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EXACT RADIATION CONDITIONS ON SPHEROIDAL BOUNDARIES WITH SPARSE ITERATIVE METHODS FOR EFFICIENT COMPUTATION OF EXTERIOR ACOUSTICS

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Abstract

Exact Dirichlet-to-Neumann (DtN) maps are derived on spheroidal boundaries for finite element implementation. The use of spheroidal boundaries enables the efficient solution of scattering from elongated objects. Comparisons of sparse preconditioned iterative solvers, including BiCG-Stab and QMR with Jacobi and SSOR preconditioning are discussed. Matrix-vector products are computed efficiently in block form using Level-2 BLAS and the special product decomposition structure of the DtN matrix. Preconditioners based on the sparse matrix partition associated with the interior mesh and local part of the radiation boundary operator are used to accelerate convergence of the Krylov-subspace iterative solution methods. In this way, the full DtN matrix block is never assembled. Three-dimensional numerical examples demonstrate the efficiency and accuracy of the boundary treatments for high-frequency scattering from elongated structures.

INTRODUCTION

We consider time-harmonic scattering in an infinite three-dimensional region surrounding an object with surface $\mathcal{S}$. For computation, the unbounded region is truncated by an artificial boundary Γ. We assume that Γ is a surface defined by a prolate spheroid in three-dimensions. We then denote by Ω, the finite subdomain bounded by $\partial\Omega = \Gamma \cup \mathcal{S}$, see Figure 1. Within

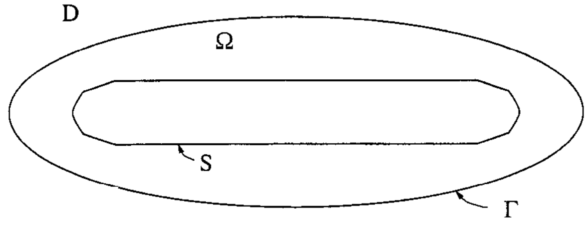

Fig. 1: Illustration of the computational domain Ω, with exterior region $\mathcal{D}$.

Ω, the solution $\phi(\boldsymbol{x}) : \Omega \mapsto \mathbb{C}$, satisfies the Helmholtz equation,

$$\nabla^2\phi + k^2\phi = 0, \qquad \boldsymbol{x} \in \Omega \tag{1}$$

$$\frac{\partial\phi}{\partial n} + \gamma\,\phi = g, \qquad \boldsymbol{x} \in \mathcal{S} \tag{2}$$

$$\frac{\partial\phi}{\partial n} = -M\,\phi, \qquad \boldsymbol{x} \in \Gamma \tag{3}$$

Here, k is the wavenumber, g is prescribed data on the scatterer $\mathcal{S}$. Equation (3) defines the Dirichlet-to-Neumann (DtN) map relating Dirichlet data to the normal derivative of the solution on Γ. The DtN operator M represents the impedance of the exterior region restricted to the boundary Γ, such that the solution satisfies the Sommerfeld radiation condition at infinity. In general the DtN map may be modified to include both a local part and a nonlocal part. For spherical truncation boundaries, several accurate and efficient methods for representing the impedance of the far-field are well understood, including the Dirichlet-to-Neumann (DtN) map, infinite elements, and absorbing layers. However for elongated scatterers, the use of spherical boundaries becomes expensive since a large computational domain must be used. To overcome this problem, spheroidal boundaries may be used to reduce the size of the computational domain, allowing for efficient solution of scattering from elongated objects. Axisymmetric finite difference implementations of both DtN and modified DtN conditions on prolate spheroidal boundaries based on first- and second- order differential operators which annihilate radial terms in a generalized multipole expansion are given in [1]. Here, we modify the DtN conditions in a form suitable for finite element implementation. The modified DtN conditions are more accurate than the DtN boundary condition, yet require no extra memory and little extra cost. Our modified DtN map differs from that used in [1] in that second-order radial derivatives are replaced in favor of second-order tangential derivatives – allowing C^0 regularity in the variational equation.

