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FINITE ELEMENT FORMULATION OF EXACT DIRICHLET-TO-NEUMANN RADIATION CONDITIONS ON ELLIPTIC AND SPHEROIDAL BOUNDARIES

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ABSTRACT

Exact Dirichlet-to-Neumann (DtN) radiation boundary conditions are derived in elliptic and spheroidal coordinates and formulated in a finite element method for the Helmholtz equation in unbounded domains. The DtN map matches the first N wave harmonics exactly at the artificial boundary. The use of elliptic and spheroidal boundaries enables the efficient solution of scattering from elongated objects in two- and three- dimensions respectively. Modified DtN conditions based on first and second order local boundary operators are also derived in elliptic and spheroidal coordinates, in a form suitable for finite element implementation. The modified DtN conditions are more accurate than the DtN boundary condition, yet require no extra memory and little extra cost. Direct implementation involves non-local spatial integrals leading to a dense, fully populated submatrix. A matrix-free interpretation of the non-local DtN map for elliptic and spheroidal boundaries, suitable for iterative solution of the resulting complex-symmetric system is described. For both the DtN and modified DtN conditions, we describe efficient and effective SSOR preconditioners with Eisenstat's trick based on the matrix partition associated with the interior mesh and local boundary operator. Numerical examples of scattering from elliptic and spheroidal boundaries are computed to demonstrate the efficiency and accuracy of the boundary treatments for elongated structures.

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INTRODUCTION

When modeling radiation and scattering from structures in a medium which extends to infinity with a domain based computational method such as the finite element method, the far-field is truncated at an artificial boundary surrounding the source of radiation. The impedance of the far-field is then represented on this boundary by either radiation boundary conditions, infinite elements, or absorbing sponge layers. Survey articles of various boundary treatments are given in (Tsynkov, 1998). If accurate boundary treatments are used, the finite computational region can be reduced so that the truncation boundary is relatively close to the radiator, and fewer elements than otherwise would be possible may be used, resulting in considerable savings in both cpu time and memory.

For time-harmonic scattering governed by the Helmholtz equation, several accurate and efficient methods for representing the impedance of the far-field are well understood, including the Dirichlet-to-Neumann (DtN) map on a circular or spherical boundary (Pearson, 1989; Keller, 1989; Harari, 1992), and infinite elements (Burnett, 1994; Astley, 1998). The DtN map relates Dirichlet to Neumann data and matches the first N wave harmonics exactly at the artificial boundary. DtN conditions in cylindrical and spherical coordinates are derived in (Pearson, 1989; Keller, 1989). In (Thompson, 1994), and independently in (Grote, 1995), exact DtN radiation conditions were first constructed for elliptic and spheroidal boundaries. The use of elliptic and spheroidal boundaries enables the efficient solution of scattering from elongated objects in two- and three- dimensions respectively. Finite difference implementations of both the DtN and modified DtN based on first and second order operators which annihilate radial terms in a generalized multipole expansion are given in (Grote, 1995). Numerical results using the DtN condition in the finite element method on elliptical boundaries are reported in (Ben-Porat, 1995).

In this paper, we derive modified DtN conditions based on first and second order local radiation boundary conditions for elliptic and spheroidal boundaries in a form suitable for finite element implementation using standard C^0 regularity on the radiation boundary. The modified DtN conditions are more accurate than the DtN boundary condition, yet require no extra memory and little extra cost. Direct implementation in the finite element method involves non-local spatial integrals leading to a dense, fully populated submatrix. When the problem size is large, the computational cost associated with the storage and factorization becomes expensive. A matrix-free interpretation of the non-local DtN map for elliptic and spheroidal boundaries, suitable for iterative solution is described based on extensions of the procedures given in (Malhotra, 1996; Oberai, 1998) for circular and spherical DtN maps. For both the DtN and modified DtN conditions for elliptic and spheroidal boundaries, we show how the SSOR preconditioner with Eisenstat's trick based on the matrix partition associated with the discretization of the interior mesh and local boundary operator provides an efficient and effective preconditioner for the resulting complex-symmetric system. Numerical examples of scattering from elliptic and spheroidal objects are computed and compared to analytical solutions to demonstrate the efficiency and accuracy of the boundary treatments for elongated structures.

THE EXTERIOR BOUNDARY-VALUE PROBLEM

We consider time-harmonic scattering and radiation of waves in an infinite $d=2$ or $d=3$ -dimensional region $\mathcal{R} \subset R^d$, surrounding an object with surface S. For computation, the unbounded region $\mathcal{R}$ is truncated by an artificial boundary Γ. We assume that Γ is a surface defined by separable coordinates, i.e., an ellipse in two-dimensions ($d=2$), or a prolate spheroid in three-dimensions ($d=3$). We then denote by $\Omega \subset \mathcal{R}$, the finite subdomain bounded by $\partial\Omega = \Gamma \cup S$, see Figure 1.

Within Ω, the solution $\phi(\boldsymbol{x}) : \Omega \mapsto C$, satisfies the Helmholtz equation,

$$\nabla^2\phi + k^2\phi = -f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \tag{1}$$

subject to an impedance condition on the surface S:

$$\beta\frac{\partial\phi}{\partial n} + \gamma\phi = g(\boldsymbol{x}), \qquad \boldsymbol{x} \in S \tag{2}$$

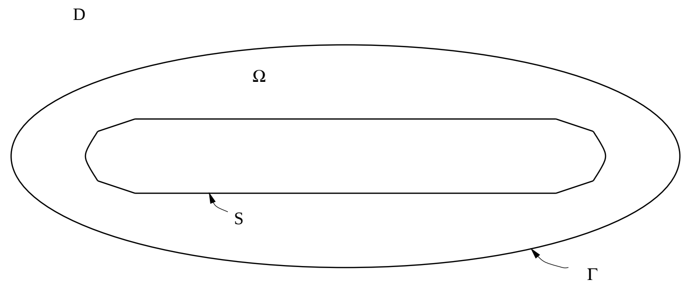

Figure 1. ILLUSTRATION OF UNBOUNDED REGION $\mathcal{R} \subset R^d$ SURROUNDING A SCATTERER S. THE COMPUTATIONAL DOMAIN $\Omega \subset \mathcal{R}$ IS SURROUNDED BY AN ELLIPTIC OR SPHEROIDAL BOUNDARY Γ WITH EXTERIOR REGION $\mathcal{D} = \mathcal{R} - \Omega$.

and supplemented by a nonreflecting boundary condition

$$\frac{\partial\phi}{\partial n} = M(\phi), \quad \boldsymbol{x} \in \Gamma \tag{3}$$

The operator M exactly represents the exterior impedance on the boundary Γ, such that the solution satisfies the Sommerfeld radiation condition at infinity.

Here k is the wavenumber, and β, γ and g are functions defined on S. The source f is assumed to be confined to the computational domain Ω, so that in the exterior region $\mathcal{D} = \mathcal{R} - \Omega$, i.e., the infinite region outside Γ, the scalar field $\phi(\boldsymbol{x})$ satisfies the homogeneous form of the Helmholtz equation,

$$\nabla^2\phi + k^2\phi = 0, \qquad \boldsymbol{x} \in \mathcal{D} \tag{4}$$

The DtN on Prolate Spheroidal Boundaries

In three-dimensions, we introduce prolate spheroidal coordinates $\boldsymbol{x} = \boldsymbol{x}(\mu, \theta, \varphi)$, $0 \le \theta < \pi$, and $0 \le \varphi < 2\pi$, such that

$$x = b\sin\theta\cos\varphi \tag{5}$$
$$y = b\sin\theta\sin\varphi \tag{6}$$
$$z = a\cos\theta \tag{7}$$

$$a = f\cosh\mu, \qquad b = f\sinh\mu \tag{8}$$

where a and b are the semimajor and semiminor axis of an ellipse respectively, and $f = \sqrt{a^2 - b^2}$ is the semi-interfocal distance. The spheroid is defined by a constant value of μ, with an ellipse revolving around the major z-axis. Alternatively, the spheroid may be parameterized by $\boldsymbol{x} = \boldsymbol{x}(\xi, \eta, \varphi)$, where $\xi = \cosh\mu$, and $\eta = \cos\theta$, so that $a = f\xi$, and $b = f\sqrt{\xi^2 - 1}$.

