Proceedings of DETC'97
1997 ASME Design Engineering Technical Conferences
September 14-17, 1997, Sacramento, California

DETC97/VIB-3841

IMPLEMENTATION OF NON-REFLECTING BOUNDARIES IN A SPACE-TIME FINITE ELEMENT METHOD FOR STRUCTURAL ACOUSTICS

Lonny L. Thompson
Computational Mechanics Laboratory
Department of Mechanical Engineering and Engineering Mechanics
Clemson University
Clemson, South Carolina 29634
Email: lonny.thompson@ces.clemson.edu

ABSTRACT

This paper examines the development and implementation of second-order accurate non-reflecting boundary conditions in a time-discontinuous Galerkin finite element method for structural acoustics in unbounded domains. The formulation is based on a multi-field space-time variational equation for both the acoustic fluid and elastic solid together with their interaction. This approach to the modeling of the temporal variables allows for the consistent use of high-order accurate adaptive solution strategies for unstructured finite elements in both time and space. An important feature of the method is the incorporation of temporal jump operators which allow for discretizations that are discontinuous in time. Two alternative approaches are examined for implementing non-reflecting boundaries within a time-discontinuous Galerkin finite element method; *direct* implementation of the exterior acoustic impedance through a weighted variational equation in time and space, and *indirectly* through a decomposition into two equations involving an auxiliary variable defined on the non-reflecting boundary. The idea for the indirect approach was originally developed in (Kallivokas, 1991) in the context of a standard semi-discrete formulation. Extensions to general convex boundaries are also discussed – numerical results are presented for acoustic scattering from an elongated structure using a first-order accurate boundary condition applied to an elliptical absorbing boundary.

INTRODUCTION

Recently, a class of high-order accurate space-time finite element methods for transient structural acoustics in unbounded domains has been developed (Thompson, 1996a; Thompson, 1996b; Thompson, 1996c; Thompson, 1996d). An important feature of the space-time formulation is the incorporation of temporal jump operators which allow for finite element basis functions that are discontinuous in time. In (Thompson, 1995a) a multi-field extension is given where independent interpolation functions are used for both the acoustic velocity potential and its time derivative, i.e. the acoustic pressure. The use of a multi-field formulation was first introduced in (Hughes, 1988) in the context of elastodynamics, and allows for greater flexibility over the choice of finite element basis functions used and eliminates a minimum requirement of quadratic basis functions present in a single-field formulation. The multi-field formulation reverts to the single-field formulation with certain choices of basis functions. Time-discontinuous Galerkin space-time finite element methods are capable of delivering very high accuracies for wave propagation simulations, particularly for problems involving sharp gradients in the solution which typically arise in the vicinity of fluid-structure interfaces and near inhomogeneities such as stiffeners, structural joints, and material discontinuities. In these problems, solutions obtained with standard numerical methods may have difficulty resolving the discontinuities in the physical solution – in the case of standard time integrators, large spurious oscillations may appear which pollute the entire solution. In addition, for problems involving the propagation of pulses with broadband frequencies over a large distance, commonly used second-order accurate numerical algorithms may exhibit significant dispersion errors causing misrepresentation of arrival time and directionality at a distant target.

When space-time finite element methods are used to solve the structural acoustics problem in unbounded domains, a fluid truncation boundary is introduced where radiation (non-reflecting) boundary conditions are applied to transmit outgoing waves. If accurate non-reflecting boundary conditions are used, fewer fluid elements are needed and considerable cost savings will result. Therefore there is a need for the implementation of high-order accurate non-reflecting (absorbing) boundary conditions which eliminate or minimize reflection of outgoing waves and that also preserve the data structure of the space-time finite element method. In (Thompson, 1996a) we indicated how the time-discontinuous space-time finite element method provides a natural variational setting for the implementation of local in time non-reflecting boundary conditions. In (Thompson, 1996b), a new sequence of high-order accurate and local in time non-reflecting boundary conditions based on an exact representation of the acoustic impedance were developed.

In this paper, we clarify the space-time formulation given in (Thompson, 1995a), and focus on the development and consistent implementation of a second-order accurate non-reflecting boundary condition within the multi-field space-time variational equation for the coupled structural acoustics problem in unbounded domains. Two alternative approaches for implementing the non-reflecting boundary condition are examined: in the first, the non-reflecting boundary operator is implemented *directly* as a 'natural' boundary condition in the space-time variational equation, i.e. it is enforced weakly in both space and time; in the second, the non-reflecting boundary operator is implemented *indirectly* through a decomposition into two equations involving an auxiliary variable. The first equation is implemented as a 'natural' boundary condition, while the second equation is incorporated through a consistent Galerkin method. The idea for the indirect approach was originally given in (Kallivokas, 1991; Kallivokas, 1995) in the context of a standard semi-discrete finite element formulation. The advantage of the *indirect* approach is the resulting symmetry of the formulation in the spatial dimension, at the expense of solving for an additional variable on the nonreflecting boundary. Extensions of the formulation to general convex boundaries is also discussed – as an example, numerical results are presented for acoustic scattering from an elongated structure using a first-order accurate boundary condition applied to an elliptical boundary.

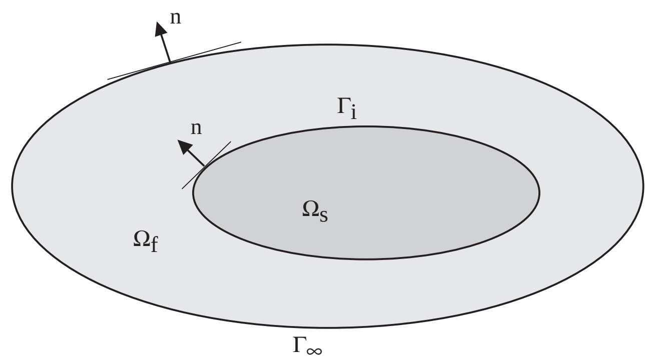

Figure 1. Coupled system for the exterior fluid-structure interaction problem, with artificial boundary Γ_∞ enclosing the finite computational domain $\Omega = \Omega_f \cup \Omega_s$.