THE DTN CONDITION ON SPHEROIDAL BOUNDARIES

In prolate spheroidal coordinates, the Cartesian coordinates may be parameterized by $\boldsymbol{x} = \boldsymbol{x}(\mu, \theta, \varphi)$, $0 \leq \mu < \infty$, $0 \leq \theta < \pi$, and $0 \leq \varphi < 2\pi$, such that $x = b\sin\theta\cos\varphi, y = b\sin\theta\sin\varphi, z = a\cos\theta$ where $a = f\cosh\mu$, $b = f\sinh\mu$, are the semimajor and semiminor axis of an ellipse respectively, and f is the semi-interfocal distance. The spheroid is defined by a constant value of μ, with a confocal ellipse revolving around the major z-axis. Let $\xi = \cosh\mu$, and $\eta = \cos\theta$, so that $a = f\xi$, and $b = f\sqrt{\xi^2 - 1}$. then the spheroid may also be parameterized by $\boldsymbol{x} = \boldsymbol{x}(\xi, \eta, \varphi)$, $1 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, and $0 \leq \varphi < 2\pi$. Solutions to (1) are obtained using separation of variables and eigenfunction expansions. Let $c = kf$ be a normalized wavenumber, then we denote by $S_{mn}(c, \eta)$ the angular prolate spheroidal wave functions of the first kind. They form a complete orthogonal set over the interval $-1 \leq \eta \leq 1$. $R^{(p)}_{mn}(c, \xi)$, $p = 1, 2, 3$ are the radial prolate spheroidal wave functions of the first, second, and third kind respectively. The symmetry of the DtN map M, follows from the Green's function for the problem in the exterior region $\mathcal{D}$. Here we write the Green's function for a spheroidal boundary Γ, using an expansion of prolate spheroidal wave functions. Let $G(\boldsymbol{x} \mid \boldsymbol{x}')$ be the Dirichlet Green's function in the exterior region $\mathcal{D}$, which satisfies,

$$(\nabla^2 + k^2)G(\boldsymbol{x} \mid \boldsymbol{x}') = -\delta(\boldsymbol{x} - \boldsymbol{x}'); \quad G(\boldsymbol{x} \mid \boldsymbol{x}') = 0, \ \boldsymbol{x}' \in \Gamma \tag{4}$$

and the Sommerfeld condition at infinity. The Helmholtz boundary integral equation associated with this Green's function is,

$$\phi(\boldsymbol{x}) = \int_\Gamma \frac{\partial G(\boldsymbol{x} \mid \boldsymbol{x}')}{\partial n'} \phi(\boldsymbol{x}')\, d\Gamma' \tag{5}$$

Taking the normal derivative of (5), and setting $\boldsymbol{x} \in \Gamma$, we obtain the DtN map expressed in terms of the Green's function:

$$\frac{\partial \phi}{\partial n} = \int_\Gamma m(\boldsymbol{x} \mid \boldsymbol{x}')\, \phi(\boldsymbol{x}')\, d\Gamma', \quad m(\boldsymbol{x} \mid \boldsymbol{x}') = \frac{\partial^2 G(\boldsymbol{x} \mid \boldsymbol{x}')}{\partial n \partial n'}, \qquad \boldsymbol{x} \in \Gamma \tag{6}$$

The symmetry of the DtN kernel m, implies the symmetry of the bilinear form appearing in the Galerkin variational equation and this in turn implies the symmetry of the resulting finite element matrix equations. From (6), it is clear that the DtN map is an integral operator coupling all points over the artificial boundary Γ. To obtain the DtN map on a spheroidal boundary defined by $\xi = \xi_0$, we expand the Dirichlet Green's function in an infinite series of spheroidal wave harmonics [2],

$$G(\boldsymbol{x} \mid \boldsymbol{x}') = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' G_{mn}(\xi \mid \xi') \psi_{mn}(\eta, \varphi \mid \eta', \varphi') \tag{7}$$

$$\psi_{mn} := \frac{1}{\pi N_{mn}} S_{mn}(c, \eta)\, S_{mn}(c, \eta') \cos m(\varphi - \varphi'), \tag{8}$$