The metrics for a prolate coordinate system are given by,

$$h_\xi = f\sqrt{(\xi^2-\eta^2)/(\xi^2-1)} \tag{9}$$
$$h_\eta = f\sqrt{(\xi^2-\eta^2)/(1-\eta^2)} \tag{10}$$
$$h_\varphi = f\sqrt{(1-\eta^2)(\xi^2-1)} \tag{11}$$

In prolate spheroidal coordinates the Helmholtz equation (4) may be written as,

$$\left[f(\xi^2-1)\frac{\partial^2}{\partial\xi^2} + 2f\xi\frac{\partial}{\partial\xi} + \Delta_\Gamma + k^2 h_\xi h_\eta h_\varphi\right]\phi = 0 \tag{12}$$

where

$$\Delta_\Gamma\,\phi := \frac{\partial}{\partial\eta}\left[\frac{h_\xi h_\varphi}{h_\eta}\frac{\partial\phi}{\partial\eta}\right] + \frac{h_\xi h_\eta}{h_\varphi}\frac{\partial^2\phi}{\partial\varphi^2} \tag{13}$$

is the surface Laplacian, and

$$f(\xi^2-1) = \frac{h_\eta h_\varphi}{h_\xi}$$

We choose the artificial boundary Γ to be a prolate spheroid defined by a constant radial coordinate $\xi_0 = \cosh\mu_0$. The solution to the exterior radiation problem in the region $\xi \geq \xi_o = \cosh\mu_o$ can be expressed as an expansion in terms of orthogonal eigenfunctions:

$$\phi(\xi,\eta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \frac{R_{mn}(c,\xi)}{R_{mn}(c,\xi_0)}\,\{a_{mn}\,\psi^c_{mn}(c,\eta,\varphi) + b_{mn}\,\psi^s_{mn}(c,\eta,\varphi)\} \tag{14}$$

where $c = kf$ is a normalized wavenumber, and $R_{mn}(c,\xi)$ are the radial prolate spheroidal wave functions of the third kind (Abramowitz,1968; Flammer, 1957). In (14), the prime after the sum indicates that terms with $m = 0$ are multiplied by 1/2.

The orthogonal eigenfunctions are defined by,

$$\psi^c_{mn}(c,\eta,\varphi) := \frac{1}{\sqrt{\pi N_{mn}}}S_{mn}(c,\eta)\cos m\varphi \tag{15}$$
$$\psi^s_{mn}(c,\eta,\varphi) \;:\; \frac{1}{\sqrt{\pi N_{mn}}}S_{mn}(c,\eta)\sin m\varphi \tag{16}$$

where $S_{mn}(c,\xi)$ are the angular prolate spheroidal wave functions of the first kind, and

$$a_{mn}(c) = \int_{-\pi}^{\pi}\int_{-1}^{1}\phi(\xi_0,\eta,\varphi)\,\psi^c_{mn}(c,\eta,\varphi)\;d\eta d\varphi \tag{17}$$

$$b_{mn}(c) = \int_{-\pi}^{\pi}\int_{-1}^{1}\phi(\xi_0,\eta,\varphi)\,\psi^s_{mn}(c,\eta,\varphi)\;d\eta d\varphi \tag{18}$$

In the above we have used the standard normalization used by Flammer (Flammer, 1957):

$$N_{mn} = 2\sum_{l=0,1}^{\infty}{}' \frac{(l+2m)!}{(2l+2m+1)l!}(d_l^{mn})^2 \tag{19}$$

where $d_l^{mn}(c)$ are the coefficients in the expansion for the angular functions $S_{mn}(c,\eta)$ in terms of associated Legendre functions. In (19), the prime on the summation sign indicates that the summation is carried out for even l when $(n-m)$ is even and for odd l when $(n-m)$ is odd.

To derive the DtN map relating Dirichlet data to a normal derivative on Γ, we simply differentiate (14) with respect to ξ evaluated at $\xi = \xi_0 = \cosh\mu_0$, and use the definition for normal derivatives,

$$\frac{\partial\phi}{\partial n} = \frac{1}{h_\xi}\frac{\partial\phi}{\partial\xi} \tag{20}$$

The result is:

$$\frac{\partial\phi}{\partial n}(\xi_0,\eta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' Z^{(0)}_{mn}(c)\,\frac{1}{J_s}\,D_{mn}(c,\eta,\varphi) \tag{21}$$

$$D_{mn}(c,\eta,\varphi) := \int_{\Gamma'}\frac{1}{J_s'}\,\phi(\xi_0,\eta',\varphi')\,\psi_{mn}(\eta,\varphi\,|\,\eta',\varphi')\;d\Gamma' \tag{22}$$

$$\psi_{mn}(\eta,\varphi\,|\,\eta',\varphi') := \frac{1}{\pi N_{mn}}S_{mn}(c,\eta)\,S_{mn}(c,\eta')\cos m(\varphi-\varphi') \tag{23}$$

$$Z^{(0)}_{mn}(c) = f(\xi_0^2-1)\frac{R'_{mn}(c,\xi_0)}{R_{mn}(c,\xi_0)} \tag{24}$$

In the above, $d\Gamma = J_s\,d\eta\,d\varphi$, where $J_s = h_\eta\,h_\varphi$ is the surface jacobian evaluated at $\xi_0 = \cosh\mu_0$. This DtN condition was first-derived in (Thompson, 1994), and independently in (Grote, 1995); in the later, a different normalization factor was used to scale the angular spheroidal functions, and the condition is left in terms of a radial derivative with respect to μ. In practice the sum over n, is truncated at a finite value N. For fixed N, the harmonics $n > N$ are evaluated with a homogeneous Neumann boundary condition on Γ. As a result, the accuracy of the harmonics

$n > N$ are poorly represented and a minimum value $N_{min} \geq c\xi$ is required to ensure uniqueness (see (Harari, 1992) for values of N_{min} for spherical coordinates).

To eliminate the bound for N_{min}, a modified DtN may be formulated by generalizing the normal derivative applied to the harmonic expansion for outgoing waves (14), with a local differential operator representing an approximate radiation boundary condition (Grote, 1995). The resulting modified DtN condition is unique for any choice of N, and approximates the harmonics $n > N$ with greater accuracy than the original DtN condition.

Local boundary conditions are easily constructed by extending the procedures employed in (Bayliss, 1982) for a circle or sphere, where radial terms in a multipole expansion for outgoing waves are annihilated. The generalization to spheroidal coordinates is given by the asymptotic expansion given by (Holford; Burnett, 1994):

$$\phi \sim \frac{\exp(ic\xi)}{c\xi} \sum_{j=0}^{\infty} \frac{g_j(\theta, \varphi; c)}{(c\xi)^j} \tag{25}$$

where $c = kf$, is the normalized wavenumber and ξ is the radial coordinate in spheroidal coordinates.

Here a sequence of local operators which annihilate radial terms in the expansion (25) is constructed as a product of normalized radial derivatives:

$$B_j = L_j(L_{j-1}(\cdots(L_2(L_1)))) \tag{26}$$

$$L_j = \frac{1}{f}\left(\frac{\partial}{\partial \xi} - ic + \frac{2j-1}{\xi}\right) \tag{27}$$

such that $B_j\phi = O([c\xi]^{-2j-1})$. Setting the remainder equal to zero defines the analogue of the boundary conditions derived in (Bayliss, 1982) for a sphere. The first two boundary conditions are:

$$B_1\,\phi = \frac{1}{f}\left(\frac{\partial}{\partial \xi} + \alpha_1\right)\phi = 0, \qquad \text{on } \Gamma \tag{28}$$

$$B_2\,\phi = \frac{1}{f^2}\left(\frac{\partial^2}{\partial \xi^2} + \alpha_2\frac{\partial}{\partial \xi} + \alpha_3\right)\phi = 0, \qquad \text{on } \Gamma \tag{29}$$

where

$$\begin{aligned}
\alpha_1 &= (1 - ic\xi_0)/\xi_0 \\
\alpha_2 &= (4 - 2ic\xi_0)/\xi_0 \\
\alpha_3 &= \left(2 - 4ic\xi_0 - (c\xi_0)^2\right)/\xi_0^2
\end{aligned}$$

Applying the B_1 operator to the expansion (14), evaluated at ξ_0, gives,

$$B_1[\phi]|_{\xi=\xi_0} \;=\; \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \, \frac{B_1[R_{mn}(c,\xi_0)]}{R_{mn}(c,\xi_0)} D_{mn}(c,\eta,\varphi) \tag{30}$$

where

$$B_1[R_{mn}(c,\xi_0)] = \frac{1}{f}\left(R'_{mn} + \alpha_1 R_{mn}\right) \tag{31}$$

Then dividing both sides of (30) by h_ξ, and rearranging gives the modified DtN condition in terms of the normal derivative on Γ:

$$\frac{\partial\phi}{\partial n} = -\frac{1}{J_s} z_1 \phi + \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \, Z_{mn}^{(1)}(c)\frac{1}{J_s} D_{mn}(c,\eta,\varphi) \tag{32}$$

$$Z_{mn}^{(1)}(c) = Z_{mn}^{(0)}(c) + z_1 \tag{33}$$

with constant

$$z_1 = f(\xi_0^2 - 1)\alpha_1 \tag{34}$$

When the condition (32) is truncated at the finite value N, it is exact for harmonics $n \leq N$, and approximates the harmonics $n > N$ with the local approximate condition $B_1\phi = 0$. The normal derivative form derived in (32) is convenient for finite element implementation as a 'natural' boundary condition in a Galerkin variational equation. The modified condition given in (Grote, 1995) is left in terms of a derivative with respect to μ, and is suitable for finite difference implementation.