THE UNBOUNDED STRUCTURAL ACOUSTICS PROBLEM

Consider the coupled system consisting of a structural region Ω_s surrounded by an infinite fluid region B. The interface boundary between the structure and fluid domains is denoted by Γ_i. The unit outward normal to the structure (inward normal to the fluid) on Γ_i is denoted by $\boldsymbol{n}$. The non-reflecting boundary is denoted Γ_∞ and positioned such that the original fluid region B is divided into a bounded interior domain Ω_f and an exterior domain Ω_∞ such that $B = \Omega_f \cup \Omega_\infty$; see Fig. 1. The structure is assumed to be governed by the equations of elastodynamics while the fluid equations are taken under the usual linear acoustic assumptions of an inviscid, compressible fluid with small disturbance. The momentum equations for the fluid are

$$\nabla p + \rho_f \dot{\boldsymbol{v}} = \boldsymbol{0} \tag{1}$$

where $p(\boldsymbol{x}, t)$ is the acoustic pressure, $\boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid particle velocity, and $\rho_f(\boldsymbol{x}) > 0$ is the density of the fluid. A superimposed dot indicates partial differentiation with respect to time t. The constitutive behavior of the fluid is assumed to be

$$\dot{p} + K_f \nabla \cdot \boldsymbol{v} = 0 \tag{2}$$

where $K_f = \rho_f c^2$ is the bulk modulus and c is the acoustic wave speed. From the assumption of an irrotational acoustic fluid, the velocity can be written as the gradient of the velocity potential ϕ as $\boldsymbol{v} = \nabla\phi$. Consequently, pressure is related to the velocity potential by $p = -\rho_f \dot{\phi}$. On the structural interface Γ_i, the normal component of the fluid velocity is assumed to be equivalent to the motion of the structural surface. Projecting the velocity normal to the structure gives the fluid-structure coupling: $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v}_s \cdot \boldsymbol{n}$ where $\boldsymbol{v}_s(\boldsymbol{x}, t)$ is the structural velocity vector. The influence of the fluid pressure acting on the structure is given by the normal traction $\boldsymbol{\sigma}\cdot\boldsymbol{n} = -p\boldsymbol{n}$ where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The stress is assumed to be related to the structural displacement vector $\boldsymbol{u}_s(\boldsymbol{x},t)$ through a linear constitutive relation of the form:

$$\boldsymbol{\sigma}(\boldsymbol{u}_s) = \boldsymbol{C} : \nabla^s \boldsymbol{u}_s \tag{3}$$

where $\nabla^s \boldsymbol{u}_s$ is the symmetric gradient and $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{x})$ is the fourth-order tensor of elastic coefficients; assumed to satisfy the usual pointwise stability and major and minor symmetry properties. The equations of motion for the structure are

$$\nabla\cdot\boldsymbol{\sigma} = \rho_s \dot{\boldsymbol{v}}_s \tag{4}$$

where $\rho_s(\boldsymbol{x}) > 0$ is the structural density, and $\boldsymbol{v}_s(\boldsymbol{x},t) = \dot{\boldsymbol{u}}_s(\boldsymbol{x},t)$. The drivers for the problem are the initial conditions:

$$\boldsymbol{u}_s(\boldsymbol{x},0) = \boldsymbol{u}_s^0(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega_s \tag{5}$$

$$\boldsymbol{v}_s(\boldsymbol{x},0) = \boldsymbol{v}_s^0(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega_s \tag{6}$$

$$\phi(\boldsymbol{x},0) = \phi^0(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega_f \tag{7}$$

$$p(\boldsymbol{x},0) = p^0(\boldsymbol{x}) \qquad \boldsymbol{x}\in\Omega_f \tag{8}$$

and any internal sources defined in Ω.

TIME-DISCONTINUOUS GALERKIN FE FORMULATION

The development of the space-time method proceeds by considering a partition of the time interval, $I =]0,T[$, of the form: $0 = t_0 < t_1 < \cdots < t_N = T$, with $I_n =]t_n, t_{n+1}[$ and $\Delta t_n = t_{n+1} - t_n$. Using this notation, $Q_n^s = \Omega_s \times I_n$, and $Q_n^f = \Omega_f \times I_n$ are the nth space-time slabs for the structure and fluid respectively. For the nth space-time slab, the spatial domain is subdivided into $(n_{el})_n$ elements, and the interior of the e^{th} element is defined as Q_n^e. Figure 2 shows an illustration of two consecutive space-time slabs Q_{n-1} and Q_n. Within each space-time element, the trial solution and weighting function are approximated by basis functions which depend on both $\boldsymbol{x}$ and t. These functions are assumed $C^0(Q_n)$ continuous throughout each space-time slab, but are allowed to be discontinuous across the interfaces of the slabs. The space of finite element basis functions for the multi-field representation for the fluid are stated in terms of *independent* trial velocity potential ϕ^h, and trial pressure p^h, variables:

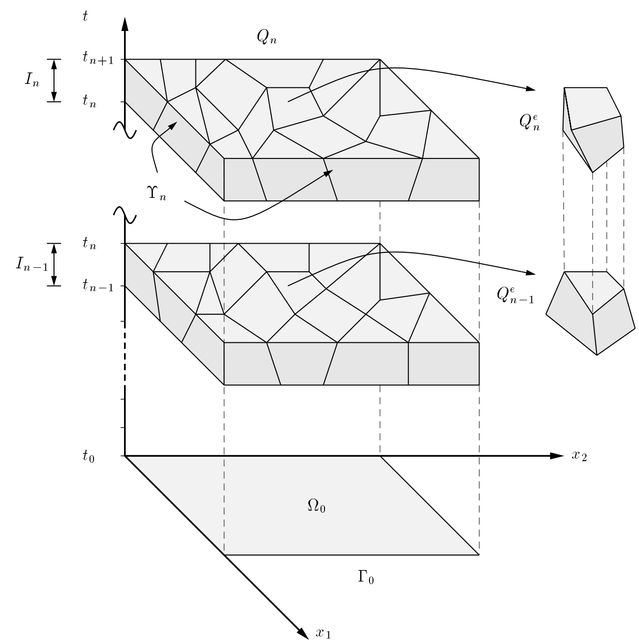

Figure 2. Illustration of two consecutive space-time slabs with unstructured finite element meshes within a slab.

Trial velocity potential:

$$T_1^h = \left\{ \phi^h \middle| \phi^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n^f\Big),\ \phi^h\Big|_{Q_n^{fe}} \in P^k(Q_n^{fe}) \right\}$$

Trial pressure:

$$T_2^h = \left\{ p^h \middle| p^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n^f\Big),\ p^h\Big|_{Q_n^{fe}} \in P^l(Q_n^{fe}) \right\}$$

where P^k denotes the space of kth-order polynomials and C^0 denotes the space of continuous functions. Similar collections of finite element basis functions for the approximation of independent structural displacements $\boldsymbol{u}_s^h$ and velocities $\boldsymbol{v}_s^h$ are defined. Independent solution variables allows for the use of first-order, $k = l = 1$, polynomials to be implemented in the multi-field space-time finite element method. An important component in the success of the space-time method is the incorporation of discontinuous temporal jump terms at each space-time slab interface; for a function w^h, a jump operator is defined as,

$$[\![w^h(t_n)]\!] = w^h(\boldsymbol{x}, t_n^+) - w^h(\boldsymbol{x}, t_n^-)$$

These jump operators are enforced in a weighted integral sense across time slabs and are crucial for establishing an unconditionalally stable algorithm for unstructured space-time finite element discretizations using high-order interpolations.