In the above, N_{mn} are normalization constants associated with S_{mn}. G_{mn} is defined in terms of the radial wave functions $R^{(1)}_{mn}$, and $R^{(3)}_{mn}$. The solution to the Helmholtz equation in the exterior region $\xi \geq \xi_0 = \cosh \mu_0$ is obtained from (5) by taking the derivative of the Green's function with respect to n', evaluated at $\xi' \to \xi_0$, and using the Wronskian relation for independent solutions to the radial wave equation. The result is,

$$\phi(\xi, \eta, \varphi) \;=\; \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{R^{(3)}_{mn}(c, \xi)}{R^{(3)}_{mn}(c, \xi_0)} \int_{\mathcal{A}} \phi(\xi_0, \eta', \varphi')\, \psi_{mn}(\eta, \varphi \mid \eta', \varphi')\; d\mathcal{A}' \tag{9}$$

where $d\mathcal{A} = d\eta\, d\varphi = \sin\theta\, d\theta\, d\varphi$. To obtain the DtN map relating normal derivatives to Dirichlet data on Γ, we simply differentiate (9) with respect to n evaluated at $\xi \to \xi_0$. The result is given by (6), with the DtN kernel,

$$m(\boldsymbol{x} \mid \boldsymbol{x}') = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' Z^{(0)}_{mn} \frac{1}{J_s J_s'} \psi_{mn}(\eta, \varphi \mid \eta', \varphi'), \quad Z^{(0)}_{mn}(c) = f(\xi_0^2 - 1) \frac{R^{(3)\prime}_{mn}(c, \xi_0)}{R^{(3)}_{mn}(c, \xi_0)} \tag{10}$$

and $J_s = h_\eta\, h_\varphi$ is the surface Jacobian for a prolate spheroid.

THE MODIFIED DTN CONDITIONS

For numerical computation, the summation over n, in the infinite series defining the DtN map is truncated at a finite value N. Use of the the truncated DtN map on the spheroidal boundary Γ, will exactly represent all radial harmonics for $n \leq N$. For $n > N$, the harmonics are evaluated with an incorrect homogeneous Neumann boundary condition on Γ. As a result, resonance will occur at discrete eigenvalues k_j. To eliminate resonance, a modified DtN is formulated by generalizing the normal derivative applied to the harmonic expansion for outgoing waves (9), with a local operator defining an approximate radiation boundary condition [1]. The resulting modified DtN condition is unique for any choice of N, and approximates the harmonics $n > N$ with greater accuracy than the original DtN condition. A sequence of local boundary conditions are easily constructed in spheroidal coordinates by extending the procedures employed in for a circle or sphere, where radial terms in a multipole expansion for outgoing waves are annihilated. Applying the first differential operator $B_1\phi = \phi_{,n} + 1/J_s)\, z_1 = 0$ to the expansion (9), evaluated at ξ_0, and rearranging leads to the modified DtN condition,

$$\frac{\partial \phi}{\partial n} = -\frac{1}{J_s} z_1 \phi + \frac{1}{J_s} \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \, Z^{(1)}_{mn}(c) \int_{\mathcal{A}} \phi(\xi_0)\, \psi_{mn}\, d\mathcal{A}, \quad Z^{(1)}_{mn}(c) = Z^{(0)}_{mn}(c) + z_1, \tag{11}$$

and $z_1 = f(\xi_0^2 - 1)\,(1 - ic\xi_0)/\xi_0$. When the condition (11) is truncated at the finite value N, it is exact for harmonics $n \leq N$, and approximates the harmonics $n > N$ with the local boundary condition $B_1\phi = 0$. The second-order operator provides a more accurate boundary condition. To allow for a natural finite element implementation, we eliminate the second-order radial derivative in the differential operator in favor of tangential derivatives using the Helmholtz equation written in spheroidal coordinates, with the result [2]:

$$B_2\, \phi = \frac{\partial \phi}{\partial n} - \frac{1}{\nu J_s} S_2 \phi = 0, \quad S_2 := \left(\Delta_\Gamma - f c^2 \eta^2 - f z_2\right) \tag{12}$$

where Δ_Γ is the surface Laplacian with metrics defined for a spheroid. Then, applying the scaled operator B_2 to (9), and using the eigenvalues for the angular harmonics we derive the new condition,

$$\frac{\partial \phi}{\partial n} = \frac{1}{\nu J_s} S_2 \phi + \frac{1}{J_s} \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \, Z^{(2)}_{mn}(c) \int_{\mathcal{A}} \phi(\xi_0)\, \psi_{mn}\, d\mathcal{A}, \tag{13}$$

$$Z^{(2)}_{mn}(c) = Z^{(0)}_{mn}(c) + \frac{f}{\nu} \left[\lambda_{mn} + \frac{m^2}{\xi_0^2 - 1} + z_2\right] \tag{14}$$

$$z_2 = (\xi_0^2 - 1)\left(2 - 4ic\xi_0 - (c\xi_0)^2\right)/\xi_0^2 - c^2\xi_0^2, \quad \nu = (4 - 2ic\xi_0)/\xi_0 - \frac{2\xi_0}{\xi_0^2 - 1}. \tag{15}$$

In the above, $\lambda_{mn}(c)$ are the characteristic values of the prolate spheroidal wave functions. When the series is truncated at mode N, the harmonics $n > N$ are approximated by $B_2\phi = 0$, an improvement over the DtN condition modified with B_1.