The second-order operator B_2 provides a more accurate boundary condition. To derive the second modified DtN condition, we apply the B_2 operator defined in (29) to the expansion (14), evaluated at ξ_0, with the result,

$$B_2[\phi]|_{\xi=\xi_0} \;=\; \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \, \frac{B_2[R_{mn}(c,\xi_0)]}{R_{mn}(c,\xi_0)} D_{mn}(c,\eta,\varphi) \tag{35}$$

The B_2 operator acting on the radial functions $R_{mn}(c,\xi)$ gives,

$$B_2[R_{mn}(c,\xi_0)] = \frac{1}{f^2}\left(R''_{mn} + \alpha_2 R'_{mn} + \alpha_3 R_{mn}\right) \tag{36}$$

The functions $R_{mn}(c,\xi)$ satisfy the radial equation,

$$(\xi^2-1)R''_{mn}+2\xi R'_{mn} = \left[\lambda_{mn}(c)-(c\xi)^2+\frac{m^2}{\xi^2-1}\right]R_{mn} \quad (37)$$

In the above, $\lambda_{mn}(c)$ are the characteristic values of the prolate spheroidal wave functions.

Replacing the second-order derivatives R''_{mn} appearing in (36), using (37) at $\xi=\xi_0$, we have

$$B_2[\phi]|_{\xi=\xi_0} = \frac{\nu}{f^3(\xi_0^2-1)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' Z_{mn}^{(2)}(c)\,D_{mn}(c,\eta,\varphi) \quad (38)$$

where

$$Z_{mn}^{(2)}(c)=Z_{mn}^{(0)}(c)+\frac{f}{\nu}\left[\lambda_{mn}+\frac{m^2}{\xi_0^2-1}+z_2\right] \quad (39)$$

$$\nu=\alpha_2-\frac{2\xi_0}{\xi_0^2-1} \quad (40)$$

$$z_2=(\xi_0^2-1)\alpha_3-c^2\xi_0^2 \quad (41)$$

For ease of finite element implementation, we eliminate the second-order radial derivative in the B_2 operator defined in (29) in favor of tangential derivatives using the Helmholtz equation (12), with the result:

$$B_2[\phi]|_{\xi=\xi_0}=\frac{1}{f^2}\left[\nu\frac{\partial\phi}{\partial\xi}+\left(\alpha_3-k^2h_\xi^2\right)\phi-\frac{1}{f(\xi_0^2-1)}\Delta_\Gamma\phi\right] \quad (42)$$

Using (42) in (38), dividing both sides by $\nu h_\xi/f^2$, and rearranging gives the second modified DtN condition,

$$\begin{aligned}\frac{\partial\phi}{\partial n} &= \frac{1}{\nu J_s}\left(\Delta_\Gamma-fc^2\eta^2-fz_2\right)\phi\\ &+\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' Z_{mn}^{(2)}(c)\frac{1}{J_s}D_{mn}(c,\eta,\varphi)\end{aligned} \quad (43)$$

The normal derivative form and second-order tangential derivatives appearing in (43) are easily implemented with C^0 regularity in a standard Galerkin variational equation using integration-by-parts on the boundary Γ. The form of the modified DtN condition (43) is different than the condition given in (Grote, 1995). In (Grote, 1995), the B_2 operator is left with second-order derivatives in the radial coordinate μ, suitable for finite difference approximation.

The DtN on Elliptic Boundaries

For an elliptic artificial boundary Γ in two-dimensions, we introduce elliptic coordinates $\boldsymbol{x}=\boldsymbol{x}(\mu,\theta)$, where μ,θ are related to rectangular Cartesian coordinates x,y by

$$x=a\cos\theta, \qquad y=b\sin\theta \quad (44)$$

where a,b are the semimajor and semiminor axis defined by (8). For a constant value of μ, and $0\le\theta<2\pi$, x and y describe a confocal ellipse. The metrics for the elliptic system are given by,

$$h_\mu = h_\theta = f\sqrt{\sinh^2\mu+\sin^2\theta} \quad (45)$$

With the artificial boundary set at $\mu=\mu_0$, the solution to the Helmholtz equation in the exterior region $\mu\ge\mu_0$ may be expressed as,

$$\begin{aligned}\phi(\mu,\theta) &= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Mc_n(\mu,q)}{Mc_n(\mu_0,q)}ce_n(\theta,q)\int_0^{2\pi}\phi(\mu_0,\theta')ce_n(\theta',q)\,d\theta'\\ &+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{Ms_n(\mu,q)}{Ms_n(\mu_0,q)}se_n(\theta,q)\int_0^{2\pi}\phi(\mu_0,\theta')se_n(\theta',q)\,d\theta'\end{aligned} \quad (46)$$

In the above, $q=(kf/2)^2$ is a normalized wavenumber, ce_n and se_n represent the angular Mathieu functions, and Mc_n and Ms_n are the even and odd modified (radial) Mathieu functions of the third kind, respectively (Abramowitz,1968; McLachlan, 1947). The radial functions Mc_n and Ms_n satisfy the modified Mathieu's equation,

$$\frac{d^2y}{d\mu^2}-(\lambda-2q\cosh 2\mu)y=0 \quad (47)$$

where $\lambda(q)$ is the separation constant (characteristic value) for the Mathieu functions.

To derive the DtN map relating normal derivatives to Dirichlet data, we simply differentiate (46) with respect to μ evaluated at $\mu=\mu_0$, and use the relation,

$$\frac{\partial\phi}{\partial n}=\frac{1}{h_\mu}\frac{\partial\phi}{\partial\mu}=\frac{1}{h_\theta}\frac{\partial\phi}{\partial\mu} \quad (48)$$

The result is,

$$\frac{\partial\phi}{\partial n}(\mu_0,\theta)=\sum_{n=0}^{\infty}Zc_n^{(0)}(q)\,D_n^c(\theta,q)+\sum_{n=1}^{\infty}Zs_n^{(0)}(q)\,D_n^s(\theta,q) \quad (49)$$

where

$$Zc_n^{(0)}(q) = \frac{Mc_n'(\mu_0,q)}{Mc_n(\mu_0,q)}\ , \quad Zs_n^{(0)}(q) = \frac{Ms_n'(\mu_0,q)}{Ms_n(\mu_0,q)} \tag{50}$$

$$D_n^c(\theta,q) \;=\; \frac{1}{\pi h_\theta} ce_n(\theta,q)\int_{\Gamma'} \frac{1}{h_{\theta'}}\phi(\mu_0,\theta')ce_n(\theta',q)\,d\Gamma' \tag{51}$$

$$D_n^s(\theta,q) \;=\; \frac{1}{\pi h_\theta} se_n(\theta,q)\int_{\Gamma'} \frac{1}{h_{\theta'}}\phi(\mu_0,\theta')se_n(\theta',q)\,d\Gamma' \tag{52}$$

where $d\Gamma = h_\theta d\theta$.

This DtN condition was first derived in (Thompson, 1994), and independently in (Ben-Porat, 1995) and (Grote, 1995); in the later, the condition is left in terms of a radial derivative with respect to μ, suitable for finite difference implementations. Numerical results using the DtN condition in the finite element method on elliptical boundaries are reported in (Ben-Porat, 1995).