The strain-energy inner product for the structure is denoted by

$$a(\delta \boldsymbol{u}_s, \boldsymbol{u}_s)_{\Omega_s} = \int_{\Omega_s} \nabla \delta \boldsymbol{u}_s \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}_s)\, d\Omega \tag{9}$$

Standard L_2 inner products are defined as

$$(\delta \boldsymbol{u}, \boldsymbol{u})_{\Omega} = \int_{\Omega} \delta \boldsymbol{u} \cdot \boldsymbol{u}\, d\Omega \tag{10}$$

and equipped with norm $||\boldsymbol{u}||_{\Omega} = (\boldsymbol{u}, \boldsymbol{u})_{\Omega}^{1/2}$ A delta refers to the variation of the function, i.e. the corresponding weighting function.

A MULTI-FIELD SPACE-TIME VARIATIONAL EQUATION

A multi-field space-time variational equation for the exterior structural acoustics problem is obtained from a weighted residual of the governing equations and incorporates time-discontinuous jump terms. For efficiency the method is applied in one space-time slab at a time; data from the end of the previous slab are employed as initial conditions for the current slab. The statement of the time-discontinuous Galerkin method for the multi-field formulation is: Within each space-time slab, $n = 0, 1, \ldots, N-1$; the objective is to find $\boldsymbol{U}_f^h := \{\phi^h, p^h\} \in T_1^h \times T_2^h$ and $\boldsymbol{U}_s^h := \{\boldsymbol{u}_s^h, \boldsymbol{v}_s^h\} \in S_1^h \times S_2^h$, such that for all weighting functions $\delta \boldsymbol{U}_f^h := \{\delta\phi^h, \delta p\} \in T_1^h \times T_2^h$, and $\delta \boldsymbol{U}_s^h := \{\delta \boldsymbol{u}_s^h, \delta \boldsymbol{v}_s^h\} \in S_1^h \times S_2^h$, the following coupled variational equation is satisfied:

$$B_f(\delta \boldsymbol{U}_f^h, \boldsymbol{U}_f^h)_n + B_s(\delta \boldsymbol{U}_s^h, \boldsymbol{U}_s^h)_n + \left(\delta p^h, \boldsymbol{v}^h \cdot \boldsymbol{n}\right)_{(\Upsilon_\infty)_n}$$

$$= \left(\delta p^h, \boldsymbol{v}_s^h \cdot \boldsymbol{n}\right)_{(\Upsilon_i)_n} - \left(\delta \boldsymbol{v}_s^h \cdot \boldsymbol{n}, p^h\right)_{(\Upsilon_i)_n} \tag{11}$$

with

$$\begin{aligned} B_f(\delta \boldsymbol{U}_f^h, \boldsymbol{U}_f^h)_n := &\left(\delta p^h, K_f^{-1} \dot{p}^h\right)_{Q_n^f} - (\nabla \delta p^h, \boldsymbol{v}^h)_{Q_n^f} \\ &+ (\delta \boldsymbol{v}^h, L_f \boldsymbol{U}_f^h)_{\tilde{Q}_n^f} \\ &+ \left(\delta p^h(t_n^+), K_f^{-1} [\![p^h(t_n)]\!]\right)_{\Omega_f} \\ &+ \left(\delta \boldsymbol{v}^h(t_n^+), \rho_f [\![\boldsymbol{v}^h(t_n)]\!]\right)_{\Omega_f} \end{aligned} \tag{12}$$

$$\begin{aligned} B_s(\delta \boldsymbol{U}_s^h, \boldsymbol{U}_s^h)_n := &\left(\delta \boldsymbol{v}_s^h, \rho_s \dot{\boldsymbol{v}}_s^h\right)_{Q_n^s} + a(\delta \boldsymbol{v}_s^h, \boldsymbol{u}_s^h)_{Q_n^s} \\ &+ a(\delta \boldsymbol{u}_s^h, L_s \boldsymbol{U}_s^h)_{\tilde{Q}_n^s} \\ &+ \left(\delta \boldsymbol{v}_s^h(t_n^+), \rho_s [\![\boldsymbol{v}_s^h(t_n)]\!]\right)_{\Omega_s} \\ &+ a\left(\delta \boldsymbol{u}_s^h(t_n^+), [\![\boldsymbol{u}_s^h(t_n)]\!]\right)_{\Omega_s} \end{aligned} \tag{13}$$

in which $\boldsymbol{v}^h = \nabla\phi^h$, $\delta\boldsymbol{v}^h = \nabla\delta\phi^h$, and

$$L_f \boldsymbol{U}_f^h = \rho_f \dot{\boldsymbol{v}}^h + \nabla p^h, \quad \text{and} \quad L_s \boldsymbol{U}_s^h = \dot{\boldsymbol{u}}_s^h - \boldsymbol{v}_s^h$$

In the above expressions, a tilde refers to integration over element interiors. $B_f(\cdot,\cdot)_n$ and $B_s(\cdot,\cdot)_n$ are bilinear forms for the fluid and structure respectively. Fluid-structure interaction is accomplished through the coupling operators defined on the fluid-structure interface $(\Upsilon_i)_n := \Gamma_i \times I_n$. The exterior acoustic impedance is represented through the normal velocity defined on the fluid truncation boundary $(\Upsilon_\infty)_n : \Gamma_\infty \times I_n$.

The positive form of coupled variational equation follows directly from a stability (coercivity) result derived in (Thompson, 1995a):

$$\mathbb{E}(t_{n+1}^-) \le \mathbb{E}(t_n^-), \quad \forall\, \Delta t_n > 0, \tag{14}$$

and $n = 0, 1, \ldots N-1$. Eq. (14) states that the computed total energy for the system

$$\mathbb{E}(\boldsymbol{U}_f^h, \boldsymbol{U}_s^h) := E_f(\boldsymbol{U}_f^h) + E_s(\boldsymbol{U}_s^h) \tag{15}$$

$$E_s(\boldsymbol{U}_s^h) = \frac{1}{2}(\boldsymbol{v}_s^h, \rho_s \boldsymbol{v}_s^h)_{\Omega_s} + \frac{1}{2} a(\boldsymbol{u}_s^h, \boldsymbol{u}_s^h)_{\Omega_s} \tag{16}$$

$$E_f(\boldsymbol{U}_f^h) \quad = \quad \frac{1}{2}\,||K_f^{-1/2}\,p^h||^2_{\Omega_f} + \frac{1}{2}\,||\rho_f^{1/2}\boldsymbol{v}^h||^2_{\Omega_f} \qquad (17)$$

at the end of a time step, is always less than or equal to the total energy at the previous time step, for arbitrary step sizes. This result implies that *the space-time formulation presented is unconditionally stable*. Accuracy can be increased in both space and time by simply increasing the order of the polynomial used in the finite element approximation.