VARIATIONAL FORMULATION

The weak form for the exterior Helmholtz problem defined by (1) - (3) may be stated as: Find: $\phi(\boldsymbol{x}) \in \mathcal{T}$, such that $\forall$ admissible weighting functions $\bar{\phi} \in \mathcal{V}$, the following variational equation is satisfied,

$$(\nabla\bar{\phi}\,,\, \nabla\phi)_\Omega \;-\; k^2\, (\bar{\phi}\,,\, \phi)_\Omega + (\gamma\,\bar{\phi}\,,\, \phi)_S + (\bar{\phi}\,,\, M\phi)_\Gamma = (\bar{\phi}\,,\, g)_S \tag{16}$$

The nonreflecting boundary operator in the variational equation is composed a nonlocal part, and in the case of the modified conditions, a local part:

$$(\bar{\phi}\,,\,M\phi)_\Gamma = B_\Gamma^{(j)}(\bar{\phi}\,,\,\phi) + Z_\Gamma^{(j)}(\bar{\phi}\,,\,\phi) \tag{17}$$

The local operator associated with B_1, B_2 follows from (11) and (13), respectively. For B_2,

$$B_\Gamma^{(2)}(\bar{\phi}\,,\,\phi) := -\frac{1}{\nu}\int_\Gamma \frac{1}{J_s}(z_2 + c^2\eta^2)\,\bar{\phi}\,\phi\; d\Gamma - \frac{1}{\nu}\int_\Gamma h_\xi\, \nabla^s\bar{\phi}\,\cdot\,\nabla^s\phi\; d\Gamma \tag{18}$$

The nonlocal part of the boundary operator is given by,

$$Z_\Gamma^{(j)}(\bar{\phi}\,,\,\phi) := -\frac{1}{\pi}\sum_{n=0}^{\infty}\;{\sum_{m=0}^{n}}{}'\;\frac{Z_{mn}^{(j)}}{N_{mn}}\left\{\bar{a}_{mn}\cdot a_{mn} + \bar{b}_{mn}\cdot b_{mn}\right\} \tag{19}$$

with $Z_{mn}^{(j)}$, $j = 0, 1, 2$, and inner-products,

$$a_{mn} := (\phi(\xi_0)\,,\,S_{mn}(c,\eta)\cos m\varphi)_{\mathcal{A}}, \qquad b_{mn} := (\phi(\xi_0)\,,\,S_{mn}(c,\eta)\sin m\varphi)_{\mathcal{A}} \tag{20}$$

We apply the standard Galerkin approximation $\phi^h(\boldsymbol{x}) = \boldsymbol{N}^T(x)\,\boldsymbol{d}$, where $\boldsymbol{N} \in \mathbb{R}^{N_{dof}}$ is a column vector of standard C^o basis functions, and $\boldsymbol{d} \in \mathbb{C}^{N_{dof}}$ is a column vector containing the nodal values of ϕ^h. Here, N_{dof} is the total number of unknowns in the finite element model. Using this approximation in (16), we arrive at the following discrete system, $(\boldsymbol{K}_S + \boldsymbol{K}_\Gamma)\boldsymbol{d} = \boldsymbol{f}$, where the global array is an indefinite complex-symmetric (non-Hermitian) matrix composed of a sparse part $\boldsymbol{K}_S$ associated with the discretization of the Helmholtz equation in Ω, and the local radiation boundary operator $B_\Gamma^{(j)}$, and a full/dense part $\boldsymbol{K}_\Gamma$, associated with the nonlocal DtN operator $Z_\Gamma^{(j)}$. Let N_Γ denote the number of unknowns on the boundary Γ.. Projecting the angular harmonics onto the finite-dimensional basis using, $S_{mn}(c,\eta)\cos m\varphi = \boldsymbol{N}^T(\theta,\varphi)\,\boldsymbol{\psi}_{mn}^c$, where $\boldsymbol{\psi}_{mn} = \{\psi_i\}_{mn}$, $i = 1\,,\,2\,,\,\cdots\,,\,N_\Gamma$, is a vector containing the nodal values of the harmonic. A similar interpolation is used for the odd functions. The matrix $\boldsymbol{Z}_\Gamma$ is defined by a summation of rank-1 updates for even, $\boldsymbol{\psi}_{mn}^c$, and odd, $\boldsymbol{\psi}_{mn}^s$ vectors,