For large values of μ, the solution admits the asymptotic expansion

$$\phi \sim \frac{\exp(ika)}{\sqrt{ka}} \sum_{j=0}^{\infty} \frac{g_j(\theta;k)}{(ka)^j} \tag{53}$$

Local boundary conditions for an elliptic boundary are easily constructed from expansion (53) by extending the procedures employed in (Bayliss, 1982) for polar to elliptic coordinates, with the first two of the series given by,

$$B_1\,\phi = \frac{1}{f\sinh\mu}\left(\frac{\partial}{\partial\mu} + \beta_1\right)\phi \;=\; 0, \qquad \text{on } \Gamma \tag{54}$$

$$B_2\,\phi = \frac{1}{f^2\sinh^2\mu}\left(\frac{\partial^2}{\partial\mu^2} + \beta_2\frac{\partial}{\partial\mu} + \beta_3\right)\phi \;=\; 0, \qquad \text{on } \Gamma \tag{55}$$

where

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\tanh\mu_0 - ikf\sinh\mu_0 \\
\beta_2 &= 3\tanh\mu_0 - 2ikf\sinh\mu_0 - \coth\mu_0 \\
\beta_3 &= \frac{3}{4}\tanh^2\mu_0 - k^2f^2\sinh^2\mu_0 - 3ikf\sinh\mu_0\tanh\mu_0
\end{aligned}$$

Applying the B_1 operator to the expansion (46), evaluated at μ_0, and after rearranging gives the first modified DtN condition,

$$\frac{\partial\phi}{\partial n} = -\frac{\beta_1}{h_\theta}\phi + \sum_{n=0}^{\infty} Zc_n^{(1)}(q)\,D_n^c(\theta,q) + \sum_{n=1}^{\infty} Zs_n^{(1)}(q)\,D_n^s(\theta,q) \tag{56}$$

$$Zc_n^{(1)}(q) = Zc_n^{(0)}(q) + \beta_1\ , \quad Zs_n^{(1)}(q) = Zs_n^{(0)}(q) + \beta_1 \tag{57}$$

To derive the second modified DtN condition, we apply the B_2 operator to both sides of (46) evaluated at μ_0. To obtain a form suitable for finite element implementation, we replace the second derivatives of Mc_n and Ms_n in the DtN kernel using (47), and the second-order radial derivatives of ϕ in favor of angular derivatives using the Helmholtz equation written in elliptic coordinates, with the result,

$$\begin{aligned}
\frac{\partial\phi}{\partial n} \;=\; & \frac{1}{\beta_2 h_\theta}\left(\frac{\partial^2}{\partial\theta^2} - \beta_3 + k^2h_\theta^2\right)\phi \\
& + \sum_{n=0}^{\infty} Zc_n^{(2)}(q)\,D_n^c(\theta,q) + \sum_{n=1}^{\infty} Zs_n^{(2)}(q)\,D_n^s(\theta,q)
\end{aligned} \tag{58}$$

$$Zc_n^{(2)}(q) = Zc_n^{(0)}(q) + \frac{1}{\beta_2}(\lambda_{Cn} - 2q\cosh 2\mu_0 + \beta_3) \tag{59}$$

$$Zs_n^{(2)}(q) = Zs_n^{(0)}(q) + \frac{1}{\beta_2}(\lambda_{Sn} - 2q\cosh 2\mu_0 + \beta_3) \tag{60}$$

In the above, $\lambda_{Cn}(q)$ and $\lambda_{Sn}(q)$ are the separation constants for Mc_n and Ms_n, respectively. In (Grote, 1995), the B_2 operator is left with second-order derivatives in the radial coordinate μ, and is not suitable for direct finite element implementation.

FINITE ELEMENT FORMULATION

The weak form for the exterior Helmholtz problem defined by (1) - (3) may be stated as: Find: $\phi(\boldsymbol{x})$ in $\mathcal{T}$, such that for all admissible weighting functions $\bar{\phi}$ in $\mathcal{V}$, the following variational equation is satisfied,

$$K_\Omega(\bar{\phi},\phi) + K_\Gamma(\bar{\phi},\phi) = F(\bar{\phi}) \tag{61}$$

with inner products $(\cdot,\cdot) : \mathcal{V}\times\mathcal{T} \to C$ defined by the sesquilinear forms,

$$K_\Omega(\bar{\phi},\phi) := \int_\Omega (\nabla\bar{\phi}\cdot\nabla\phi - k^2\bar{\phi}\phi)\,d\Omega + \int_S \frac{\gamma}{\beta}\bar{\phi}\phi\,dS \tag{62}$$

$$K_\Gamma(\bar{\phi},\phi) := -\int_\Gamma \bar{\phi}M(\phi)\,d\Gamma \tag{63}$$

and conjugate linear form $(\cdot) : \mathcal{V} \to C$ defined by,

$$F(\bar{\phi}) := \int_\Omega \bar{\phi}\,f\,d\Omega + \int_S \bar{\phi}\frac{g}{\beta}\,dS \tag{64}$$

The function spaces are defined by,

$$T := \{\phi | \phi \in H^1(\Omega);\ \text{If } \beta = 0,\ \text{then } \phi(\boldsymbol{x}) = g/\alpha, \boldsymbol{x} \in S\} \quad (65)$$

$$V := \{\bar{\phi} | \bar{\phi} \in H^1(\Omega);\ \text{If } \beta = 0,\ \text{then } \bar{\phi}(\boldsymbol{x}) = 0, \boldsymbol{x} \in S\} \quad (66)$$

Here H^1 denotes the Sobolev space of degree one.

The nonreflecting boundary operator K_Γ in the variational equation is composed a nonlocal part, and in the case of the modified conditions, a local part:

$$K_\Gamma(\bar{\phi}, \phi) := -\int_\Gamma \bar{\phi} M(\phi)\, d\Gamma = B_\Gamma^{(j)}(\bar{\phi}, \phi) + Z_\Gamma^{(j)}(\bar{\phi}, \phi) \quad (67)$$

The local part associated with the DtN is zero,

$$B_\Gamma^{(0)}(\bar{\phi}, \phi) := 0 \quad (68)$$

In prolate spheroidal case, the local part of the modified DtN operators using B_1 and B_2, respectively are,

$$B_\Gamma^{(1)}(\bar{\phi}, \phi) := z_1 \int_\Gamma \frac{1}{J_s} \bar{\phi}\phi\, d\Gamma \quad (69)$$

$$B_\Gamma^{(2)}(\bar{\phi}, \phi) := \frac{f}{\nu} \int_\Gamma \frac{1}{J_s}(z_2 + c^2\eta^2)\, \bar{\phi}\phi\, d\Gamma + \frac{1}{\nu}\int_\Gamma h_\xi\, \nabla^s\bar{\phi} \cdot \nabla^s\phi\, d\Gamma \quad (70)$$

For the case of the modified B_2 condition, we have used integration-by-parts over a closed spheroidal surface Γ:

$$\int_\Gamma \frac{1}{J_s} \bar{\phi} \Delta_\Gamma \phi\, d\Gamma = -\int_\Gamma h_\xi\, \nabla^s\bar{\phi} \cdot \nabla^s\phi\, d\Gamma \quad (71)$$

where the gradient is defined by,

$$\nabla^s\phi := \frac{1}{h_\eta}\frac{\partial\phi}{\partial\eta}\boldsymbol{e}_\eta + \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi}\boldsymbol{e}_\varphi \quad (72)$$

To obtain the nonlocal part for prolate spheroidal boundary, we make use of the following result,

$$\int_\Gamma \frac{1}{J_s}\bar{\phi} D_{mn}(c,\eta,\varphi)\, d\Gamma = (\bar{\phi}, \psi^c_{mn})_\Gamma \cdot (\phi, \psi^c_{mn})_\Gamma + (\bar{\phi}, \psi^s_{mn})_\Gamma \cdot (\phi, \psi^s_{mn})_\Gamma \quad (73)$$

where the inner-products over Γ are defined by,

$$(\phi, \psi^c_{mn})_\Gamma := \int_\Gamma \frac{1}{J_s}\phi\, \psi^c_{mn}(c,\eta,\varphi)\, d\Gamma \quad (74)$$

$$(\phi, \psi^s_{mn})_\Gamma := \int_\Gamma \frac{1}{J_s}\phi\, \psi^s_{mn}(c,\eta,\varphi)\, d\Gamma \quad (75)$$

The nonlocal part for prolate spheroidal case is then given by,

$$-Z_\Gamma^{(j)}(\bar{\phi}, \phi) := \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' Z_{mn}^{(j)} \{(\bar{\phi}, \psi^c_{mn})_\Gamma \cdot (\phi, \psi^c_{mn})_\Gamma + (\bar{\phi}, \psi^s_{mn})_\Gamma \cdot (\phi, \psi^s_{mn})_\Gamma\} \quad (76)$$

where $Z_{mn}^{(j)}$, $j = 0, 1, 2$ are the kernels defined in (24) for the DtN, and (33),(39) for the modified DtN associated with B_1 and B_2, respectively.