For additional stability, local residuals of the governing differential equations in the form of least-squares may be added to the discontinuous Galerkin variational equations. Stabilized methods of this type are referred to as Galerkin Least Squares (GLS) methods, and in the context of transient wave propagation, may be designed to provide numerical dissipation of unresolved high-frequencies without degrading the accuracy of the underlying Galerkin method. GLS methods have also been used to improve the accuracy for the related reduced wave equation (Helmholtz equation) governing time-harmonic acoustics in the frequency domain, (Harari, 1991; Thompson, 1995b).

If the finite element approximation for the acoustic pressure is selected such that, $p^h = -\rho_f\dot{\phi}^h$, which implies that the residual $L_f\boldsymbol{U}_f^h = \rho_f\nabla\dot{\phi}^h + \nabla p^h = 0$, and similarly if the structural velocity is the time derivative of the structural displacement, $\boldsymbol{v}_s^h = \dot{\boldsymbol{u}}_s^h$, then the multi-field formulation (11) specializes to the single-field formulation presented in (Thompson, 1996a). This simplification occurs when the temporal order of approximation for p^h and $\boldsymbol{v}_s^h$ is one order less than that used for ϕ^h and $\boldsymbol{u}_s^h$. For the single field formulation, quadratic interpolation in the time-dimension is required to resolve the second-order time derivatives appearing on ϕ^h and $\boldsymbol{u}_s^h$ in the simplified variational equation.

EXACT NON-REFLECTING BOUNDARY CONDITIONS

For large-scale simulations the use of high-order accurate radiation boundary conditions is essential to allow the fluid truncation boundary to be placed close to the scatterer and thereby minimizing the mesh and matrix problem size. An exact non-reflecting boundary condition may be obtained by taking advantage of the fact that an outgoing wave solution can be written as a series of wave harmonics with respect to a separable coordinate system. In the frequency domain, i.e., the time-harmonic problem, this idea has been exploited by several researchers to derive exact non-reflecting boundary conditions; see e.g. the Dirichlet-to-Neumann (DtN) impedance operator derived in (Keller, 1989). The DtN operator is a nonlocal (integral) and frequency dependent boundary condition applied on a separable boundary Γ_∞.

Let $\omega > 0$ be the circular frequency, so that $k = \omega/c$, is the acoustic wavenumber. For a spherical truncation boundary Γ_∞ of radius $r = R$ with circumferential angle $0 \leq \theta < 2\pi$, polar angle $0 \leq \varphi < \pi$, and unit outward normal $\boldsymbol{n}$, the exact representation of the exterior acoustic impedance restricted to Γ_∞ is,

$$\boldsymbol{v}(R,\theta,\varphi)\cdot\boldsymbol{n} = \sum_{n=0}^{N-1}\alpha_n(\hat{k})\int_{\Gamma_\infty} s_n(\theta,\varphi,\theta',\varphi')\;\phi(R,\theta',\varphi')\;d\Gamma' \qquad (18)$$

where the DtN kernels $s_n,\ n = 0,1,2,\cdots$ are given by,

$$s_n = {\sum_{j=0}^{n}}'\,\frac{(2n+1)(n-j)!}{2\pi R^2(n+j)!}P_n^j(\cos\varphi)P_n^j(\cos\varphi')\cos j(\theta-\theta') \qquad (19)$$

with impedance coefficients,

$$\alpha_n(\hat{k}) = \frac{kh_n'(\hat{k})}{h_n(\hat{k})} \qquad (20)$$

where $\hat{k} = kR$, is a nondimensional wavenumber, $d\Gamma = R^2\sin\varphi d\theta d\varphi$. The functions P_n^j are associated Legendre functions of the first kind, and h_n are spherical Hankel functions of the first kind of order n. The prime on h_n indicates differentiation with respect to its argument. The acoustic impedance (18) represents an exact boundary condition for the exterior problem when $N = \infty$. A direct time-dependent counterpart to (18) can be obtained through a convolution integral in time resulting in a boundary condition that is non-local in both space and time dimensions. Unfortunately implementation of such a boundary condition in a numerical method requires storage of all previous solutions up to the current time step; a property that makes its use impractical for large-scale computations over long time intervals. Note that this limitation of the time convoluted DtN operator is also shared with the Kirchoff boundary integral representation.

In order to circumvent the difficulty of having to implement a temporal convolution integral, time-dependent boundary conditions have been derived which replace the temporal integral with local temporal derivatives; (Thompson, 1996b). Two alternative sequences were derived; the first retains the nonlocal spatial integral of the DtN map (18), while replacing the time-convolution with higher-order local time derivatives (local in time and nonlocal in space version), while the second involves only time and spatial derivatives (local in time and local in space version).

The local in time and space version is obtained by first localizing the acoustic impedance relation (18) in the frequency domain, followed by an inverse Fourier transform. In (Thompson, 1996b) it is shown that when the solution on the boundary Γ_∞ contains only a finite number of spherical harmonics, then such a transformation gives an exact time-dependent counterpart which is local in both space $\boldsymbol{x}$ and time t. The transformation starts with the ideas of Givoli and Keller (Givoli, 1990), where a spatially local counterpart to the non-local *DtN* map was obtained in two-dimensions. The extension to three-dimensions was given by Harari (Harari, 1991). The development proceeds by recognizing that the spherical harmonics can be interpreted as eigenfunctions of the Laplace-Beltrami operator

$$\Delta_\Gamma := \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} \tag{21}$$

Using this result, the DtN map (18) can be written in the following localized form:

$$\boldsymbol{v}\cdot\boldsymbol{n} \quad = \quad \sum_{m=0}^{N-1}\beta_m(\hat{k})\,(-\Delta_\Gamma)^m\phi \qquad \text{on } \Gamma_\infty \tag{22}$$

where values of the coefficients $\beta_m(\hat{k})$ are obtained by solving the $N\times N$ linear algebraic system,

$$\sum_{m=0}^{N-1}[n(n+1)]^m\beta_m(\hat{k}) \quad = \quad \alpha_n(\hat{k}), \qquad n=0,1,\cdots,N-1 \tag{23}$$

Since this sequence follows directly from the truncated *DtN* map, these radiation boundary operators annilate the first N spherical harmonics for the outgoing solution on a spherical boundary Γ_∞.

