomposition

$$
T^* = \lim_{i \to \infty} T^{(i)} = \lim_{i \to \infty} \left(S^{(i)}\right)^{-1} J^{(i)} S^{(i)} = (S^*)^{-1} J^* S^*. \tag{33}
$$

We may now apply the convergence $S^{(i)} \to S^*$, $U^{(i)} \to U^*$, $J^{(i)} \to J^*$, and $\left(S^{(i)}\right)^{-1} \to (S^*)^{-1}$ that were determined in the course of formulating Equation (33), to the sequence $\left\{\left(T^{(i)}\right)^{\#}\right\}$ in order to obtain

$$
\lim_{i \to \infty} \left(T^{(i)}\right)^{\#} = (S^*)^{-1} \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & (I - U^*)^{-1} \end{array} \right) S^* = (T^*)^{\#},
$$

where $\left\{\left(I - U^{(i)}\right)^{-1}\right\} \to (I - U^*)^{-1}$ results from Proposition 4. $\qquad\square$

Acknowledgements

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