For the two-dimensional elliptic boundary, the local part associated with the DtN, and modified DtN operators using B_1 and B_2, respectively are,

$$B_\Gamma^{(0)}(\bar{\phi}, \phi) := 0 \quad (77)$$

$$B_\Gamma^{(1)}(\bar{\phi}, \phi) := \beta_1 \int_\Gamma \frac{1}{h_\theta}\bar{\phi}\phi\, d\Gamma \quad (78)$$

$$B_\Gamma^{(2)}(\bar{\phi}, \phi) := \frac{1}{\beta_2}\int_\Gamma \frac{1}{h_\theta}(\beta_3 - k^2 h_\theta^2)\, \bar{\phi}\phi\, d\Gamma + \frac{1}{\beta_2}\int_\Gamma \frac{1}{h_\theta}\frac{\partial\bar{\phi}}{\partial\theta}\frac{\partial\phi}{\partial\theta}\, d\Gamma \quad (79)$$

The form of the nonlocal part associated with elliptic boundary is similar to the prolate spheroidal case, with the result:

$$Z_\Gamma^{(j)}(\bar{\phi}, \phi) := -\frac{1}{\pi}\sum_{n=0}^{\infty} Zc_n^{(j)}\, (\bar{\phi}, ce_n)_\Gamma \cdot (\phi, ce_n)_\Gamma - \frac{1}{\pi}\sum_{n=1}^{\infty} Zs_n^{(j)}\, (\bar{\phi}, se_n)_\Gamma \cdot (\phi, se_n)_\Gamma \quad (80)$$

where $Zc_n^{(j)}$ and $Zs_n^{(j)}$, $j = 0, 1, 2$ are the kernels defined in (50) for the DtN, and (57),(60) for the modified DtN respectively. The inner products over Γ are defined by

$$(\phi, ce_n)_\Gamma := \int_\Gamma \frac{1}{h_\theta}\phi\, ce_n(\theta, q)\, d\Gamma \quad (81)$$

$$(\phi, se_n)_\Gamma := \int_\Gamma \frac{1}{h_\theta}\phi\, se_n(\theta, q)\, d\Gamma \quad (82)$$

Finite Element Discretization

To obtain a finite element approximation to the solution of the variational equation (61), the domain Ω is discretized into a finite number of subdomains (elements), and we apply the standard Galerkin approximation,

$$\phi(\boldsymbol{x}) \approx \phi^h(\boldsymbol{x}) = \boldsymbol{N}(x)\,\boldsymbol{d} \quad (83)$$

$$\bar{\phi}(\boldsymbol{x}) \approx \bar{\phi}^h(\boldsymbol{x}) = \boldsymbol{N}(x)\,\bar{\boldsymbol{d}} \quad (84)$$

where $\boldsymbol{N} \in R^{N_{dof}}$ is a row vector of standard C^o basis functions with compact support associated with each node, and $\boldsymbol{d} \in C^{N_{dof}}$ is a column vector containing the nodal values of ϕ^h. Here, N_{dof} is the total number of unknowns in the finite element model, and the superscript h denotes a finite-dimensional basis.

Using this approximation in (61), we arrive at the following system of linear algebraic equations:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f} \tag{85}$$

where the global array $\boldsymbol{K} \in C^{N_{dof} \times N_{dof}}$ is an indefinite complex-symmetric matrix:

$$\boldsymbol{K} = (\boldsymbol{K}_\Omega + \boldsymbol{B}_\Gamma) + \boldsymbol{Z}_\Gamma = \boldsymbol{K}_1 + \boldsymbol{K}_2 \tag{86}$$

composed of a sparse/banded part $\boldsymbol{K}_1$ associated with the discretization of the Helmholtz equation in Ω, and the local radiation boundary operator $B_\Gamma^{(j)}$: $\boldsymbol{K}_1 = \boldsymbol{K}_\Omega + \boldsymbol{B}_\Gamma$, and a full/dense part associated with the nonlocal DtN operator $Z_\Gamma^{(j)}$: $\boldsymbol{K}_2 = \boldsymbol{Z}_\Gamma$. In the above,

$$\boldsymbol{K}_\Omega = K_\Omega(\boldsymbol{N}^T, \boldsymbol{N}) \tag{87}$$

$$\boldsymbol{B}_\Gamma = B_\Gamma^{(j)}(\boldsymbol{N}^T, \boldsymbol{N}) \tag{88}$$

$$\boldsymbol{Z}_\Gamma = Z_\Gamma^{(j)}(\boldsymbol{N}^T, \boldsymbol{N}) \tag{89}$$

Let N_Γ denote the number of unknowns on the boundary Γ. With the unknowns on Γ numbered last, then $\boldsymbol{Z}_\Gamma$ has zeros everywhere except for a fully populated $N_\Gamma \times N_\Gamma$ block on the lower partition. In particular, for a prolate spheroidal boundary, we introduce the finite element interpolation for the test and weighting functions on Γ,

$$\phi^h(\xi_0, \theta, \varphi) = \sum_{I=1}^{N_\Gamma} N_I(\theta, \varphi)\, d_I = \boldsymbol{N}(\theta, \varphi)\, \boldsymbol{d} \tag{90}$$

so that the matrix $\boldsymbol{Z}_\Gamma$ is defined by the product decomposition,

$$\boldsymbol{Z}_\Gamma = -\sum_{n=0}^{N} \sum_{m=0}^{n}{}' Z_{mn}^{(j)} \left\{ \boldsymbol{c}_{mn} \boldsymbol{c}_{mn}^T + \boldsymbol{s}_{mn} \boldsymbol{s}_{mn}^T \right\} \tag{91}$$

where

$$\boldsymbol{c}_{mn} = (\boldsymbol{N}^T, \psi_{mn}^c)_\Gamma = \int_\Gamma \frac{1}{J_s} \boldsymbol{N}^T \psi_{mn}^c(c, \eta, \varphi)\, d\Gamma \tag{92}$$

$$\boldsymbol{s}_{mn} = (\boldsymbol{N}^T, \psi_{mn}^s)_\Gamma = \int_\Gamma \frac{1}{J_s} \boldsymbol{N}^T \psi_{mn}^s(c, \eta, \varphi)\, d\Gamma \tag{93}$$

A similar form is obtained for the elliptic case. The summation over n is truncated at a finite number N. Let N_T denote the total number of harmonics included in the DtN or modified DtN condition, then $N_T = 2N+1$, and $N(N+1)$, for elliptic and prolate spheroidal boundary, respectively. The storage of the dense matrix $\boldsymbol{Z}_\Gamma$ in the unknowns on Γ, associated with the nonlocal DtN operator, requires $O(N_\Gamma^2)$ complex numbers. The storage for the sparse matrix $\boldsymbol{K}_1$ is $O(N_{dof})$, so that the total storage required for $\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2$ is $O(N_{dof} + N_\Gamma^2)$. For large models, the fully populated submatrix of $\boldsymbol{K}_2$ becomes prohibitively expensive to store and factorize. However, when solving the system of equations (85) using an iterative method requiring matrix-vector products of the kind $\boldsymbol{v} = \boldsymbol{K}\,\boldsymbol{p}$, at each iteration, the special structure of the DtN matrix $\boldsymbol{K}_2$, as an multiplicative split involving the vectors $\boldsymbol{c}_{mn}$ and $\boldsymbol{s}_{mn}$ defined in (92) and (93), can be exploited to reduce storage and cost (Malhotra, 1996).