Local time-dependent counterparts to (22) have been obtained in (Thompson, 1996b) using an inverse Fourier transform procedure by first solving (23) for the coefficients β_m in terms of the impedance coefficients α_n and then using recurrence relations for the spherical Hankel functions h_n to simplify the result. For example, for $N=2$, (23) can be solved for $\beta_0=\alpha_0$ and $2\beta_1=(\alpha_1-\alpha_0)$, leading to the result:

$$\boldsymbol{v}\cdot\boldsymbol{n} = \frac{R}{z_o}p_{,n}+\frac{R}{K_f}\dot{p}+\frac{2}{z_o}p-\frac{1}{2R}(2-\Delta_\Gamma)\phi \tag{24}$$

where $z_o=\rho_f c$. When applied on a spherical boundary Γ_∞, this operator acts as a high-order accurate local boundary condition which is perfectly absorbing for the first two spherical wave harmonics of orders $n=0$ and $n=1$. As the order N is increased, i.e. more terms are used in (22), the resulting boundary conditions match more terms in the harmonic expansion for outgoing waves, and a better approximation is obtained: see (Thompson, 1996b) for expressions for the time-dependent counterparts to (22) for $N\geq 3$. These boundary conditions are implemented in the multi-field space-time finite element formulation as natural boundary conditions, i.e., they are enforced weakly in both time and space. We note that the operator defined in (24) is identical to the second-order radiation boundary condition derived by Bayliss and Turkel in (Bayliss, 1980), after second-order radial derivatives are eliminated in favor of second-order tangential derivatives through use of the wave equation in spherical coordinates. Thus, while the boundary conditions derived by Bayliss and Turkel were obtained by annilating radial terms in a multipole expansion, it is seen, that in fact, the first two boundary conditions in the sequence share the property of the localized DtN, in that they match the first two spherical harmonics for outgoing waves on a spherical boundary Γ_∞. The operator (24) also coincides with the spherical absorbing boundary condition given in (Kallivokas, 1995). For higher-order boundary conditions in the sequence beyond $N\geq 3$, the form of the boundary conditions derived in (Thompson, 1996b) differ from those given in (Bayliss, 1980) and (Kallivokas, 1995). Because the time-discontinuous formulation allows for the use of $C^0(I_n)$ interpolations to represent the high-order time derivatives, it is possible to implement these sequences of time-dependent absorbing boundary conditions up to any order desired. However for higher-order operators extending beyond $N\geq 3$, the lowest possible order of spatial continuity on the artificial boundary that can be achieved after integration by parts is $C^{N-2}(\Gamma_\infty)$. For these high-order operators a layer of boundary elements adjacent to Γ_∞, possessing high-order tangential continuity are needed, see e.g. (Givoli, 1994).

Direct Implementation

The nonreflecting boundary condition (24) may be implemented *directly* in the space-time variational equation (11) by replacing the normal velocity on Γ_∞ with the differential operators found on the right-hand-side of (24). For the discontinuous Galerkin method, consistent jump terms are added together with weak enforcement of the momentum equations on Γ_∞. Using this approach, the boundary operator is defined as,

$$\begin{aligned}
\left(\delta p^h,\boldsymbol{v}^h\cdot\boldsymbol{n}\right)_{(\Upsilon_\infty)_n} = {} & \frac{R}{K_f}\left(\delta p^h,\dot{p}^h\right)_{(\Upsilon_\infty)_n} \\
& + d_1\left(\delta p^h,p^h\right)_{(\Upsilon_\infty)_n} \\
& - d_0\left(\delta p^h,\phi^h\right)_{(\Upsilon_\infty)_n} \\
& + d_0\left(\delta\phi^h,p^h+\rho_f\dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\
& + \frac{R}{K_f}\left(\delta p^h(t_n^+),[\![p^h(t_n)]\!]\right)_{\Gamma_\infty} \\
& + d_0\left(\delta\phi^h(t_n^+),\rho_f[\![\phi^h(t_n)]\!]\right)_{\Gamma_\infty}
\end{aligned} \tag{25}$$

where

$$d_0\left(\delta p^h, \phi^h\right)_{(\Upsilon_\infty)_n} := \frac{1}{R}\left(\delta p^h, \phi^h\right)_{(\Upsilon_\infty)_n} + \frac{R}{2}\left(\nabla_\Gamma \delta p^h, \nabla_\Gamma \phi^h\right)_{(\Upsilon_\infty)_n} \tag{26}$$

$$d_1\left(\delta p^h, p^h\right)_{(\Upsilon_\infty)_n} := \frac{2}{z_o}\left(\delta p^h, p^h\right)_{(\Upsilon_\infty)_n} + \frac{R}{z_o}\left(\delta p^h, p^h_{,r}\right)_{(\Upsilon_\infty)_n} \tag{27}$$

In the above, ∇_Γ is the tangential (surface) gradient defined by

$$R\nabla_\Gamma \phi := \phi_{,\varphi}\, \boldsymbol{e}_\varphi + \csc(\varphi)\, \phi_{,\theta}\, \boldsymbol{e}_\theta$$

Integration-by-parts has been used to relax the continuity implied by the second-order tangential (surface) derivatives appearing in Δ_Γ from $C^1(\Gamma_\infty)$ to $C^0(\Gamma_\infty)$, i.e.,

$$\begin{aligned}\left(\delta p, \Delta_\Gamma \phi\right)_{\Gamma_\infty} &= -R^2\left(\nabla_\Gamma \delta p, \nabla_\Gamma \phi\right)_{\Gamma_\infty} \\ &= -\left(\delta p_{,\varphi}, \phi_{,\varphi}\right)_{\Gamma_\infty} - \left(\delta p_{,\theta}, \csc^2(\varphi)\, \phi_{,\theta}\right)_{\Gamma_\infty}\end{aligned} \tag{28}$$

The consistent jump terms act to weakly enforce continuity of $\boldsymbol{U}_f^h$ between space-time-slabs on Γ_∞. When the finite element approximation for the acoustic pressure is selected such that, $p^h = -\rho_f \dot{\phi}^h$, (25) specializes to the implementation described in (Thompson, 1996b). Several numerical simulations involving acoustic radiation and scattering from structures using a direct implementation in terms of the acoustic velocity potential ϕ^h, are presented in (Thompson, 1996b; Thompson, 1996c; Thompson, 1996d). The results found in these papers demonstrate the effectiveness of the time-discontinuous Galerkin space-time finite element method to accurately model transient radiation and scattering from geometrically complex surfaces. The problems investigated also show the increase in accuracy obtained by using the second-order boundary condition (24) when compared to a standard first-order boundary condition.