Here, the matrix-vector product of the DtN block matrix $\boldsymbol{v} = \boldsymbol{K}_2\,\boldsymbol{p}$, may be computed in "matrix-free" form as,

$$\boldsymbol{v} = -\sum_{n=0}^{N} \sum_{m=0}^{n}{}' Z_{mn}^{(j)} \left\{ \alpha_{mn} \boldsymbol{c}_{mn} + \beta_{mn} \boldsymbol{s}_{mn} \right\} \tag{94}$$

$$\alpha_{mn} = \boldsymbol{c}_{mn}^T \boldsymbol{p} \tag{95}$$

$$\beta_{mn} = \boldsymbol{s}_{mn}^T \boldsymbol{p} \tag{96}$$

By calculating and storing the vectors $\boldsymbol{c}_{mn}$ and $\boldsymbol{s}_{mn}$, for each harmonic, the storage requirements and number of operations may be reduced to $O(N_{dof} + N_T N_\Gamma)$. Since $N_T < N_\Gamma$, the storage and cost is considerably lower than a straightforward matrix-vector product requiring storage and number of operations of $O(N_{dof} + N_\Gamma^2)$. This matrix-free representation of the DtN block matrix was first recognized in the context of circular and spherical boundaries by Malholtra (Malhotra, 1996). In (Malhotra, 1996), it was also pointed out that the matrix-vector product could be carried out at the element level, so that standard element-based data structures can be used in the presence of the DtN map. This can be accomplished by constructing element vectors $\boldsymbol{p}^e$, from the global vector $\boldsymbol{p}$, using standard local destination arrays formed by element-node connectivity data, and computing element level matrix-vector products of the form,

$$\boldsymbol{v}^e = -\sum_{n=0}^{N} \sum_{m=0}^{n}{}' Z_{mn}^{(j)} \left\{ \alpha_{mn}^e \boldsymbol{c}_{mn}^e + \beta_{mn}^e \boldsymbol{s}_{mn}^e \right\} \tag{97}$$

$$\alpha_{mn}^e = \sum_{e=1}^{N_{\Gamma_e}} \boldsymbol{c}_{mn}^{eT} \boldsymbol{p}^e \tag{98}$$

$$\beta_{mn}^{e} = \sum_{e=1}^{N_{\Gamma_e}} \boldsymbol{s}_{mn}^{eT} \boldsymbol{p}^{e} \tag{99}$$

$$\boldsymbol{c}_{mn}^{e} = (\boldsymbol{N}^{eT}, \psi_{mn}^{c})_{\Gamma_e} = \int_{\Gamma_e} \frac{1}{J_s} \boldsymbol{N}^{eT} \psi_{mn}^{c}(c,\eta,\varphi)\, d\Gamma \tag{100}$$

$$\boldsymbol{s}_{mn}^{e} = (\boldsymbol{N}^{eT}, \psi_{mn}^{s})_{\Gamma_e} = \int_{\Gamma_e} \frac{1}{J_s} \boldsymbol{N}^{eT} \psi_{mn}^{s}(c,\eta,\varphi)\, d\Gamma \tag{101}$$

Here N_{Γ_e} is the number of elements on the radiation boundary, $\Gamma_e = \Omega^e \cap \Gamma$ are element boundaries adjacent to the radiation boundary, and $\boldsymbol{N}^e$ is a row vector of element shape functions. The final result, $\boldsymbol{v}$ is obtained from standard assembly of the element vectors $\boldsymbol{v}^e$.

SSOR Preconditioner

In practice, the number of iterations required for convergence can be reduced by using a preconditioner of the form $\boldsymbol{M} = \boldsymbol{C}\boldsymbol{C}^T$, and solving the transformed system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{102}$$

where $\boldsymbol{A} = \boldsymbol{C}^{-1}\boldsymbol{K}\boldsymbol{C}^{-T}$, $\boldsymbol{x} = \boldsymbol{C}^T\boldsymbol{d}$, and $\boldsymbol{b} = \boldsymbol{C}^{-1}\boldsymbol{f}$. Straightforward solution of the preconditioned system using iterative methods requires a matrix-vector product of the preconditioned matrix $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{p}$, which involves one matrix-vector multiply $\boldsymbol{v}_2 = (\boldsymbol{K}_1 + \boldsymbol{K}_2)\boldsymbol{v}_1$ and two efficient solves $\boldsymbol{C}^T\boldsymbol{v}_1 = \boldsymbol{p}$ and $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{v}_2$.

Effective and efficient preconditioners are difficult to construct at the element level in the context of matrix-free iterations. This issue was studied using a hierarchical basis preconditioner, in conjunction with matrix-free iterative computations in (Malhotra, 1998). At the global level, an effective preconditioner for complex-symmetric systems is the SSOR preconditioner in conjunction with the QMR iterative solver (Freund, 1992; Freund, 1991), together with Eisenstat's trick for matrix-vector multiplication (Eisenstat, 1984). For systems such as (86), composed of a sparse part $\boldsymbol{K}_1$, and a full/dense part $\boldsymbol{K}_2$, storage and cost may be reduced by not assembling $\boldsymbol{K}_2$, and basing the preconditioner on $\boldsymbol{K}_1$, only (Oberai, 1998).

Extending the procedures described in (Oberai, 1998) for the first modified DtN map on a circle or sphere, to first and second modified DtN maps for elliptic and spheroidal surfaces, we take advantage of the special structure of $\boldsymbol{K}_2 = \boldsymbol{Z}_\Gamma$ defined in (91) as a product decomposition of $\boldsymbol{c}_{mn}$ and $\boldsymbol{s}_{mn}$ to save both storage and cost of the matrix-vector product $\boldsymbol{K}_2\boldsymbol{v}_1$. Without the explicit computation of the full $\boldsymbol{K}_2$ block, we define the SSOR preconditioner in terms of the symmetric, banded matrix $\boldsymbol{K}_1 = \boldsymbol{K}_\Omega + \boldsymbol{B}_\Gamma$, which is stored and factorized into,

$$\boldsymbol{K}_1 = \boldsymbol{L} + \boldsymbol{\Delta} + \boldsymbol{L}^T \tag{103}$$

where $\boldsymbol{\Delta}$ is a diagonal matrix and $\boldsymbol{L}$ is strictly lower triangular. The preconditioner only involves the local part of the modified DtN map $\boldsymbol{B}_\Gamma$, yet provides a good approximation to the complete matrix $\boldsymbol{K}_1 + \boldsymbol{Z}_\Gamma$, especially for the DtN condition modified with the local B_2 operator. For the DtN condition, with no modification with a local radiation boundary operator, $\boldsymbol{B}_\Gamma = \boldsymbol{0}$, and it is not feasible to formulate a SSOR preconditioner based on a direct additive split, since in this case, a preconditioner based on $\boldsymbol{K}_1 = \boldsymbol{K}_\Omega$ is an inappropriate approximation to the full system $\boldsymbol{K} = \boldsymbol{K}_\Omega + \boldsymbol{Z}_\Gamma$. For the DtN, a preconditioner may be constructed by adding and subtracting the local matrix $\boldsymbol{B}_\Gamma$ to $\boldsymbol{K}$, with the result, $\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2$, where $\boldsymbol{K}_1 = \boldsymbol{K}_\Omega + \boldsymbol{B}_\Gamma$, and $\boldsymbol{K}_2 = \boldsymbol{Z}_\Gamma - \boldsymbol{B}_\Gamma$. Here $\boldsymbol{K}_1$ is decomposed according to (103) for the SSOR preconditioner and $\boldsymbol{K}_2 = \boldsymbol{Z}_\Gamma - \boldsymbol{B}_\Gamma$ is not assembled when performing matrix-vector products.

For the modified DtN conditions, the matrix-vector products are performed efficiently using Eisenstat's trick, and the special structure of $\boldsymbol{Z}_\Gamma$. For the left SSOR preconditioner, $\boldsymbol{C} = L\,\boldsymbol{\Delta}^{-1/2}$, where $L = (\boldsymbol{\Delta} + \omega\boldsymbol{L})$, and $0 < \omega < 2$, (Freund, 1992; Freund, 1991). Eisenstat's trick is to write the preconditioned matrix $\boldsymbol{A}$ as (Eisenstat, 1984; Oberai, 1998):

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{C}^{-1}(\boldsymbol{K}_1 + \boldsymbol{K}_2)\boldsymbol{C}^{-T} \\
&= \boldsymbol{\Delta}^{1/2} L^{-1}(\boldsymbol{L} + \boldsymbol{\Delta} + \boldsymbol{L}^T + \boldsymbol{K}_2) L^{-T} \boldsymbol{\Delta}^{1/2} \\
&= \frac{1}{\omega}\boldsymbol{\Delta}^{1/2}\left\{ L^{-T} + L^{-1}(\boldsymbol{I} + [(\omega - 2)\boldsymbol{\Delta} + \omega\boldsymbol{K}_2] L^{-T}) \right\} \boldsymbol{\Delta}^{1/2}
\end{aligned} \tag{104}$$

so that the matrix-vector product $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{p}$ can be calculated efficiently as:

1. Solve $L^T\,\hat{\boldsymbol{p}} = \boldsymbol{\Delta}^{1/2}\boldsymbol{p}$ for $\hat{\boldsymbol{p}}$.
2. Set $\boldsymbol{p} := \boldsymbol{p} + (\omega - 2)\boldsymbol{\Delta}\hat{\boldsymbol{p}}$
   $- \omega \sum\sum Z_{mn}^{(j)} \left\{ (\boldsymbol{c}_{mn}^T \hat{\boldsymbol{p}})\,\boldsymbol{c}_{mn} + (\boldsymbol{s}_{mn}^T \hat{\boldsymbol{p}})\,\boldsymbol{s}_{mn} \right\}$.
3. Solve $L\,\tilde{\boldsymbol{p}} = \boldsymbol{p}$ for $\tilde{\boldsymbol{p}}$.
4. Set $\boldsymbol{v} = \frac{1}{\omega}\boldsymbol{\Delta}^{1/2}(\hat{\boldsymbol{p}} + \tilde{\boldsymbol{p}})$.

Since $\boldsymbol{C}$ is triangular, the SSOR preconditioner involves two efficient back-solves with L. Eisenstat's trick replaces the matrix-vector multiply with $\boldsymbol{K}_1$, with two matrix-vector multiplies with $\boldsymbol{\Delta}^{1/2}$, a significant reduction in the number of required operations. The storage and cost of computing the matrix-vector multiply $\boldsymbol{K}_2\,\hat{\boldsymbol{p}}$ is reduced from $O(N_\Gamma^2)$ to $O(N_T\,N_\Gamma)$ by exploiting the product decomposition of $\boldsymbol{K}_2$ and storing the vectors $\boldsymbol{c}_{mn}$ and $\boldsymbol{s}_{mn}$ defined in (92) and (93), respectively.

To reduce storage requirements further to $O(N\,N_\Gamma)$, the vectors $\boldsymbol{c}_{mn}$ and $\boldsymbol{s}_{mn}$ may be recomputed, as needed. In this case, we project the angular harmonics onto the finite-dimensional basis using,

$$\psi_{mn}^{c}(\theta,\varphi) = \boldsymbol{N}(\theta,\varphi)\,\boldsymbol{\psi}_{mn}^{c} \tag{105}$$

$$\psi_{mn}^{s}(\theta,\varphi) = \boldsymbol{N}(\theta,\varphi)\,\boldsymbol{\psi}_{mn}^{s} \tag{106}$$

where $\boldsymbol{\psi}_{mn} = \{\psi_{mn,l}\},\ l = 1, 2, \cdots, N_\Gamma$, is a vector containing the nodal values of the harmonic defined by (n,m) on Γ, i.e., $\psi_{mn,l} = \psi_{mn}(\theta_l, \varphi_l)$. A similar projection may be made for the elliptic coordinate θ in 2D.

Using this expansion in (92) and (93), the vectors may be approximated by

$$\boldsymbol{c}_{mn} = \boldsymbol{M}_\Gamma \boldsymbol{\psi}^c_{mn} \tag{107}$$
$$\boldsymbol{s}_{mn} = \boldsymbol{M}_\Gamma \boldsymbol{\psi}^s_{mn} \tag{108}$$

resulting in,

$$\boldsymbol{Z}_\Gamma = -\boldsymbol{M}_\Gamma \left(\sum_{n=0}^{N} {\sum_{m=0}^{n}}' Z^{(j)}_{mn} \left\{ \boldsymbol{\psi}^c_{mn} \boldsymbol{\psi}^{cT}_{mn} + \boldsymbol{\psi}^s_{mn} \boldsymbol{\psi}^{sT}_{mn} \right\} \right) \boldsymbol{M}_\Gamma \tag{109}$$

where $\boldsymbol{M}_\Gamma$ is the $N_\Gamma \times N_\Gamma$ sparse/banded symmetric matrix,

$$\boldsymbol{M}_\Gamma := (\boldsymbol{N}^T, \boldsymbol{N})_\Gamma = \int_\Gamma \frac{1}{J_s} \boldsymbol{N}^T \boldsymbol{N}\, d\Gamma \tag{110}$$

This matrix may be diagonalized using nodal (Lobotto) quadrature. In this form, the matrix $\boldsymbol{M}_\Gamma$ is stored, and the matrix-vector product $\boldsymbol{v} = \boldsymbol{K}_2 \boldsymbol{p}$, may be performed efficiently using Algorithm 3.1.1. of (Oberai, 1998). For a circular or spherical boundary Γ, the interpolant of the angular harmonics $\psi_{mn,l}$, for each mode and for each node on Γ, may be computed efficiently using recurrence relations between successive complex exponential and Legendre functions (Oberai, 1998). However, for elliptic and spheroidal boundaries, direct computation of recurrence relations for Mathieu and Spheroidal functions are not as efficient as the circular and spherical case. As a result, while the storage requirements are low, the cost in recomputing the angular functions for each harmonic and node on Γ, at every iteration, is relatively high.

NUMERICAL EXAMPLES

We first compare the accuracy of the DtN and modified DtN radiation boundary conditions as a function of the number of harmonics included in the eigenfunction expansions for the DtN map. Then, the performance of the SSOR-type preconditioners in conjunction with Eisenstat's trick is examined based on the number of QMR iterations required to achieve a given tolerence on the solution residual. Solutions obtained using the DtN conditions are denoted 'DtN', while the first and second modified DtN conditions are denoted 'MDtN(B1)' and 'MDtN(B2)', respectively. Both the DtN and MDtN conditions are exact for modes $n \leq N$, however for modes $n > N$, the DtN condition reduces to a rigid (homogeneous Neumann) condition, while the modified DtN conditions reduce to the first and second-order local radiation boundary conditions, B_1, and B_2, respectively.

Scattering of a Plane Wave by an Elliptic Cylinder

Consider scattering of a plane wave represented by $\phi^{(i)} = e^{-ikx}$, from an infinite elliptic cylinder with $f = 1$, and $\mu = \overline{\mu} = 0.1$, with major and minor radius $a = f\cosh\overline{\mu}$ and $b = f\sinh\overline{\mu}$, respectively, on which we assume a 'soft' boundary,

$$\phi = \phi^{(i)} + \phi^{(s)} = 0, \qquad \text{on } S = \{\overline{\mu} = 0.1, 0 \leq \theta \leq 2\pi\} \tag{111}$$

where the total field $\phi(\mu,\theta)$ is composed of the incident wave $\phi^{(i)}$, and the scattered wave field $\phi^{(s)}$.

For numerical computation, we set the artificial boundary Γ at $\mu_0 = 0.5$, and take advantage of symmetry to form the computational domain $\Omega = \{0.1 \leq \mu \leq 0.5, 0 \leq \theta \leq \pi,\}$. The bounded region Ω is then discretized with 40×240 standard 4-node bilinear elements evenly spaced in both the μ and θ directions. Figure 2 shows contours for the real part of the scattered solution for wavenumber $k = 4\pi$ computed using the finite element formulation of the modified DtN condition MDtN(B2) with $N = 15$ terms included in the series. For this problem, the value $N = 15$ was sufficient to obtain an exact radiation boundary condition for the fixed discretization defined by the finite element mesh.

The error in the approximate finite element solution ϕ^h is measured with a relative $L_2(\Omega)$ norm defined over the entire computational domain, i.e.

$$E = \frac{||\phi^h - \phi||_{L_2}}{||\phi||_{L_2}} \tag{112}$$

where ϕ is the exact solution.

As shown in Figure 3, the solutions obtained using the elliptic DtN and MDtN all converge to a finite error value as N is increased. This limiting error is controlled primarily by the finite element discretization of the computational domain. The results also show that when only a few terms N are included, the modified conditions yields more accurate results than the DtN condition, as expected. In particular, the MDtN(B2) condition results in the lowest error, with no additional memory and very little extra cost.

Figure 4 shows the speedup in the convergence rate of the QMR iterative solver when using the SSOR-type preconditioner based on the local, sparse/banded matrix partition $\boldsymbol{K}_1$. In the figure, the abscissa represents the iteration number n and the ordinate axis represents the relative residual $\|\boldsymbol{r}_n\|_2 / \|\boldsymbol{r}_0\|_2$. We observe that the DtN and MDtN(B1) formulations converge in approximately 1100 and 900 iterations, respectively, while the solutions with the SSOR preconditioner based on the local sparse/banded partition of the global matrix converges in only 525 iterations, a significant speedup.