Indirect Implementation

An implementation of the nonreflecting boundary condition (24) which avoids approximation of the radial derivative on Γ_∞, is also possible. This alternative formulation is based on the procedure used in (Kallivokas, 1991; Kallivokas, 1995) for the symmetrization of a second-order absorbing boundary condition in a standard semi-discrete finite element method. Dividing (24) by R and recognizing the hierarchical structure inherent in the boundary operator, Eqn. (24) can be slit in the form,

$$\frac{1}{R}\mathbb{B}_1\phi - \frac{1}{z_o}\mathbb{B}_1 p - \frac{1}{2R^2}\Delta_\Gamma \phi = 0 \tag{29}$$

where $\mathbb{B}_1$ is the first-order boundary operator defined by

$$\mathbb{B}_1\phi := \phi_{,n} + \frac{1}{R}\phi + \frac{1}{c}\dot{\phi} \tag{30}$$

By introducing an auxiliary variable ψ on Γ_∞, such that

$$\frac{1}{2R^2}\Delta_\Gamma \phi = \frac{1}{2R^2}\Delta_\Gamma\left(\frac{1}{R}\psi + \frac{1}{c}\dot{\psi}\right) \tag{31}$$

and enforcing $p = -\rho_f \dot{\phi}$, equation (29) takes on the form:

$$\frac{1}{R}\left(\mathbb{B}_1\phi - \frac{1}{2R^2}\Delta_\Gamma \psi\right) - \frac{1}{c}\left(\mathbb{B}_1\dot{\phi} - \frac{1}{2R^2}\Delta_\Gamma \dot{\psi}\right) = 0 \tag{32}$$

which is satisfied by,

$$\mathbb{B}_1\phi - \frac{1}{2R^2}\Delta_\Gamma \psi = 0 \tag{33}$$

By weakly enforcing (33) and (31), together with the momentum equations, and integrating-by-parts the terms containing tangential derivatives, the boundary operator appearing in the space-time variational equation (11) may be defined as,

$$\begin{aligned}\left(\delta p^h, \boldsymbol{v}^h \cdot \boldsymbol{n}\right)_{(\Upsilon_\infty)_n} = &-\frac{1}{R}\left(\delta p^h, \phi^h\right)_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{z_o}\left(\delta p^h, p^h\right)_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{R}\left(\delta \phi^h, p^h + \rho_f \dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{2}\left(\nabla_\Gamma \delta p^h, \nabla_\Gamma \psi^h\right)_{(\Upsilon_\infty)_n} \\ &- \frac{1}{2}\left(\nabla_\Gamma \delta \psi^h, \nabla_\Gamma p^h + \rho_f \nabla_\Gamma \dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\ &- \frac{1}{2}\left(\nabla_\Gamma \delta \dot{\psi}^h, \nabla_\Gamma \phi^h\right)_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{2R}\left(\nabla_\Gamma \delta \dot{\psi}^h, \nabla_\Gamma \psi^h\right)_{(\Upsilon_\infty)_n} \\ &+ \frac{1}{2}\left(\nabla_\Gamma \delta \dot{\psi}^h, \nabla_\Gamma \dot{\psi}^h\right)_{(\Upsilon_\infty)_n}\end{aligned} \tag{34}$$

where $\delta\psi^h$ is an auxiliary weighting function and ψ^h and $\dot{\psi}^h$ are required to vanish at $t = 0$.

If the finite element approximation for the acoustic pressure is selected such that, $p^h = -\rho_f \dot{\phi}^h$, the indirect formulation reduces to the single field formulation:

$$
\begin{aligned}
\left(\delta p^h , \boldsymbol{v}^h \cdot \boldsymbol{n}\right)_{(\Upsilon_\infty)_n} = &+\frac{\rho_f}{R}\left(\delta\dot{\phi}^h , \phi^h\right)_{(\Upsilon_\infty)_n} \\
&+ \frac{\rho_f}{c}\left(\delta\dot{\phi}^h , \dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\
&- \frac{\rho_f}{2}\left(\nabla_\Gamma \delta\dot{\phi}^h , \nabla_\Gamma \psi^h\right)_{(\Upsilon_\infty)_n} \\
&- \frac{\rho_f}{2}\left(\nabla_\Gamma \delta\dot{\psi}^h , \nabla_\Gamma \phi^h\right)_{(\Upsilon_\infty)_n} \\
&+ \frac{1}{2R}\left(\nabla_\Gamma \delta\dot{\psi}^h , \nabla_\Gamma \psi^h\right)_{(\Upsilon_\infty)_n} \\
&+ \frac{1}{2}\left(\nabla_\Gamma \delta\dot{\psi}^h , \nabla_\Gamma \dot{\psi}^h\right)_{(\Upsilon_\infty)_n}
\end{aligned}
\tag{35}
$$

For the discontinuous Galerkin method, consistent jump terms are added to these expressions. The advantage of this *indirect* implementation of the boundary condition (24) is that only spatially symmetric terms are present, thus avoiding the spatially unsymmetric operator (27) present in the direct implementation; albeit at the expense of solving for an additional auxiliary variable on Γ_∞.

GENERAL CONVEX BOUNDARIES

To accommodate highly elongated structures in an efficient way, non-reflecting boundary conditions may be generalized to other separable coordinates, e.g. elliptical coordinates for $\boldsymbol{x} \in \mathbb{R}^2$ and spheroidal coordinates for $\boldsymbol{x} \in \mathbb{R}^3$. In this case, the curvature of the non-reflecting boundary is no longer constant and will vary with position, and the derivatives must be taken with respect to the normal coordinate measured in the appropriate coordinate system. For example, consider the first-order boundary condition given in (Kallivokas, 1991; Kallivokas, 1995),

$$
\boldsymbol{v} \cdot \boldsymbol{n} - \alpha\phi - p/z_o = 0 \tag{36}
$$

In this equation, $\alpha = \kappa(\boldsymbol{x})/2$, where $\kappa(\boldsymbol{x})\,, \boldsymbol{x}$ on Γ_∞ is the variable curvature in $\mathbb{R}^2$, and $\alpha = H(\boldsymbol{x})$ is the mean curvature in $\mathbb{R}^3$. Using (36) applied to a general convex truncation boundary, the

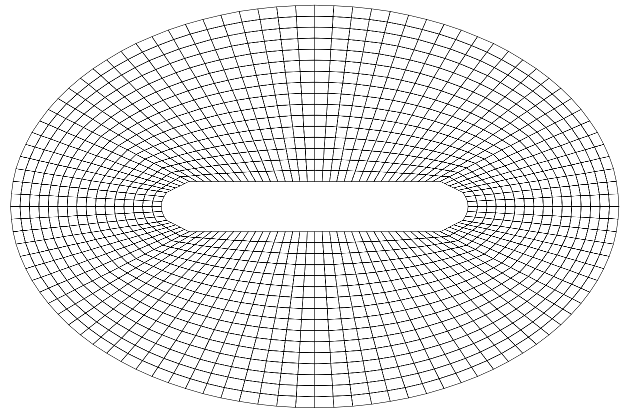

Figure 3. Spatial discretization for cylinder with conical to spherical ends enclosed by an elliptical boundary Γ_∞. (1600 quadratic elements)

multi-field, time-discontinuous boundary operator becomes,

$$
\begin{aligned}
\left(\delta p^h , \boldsymbol{v}^h \cdot \boldsymbol{n}\right)_{(\Upsilon_\infty)_n} = &\left(\delta p^h , \alpha\phi^h\right)_{(\Upsilon_\infty)_n} + \frac{1}{z_o}\left(\delta p^h , p^h\right)_{(\Upsilon_\infty)_n} \\
&- \left(\alpha\delta\phi^h , p^h + \rho_f \dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\
&- \left(\delta\phi^h(t_n^+) , \rho_f \kappa [\![\phi^h(t_n)]\!]\right)_{\Gamma_\infty}
\end{aligned}
\tag{37}
$$

Specializing to the single-field form, results in

$$
\begin{aligned}
\left(\delta p^h , \boldsymbol{v}^h \cdot \boldsymbol{n}\right)_{(\Upsilon_\infty)_n} = &-\rho_f\left(\delta\dot{\phi}^h , \alpha\phi^h\right)_{(\Upsilon_\infty)_n} \\
&+ \frac{\rho_f}{c}\left(\delta\dot{\phi}^h , \dot{\phi}^h\right)_{(\Upsilon_\infty)_n} \\
&- \left(\delta\phi^h(t_n^+) , \rho_f \alpha [\![\phi^h(t_n)]\!]\right)_{\Gamma_\infty}
\end{aligned}
\tag{38}
$$

For structures with a large aspect ratio, the use of a noncircular or nonspherical boundary Γ_∞, allows for a smaller problem domain Ω, with a corresponding increase in computational efficiency.

In Figure 4, numerical results are presented for the transient scattering from an infinite rigid cylinder, $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ_i, with conical-to-spherical end caps and a large length-to-diameter ratio, $L/d = 6.1$. This example illustrates the economy that can be achieved by using a non-reflecting boundary condition on a seperable Γ_∞ which is noncircular for structures with large aspect ratios. The driver for this problem is the source $f = \delta(x_0, y_0)\sin\omega t$ for the time interval $t \in [0,3]$, positioned inside the computational domain such that an incident wave strikes the scatterer at an oblique angle. The phase speed is set at $c = 1$ with frequency $\omega = \pi/3$. This example represents a challenging problem where the multiple-scales involving the ratio of the wavelength to cylinder diameter and cylinder length dimension play a critical role in the complexity of the resulting scattered wave field. For this example, quadratic interpolation is used for the approximation to ϕ^h and $p^h = -\rho_f \dot{\phi}^h$. Figure 3 illustrates the finite element spatial discretization of the computational domain. A total of 1600 space-time quadratic elements are used for this example. On the truncation boundary Γ_∞, the local first-order non-reflecting boundary condition (36) is applied via the variational equation (37) with the $\alpha = \kappa(\boldsymbol{x})/2$, and the curvature $\kappa(\boldsymbol{x})$ defined for an elliptical boundary $\boldsymbol{x}$ on Γ_∞. The use of a noncircular boundary Γ_∞, allows for a smaller problem domain Ω surrounding the elongated rigid scatterer, with a corresponding increase in computational efficiency. The numerical simulation starts with an initial pulse at $t = 3$. At $t = 6$ the incident pulse has expanded and has just reached the boundaries of the rigid cylinder. At the non-reflecting boundary Γ_∞, the wave front is absorbed through the boundary. As time progresses, the wave has begun to reflect off the rigid boundary, creating a complicated backscattered wave, that eventually passes through Γ_∞, leaving a quiscent solution in its wake. At $t = 9$ we observe some spurious reflection at the absorbing boundary Γ_∞ when using $\mathbb{B}_1$.

CONCLUSIONS

In this paper, a stable and high-order accurate space-time finite element method for solution of the transient acoustics problem in exterior domains has been summarized. Two alternative approaches were examined for implementing non-reflecting boundaries; *direct* implementation of the exterior acoustic impedance through a weighted variational equation in time and space, and *indirectly* through a decomposition into two equations involving an auxiliary variable defined on the non-reflecting boundary. Extensions of the formulation to general convex boundaries were also discussed. Numerical results for acoustic scattering using a first-order accurate operator applied to an elliptical boundary, demonstrated the economy that can be achieved by using a noncircular boundary surrounding an elongated structure. Using this first-order condition, we observed some spurious reflection at the absorbing boundary. While the accuracy achieved using the simple first-order boundary condition is sufficient to simulate the fundamental physics associated with the scattered field, there is some nonphysical reflection at the absorbing boundary. In order to elliminate this error in the solution, a higher-order boundary condition is needed. A derivation and formulation of a second-order absorbing boundary for general convex boundaries is given in (Kallivokas, 1991; Kallivokas, 1995), in the context of a standard semi-discrete finite element formulation. However, numerical results and use of this second-order condition in the time-domain, for general noncircular or nonspherical boundaries has not been reported in the open literature. Research efforts are underway to address the consistent and stable implementation of high-order boundary conditions for general convex boundaries, including elliptical and spheroidal boundaries, in a time-discontinuous Galerkin formulation.

The time-discontinuous Galerkin space-time method provides a natural variational setting for the incorporation of general high-order accurate non-reflecting boundary conditions possessing the property of being local in time. This is accomplished in the time-discontinuous formulation by allowing for the use of C^0 continuous finite element basis functions in time. Crucial to the unconditional stability and optimal convergence rates of the time-discontinuous Galerkin formulation is the introduction of consistent temporal jump operators across space-time slabs restricted to the nonreflecting boundary. The specific form of these operators are designed such that continuity of the solution across slabs is weakly enforced in a form consistent with the absorbing boundary conditions.

While the implementation presented in this paper have been limited to boundary conditions up to second-order, the time-discontinuous space-time finite element formulation is applicable to third and higher-order boundary conditions; research efforts are underway to address the implementational issues of the high-order boundary conditions. Third and higher-order boundary conditions involve fourth and higher-order time derivatives, which can be implemented efficiently by employing space-time elements with fourth and higher-order temporal interpolation on the face adjacent to the radiation boundary Γ_∞ and low-order temporal interpolation on the other faces. It remains to be seen what (if any) additional advantage in terms of accuracy and economy can be achieved by the implementation of the high-order operators beyond second-order.

ACKNOWLEDGMENT

The support from a CU Research Grant, and a NASA, SC Space Grant Research Enhancement Award is gratefully acknowledged.

REFERENCES

Bayliss, A., and Turkel, E., 1980, ‘Radiation Boundary Conditions for Wave-Like Equations,’ *Commun. Pure Appl. Math*, Vol. 33, pp. 707-725.

Givoli, D., and Keller, J.B., 1990, ‘Non-Reflecting Boundary Conditions for Elastic Waves,’ *Wave Motion*, Vol. 12, pp. 261-279.

Givoli, D., and Keller, J.B., 1994, ‘Special Finite Elements for use with High-Order Boundary Conditions,’ *Comp. Methods in Applied Mech. Engng.*, Vol. 119, pp. 199-213.

Hughes, T.J.R., and Hulbert, G.M., 1988, ‘Space-Time Finite Element Methods for Elastodynamics: Formulations and Error Estimates,’ *Comp. Methods in Applied Mech. Engng.*, Vol. 66, pp. 339-363.

Kallivokas, L.F., and Bielak, J., and MacCamy, R.C., 1991, 'Symmetric Local Absorbing Boundaries in Time and Space,' *ASCE J. Eng. Mech.*, Vol. 117, pp. 2027-2048.