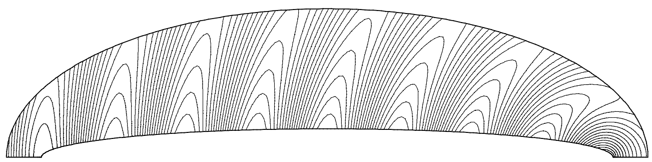

Figure 2. PLANE-WAVE SCATTERING FROM AN INFINITE ELLIPTIC CYLINDER $\mu = 0.1$ WITH 10:1 ASPECT RATIO. ARTIFICIAL BOUNDARY LOCATED AT $\mu = 0.5$. SOLUTION CONTOURS OF REAL PART OF FE SOLUTION USING ELLIPTIC MDtN(B2) WITH $N = 15$ AND WAVENUMBER $k = 4\pi$.

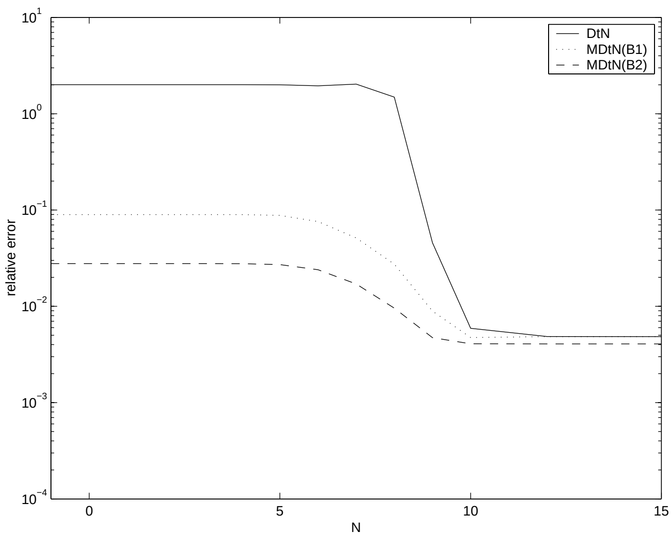

Figure 3. SCATTERING FROM AN ELLIPTIC CYLINDER. RELATIVE ERROR MEASURED IN $L_2(\Omega)$ NORM VERSUS THE NUMBER OF HARMONICS N INCLUDED IN THE DTN BOUNDARY CONDITION (49) AND MODIFIED CONDITIONS (56), (58).

Scattering of a Plane Wave by a Spheroid

We now consider the scattering of a plane wave $\phi^{(i)} = e^{-ikz}$ from the spheroid $\xi = \cosh 0.1$, on which we assume a 'soft' (homogeneous Dirichlet) boundary. With the artificial boundary Γ located at $\xi_o = \cosh 0.5$, the bounded domain is discretized with a uniform mesh of standard 4-node bilinear axisymmetric finite elements with 240×40 evenly spaced elements in $0 \leq \theta \leq \pi$, and $0.1 \leq \mu \leq 0.5$, respectively.

Solution contours for the real part of the scattered solution computed using MDtN(B2) positioned at $\xi_0 = \cosh 0.5$ are shown in Figure 5 for a normalized wavenumber $c = kf = 4\pi$. Figure 6 shows the relative $L_2(\Omega)$ error in solutions obtained using the DtN and modified DtN conditions MDtN(B1) and MDtN(B2), with N varying from 0 to 15. Similar to the two-

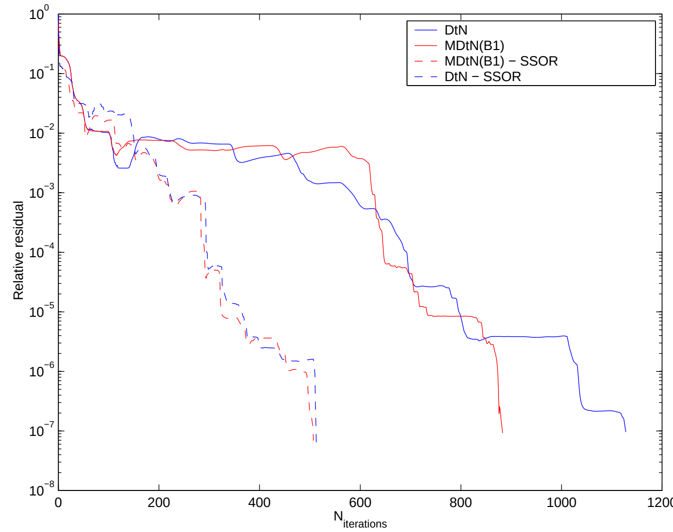

Figure 4. RATE OF CONVERGENCE OF QMR ITERATIVE SOLVER WITH AND WITHOUT SSOR PRECONDITIONING.

dimensional case, both the DtN and MDtN solutions converge to a finite error value as N is increased, and when only a few terms N are included, the MDtN(B2) condition gives the most accurate solution.

CONCLUSIONS

The DtN radiation condition relating normal derivatives and Dirichlet data on elliptic and prolate spheroidal boundaries are derived from harmonic expansions in two and three dimensions, respectively. The use of elliptic and spheroidal coordinates allows for a tight fit of the radiation boundary surrounding an elongated structure, with corresponding reduction in the size of the computational region required to model time-harmonic radiation and scattering. Modified DtN conditions based on first and second order local boundary operators are also derived in elliptic and spheroidal coordinates, in a form suitable for finite element implementation. The second modified DtN condition based on the local B_2 operator is formulated in terms of second-order tangential derivatives, which are then enforced weakly with standard C^0 regularity at the artificial boundary Γ. This condition is more accurate than the DtN boundary condition, yet requires no extra memory and little extra cost. The finite element formulation of the elliptic and spheroidal DtN conditions retain the special structure found in the circular and spherical cases; a multiplicative split defined by the outer-product decomposition of linear forms. This special structure allows for the matrix-free implementation of iterative solvers such as QMR, which do not require the explicit storage of a full/dense block relating unknowns on the boundary Γ. A SSOR preconditioner with Eisenstat's trick

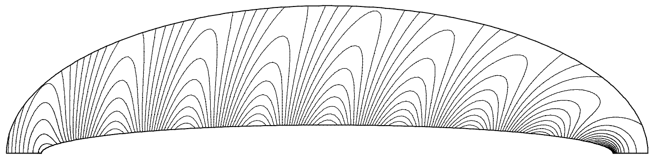

Figure 5. SCATTERING OF A PLANE WAVE FROM A PROLATE SPHEROID $\mu = 0.1$ with 10:1 ASPECT RATIO. THE ARTIFICIAL BOUNDARY IS LOCATED AT $\mu = 0.5$. SOLUTION CONTOURS FOR THE REAL PART USING THE SPHEROIDAL MDtN(B2) CONDITION WITH $N = 15$, AND NORMALIZED WAVENUMBER $c = kf = 4\pi$.

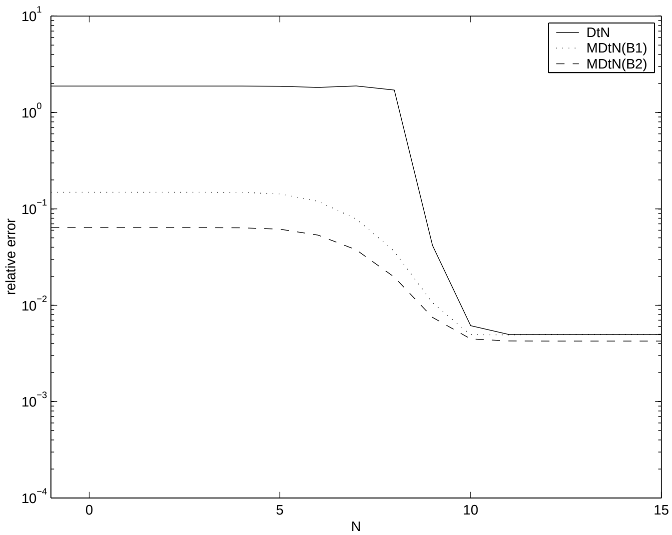

Figure 6. SCATTERING FROM A PROLATE SPHEROID. RELATIVE ERROR MEASURED IN $L_2(\Omega)$ NORM VERSUS THE NUMBER OF HARMONICS N INCLUDED IN THE DTN BOUNDARY CONDITION (21) AND MODIFIED CONDITIONS (32), (43).

based on the matrix partition associated with the discretization of the interior mesh and local boundary operator provides an efficient and effective preconditioner for the resulting complex-symmetric system.

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