Kallivokas, L.F., and Bielak, J., 1995, 'Transient and Time-Harmonic Infinite Elements for Near-Surface Computations of Three-Dimensional Structures Submerged in a Half-Space,' in *Proceedings of the Symposium on Acoustics of Submerged Structures and Transduction Systems*, 1995 Design Engineering Technical Conferences, Acoustics, Vibrations, and Rotating Machines, DE-Vol. 84-2, Vol. 3, Part B, pp. 129-140. ASME 15th Biennial Conference on Mechanical Vibration and Noise, Boston, Massachusetts, September 1995.

Keller, J.B., and Givoli, D., 1989, 'Exact Non-Reflecting Boundary Conditions,' *J. Comput. Phys.*, Vol. 82(1), pp. 172-192.

Thompson, L.L., and Pinsky, P.M., 1996, 'A Space-Time Finite Element Method for Structural Acoustics in Infinite Domains, Part I: Formulation, Stability, and Convergence,' *Comp. Methods in Applied Mech. Engng.*, Vol. 132, pp. 195-227.

Thompson, L.L., and Pinsky, P.M., 1996, 'A Space-Time Finite Element Method for Structural Acoustics in Infinite Domains, Part II: Exact Time-Dependent Non-Reflecting Boundary Conditions,' *Comp. Methods in Applied Mech. Engng.*, Vol. 132, pp. 229-258.

Thompson, L.L., and Pinsky, P.M., 1996, 'A Space-Time Finite Element Method for the Exterior Structural Acoustics Problem: Time-Dependent Radiation Boundary Conditions in Two Spatial Dimensions,' *Int. J. Numer. Methods Engng.*, Vol. 39, pp. 1635-1657.

Thompson, L.L., and Pinsky, P.M., 1996, 'A Space-Time Finite Element Method for the Exterior Acoustics Problem,' *J. Acoust. Soc. Am.*, Vol. 99(6), pp. 3297-3311.

Thompson, L.L., and Pinsky, P.M., 1995, 'A Multi-Field Space-Time Finite Element Method for Structural Acoustics,' in *Proceedings of the Symposium on Acoustics of Submerged Structures and Transduction Systems*, 1995 Design Engineering Technical Conferences, Acoustics, Vibrations, and Rotating Machines, DE-Vol. 84-2, Vol. 3, Part B, pp. pp. 44-64. ASME 15th Biennial Conference on Mechanical Vibration and Noise, Boston, Massachusetts, September 1995.

Harari, I., 1991, 'Computational Methods for Problems of Acoustics with Particular Reference to Exterior Domains,' *PhD Thesis, Stanford University*.

Thompson, L.L., and Pinsky, P.M., 1995, 'A Galerkin Least Squares Finite Element Method for the Two-Dimensional Helmholtz Equation,' *Int. J. Numer. Methods Engng.*, Vol. 38, pp. 371-397.

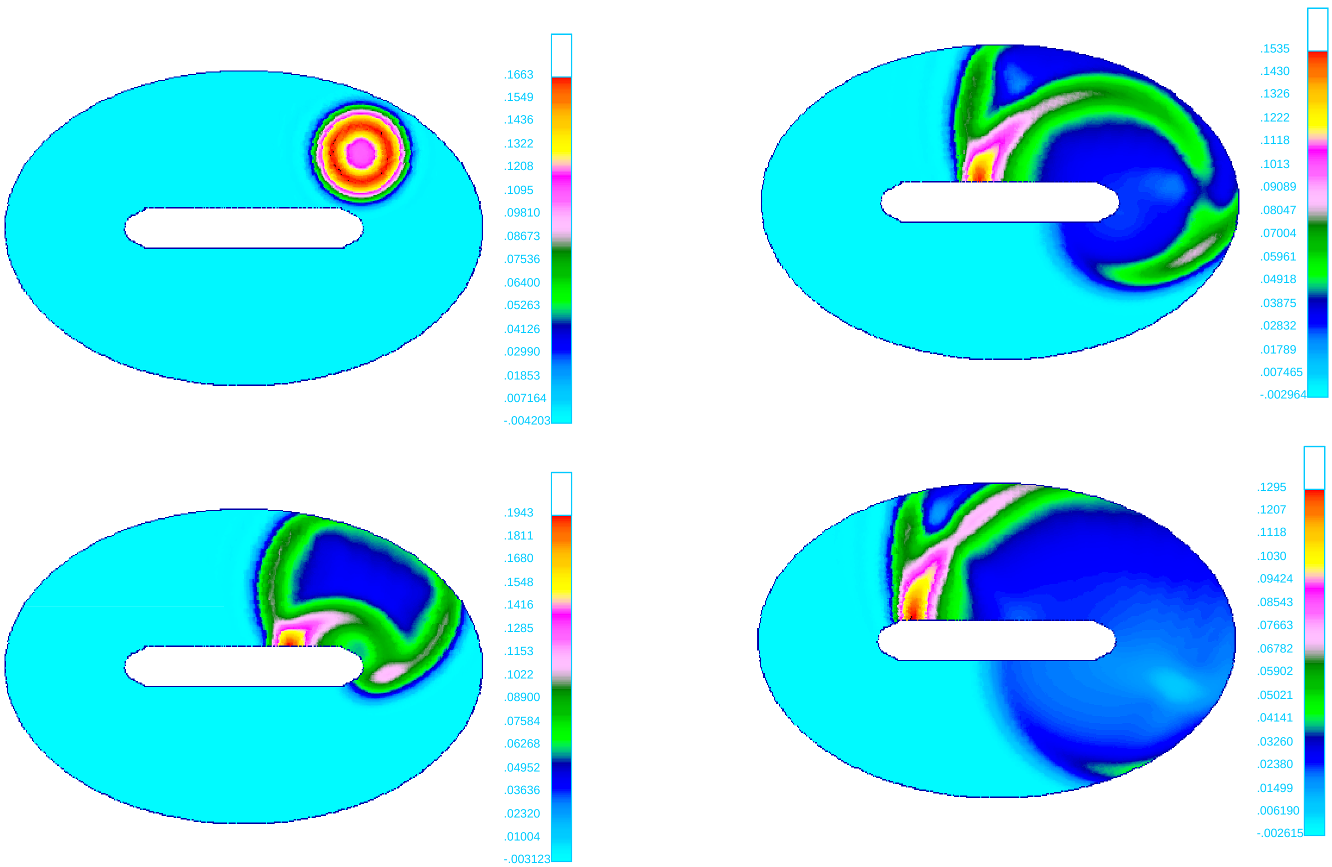

Figure 4. Space-time finite element solution of transient scattering from an elongated structure with a first-order accurate non-reflecting boundary condition $\mathbb{B}_1$ applied to an elliptical truncation boundary. Solution contours are shown at snapshots in time for $T = 6, 9, 12, 15$.