$$\boldsymbol{K}_\Gamma = -\frac{1}{\pi}\boldsymbol{M}_\Gamma\left(\sum_{n=0}^{N-1}\;{\sum_{m=0}^{n}}{}'\;\frac{Z_{mn}}{N_{mn}}\left\{\boldsymbol{\psi}_{mn}^c\,\boldsymbol{\psi}_{mn}^{cT} + \boldsymbol{\psi}_{mn}^s\,\boldsymbol{\psi}_{mn}^{sT}\right\}\right)\boldsymbol{M}_\Gamma \tag{21}$$

where $\boldsymbol{M}_\Gamma$ is the $N_\Gamma \times N_\Gamma$ sparse symmetric matrix, $\boldsymbol{M}_\Gamma := (\boldsymbol{N}\,,\,\boldsymbol{N}^T)_\Gamma$.

SPARSE ITERATIVE SOLUTION METHOD

Let $N_T = N(N{+}1)$ denote the total number of harmonics $n = 0, 1, \ldots, N{-}1$, $m = 0, 1, \ldots, n$, included in the DtN or modified DtN condition. For a direct solve, the storage of the dense matrix $\boldsymbol{K}_\Gamma$, associated with the nonlocal DtN operator, requires $O(N_\Gamma^2)$ complex numbers. The storage for the sparse matrix $\boldsymbol{K}_S$ is $O(N_{dof})$, so that the total storage required for $\boldsymbol{K} = \boldsymbol{K}_S + \boldsymbol{K}_\Gamma$ is $O(N_{dof} + N_\Gamma^2)$. For large models, the fully populated submatrix of $\boldsymbol{K}_\Gamma$ becomes expensive to store and factorize. However, when solving using an iterative method requiring matrix-vector products of the kind $\boldsymbol{v} = \boldsymbol{K}\,\boldsymbol{u}$, at each iteration, the special structure of the DtN matrix $\boldsymbol{K}_\Gamma$, as a summation of rank-1 vector updates can be exploited to avoid assembling a dense matrix, thus significantly reducing storage and cost. Here, we organize the vectors $\boldsymbol{\psi}_{mn}$ in order from $j = 1, 2, \ldots, N_T$, and store the spheroidal harmonics at each node on the boundary Γ, in the matrix $\boldsymbol{C} = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{N_T}]$, of dimension $(N_\Gamma \times N_T)$, with coefficients $[C_{ij}] = \{\psi_i\}_j$, and write the DtN matrix in the form of a generalized rank-N_T update,

$$\boldsymbol{K}_\Gamma = -\boldsymbol{M}_\Gamma\, \boldsymbol{C}\, \boldsymbol{D}\, \boldsymbol{C}^T \boldsymbol{M}_\Gamma, \tag{22}$$

where $\boldsymbol{D}$ is a $N_T \times N_T$ diagonal matrix of impedance coefficients, $Z_j \leftarrow Z_{mn}/\pi N_{mn}$, organized from $j = 1, 2, \ldots, N_T$. Using this construction, we compute the matrix-vector product of the DtN matrix $\boldsymbol{v} = \boldsymbol{K}_\Gamma\, \boldsymbol{u}$, efficiently in block form using Level-2 BLAS and the following algorithm: 1. Set $\boldsymbol{u} = \boldsymbol{M}_\Gamma \boldsymbol{u}$, 2. Set $\boldsymbol{w} = \boldsymbol{C}^T \boldsymbol{u}$, 3. Set $\boldsymbol{w} = \boldsymbol{D}\boldsymbol{w}$, 4. Set $\boldsymbol{v} = \boldsymbol{C}\boldsymbol{w}$, 5. Set $\boldsymbol{v} = \boldsymbol{M}_\Gamma \boldsymbol{v}$. Thus, we replace the assembly and matrix-vector product of the full matrix $\boldsymbol{K}_\Gamma$, of size $N_\Gamma \times N_\Gamma$, with two matrix-vector multiplies of reduced size, diagonal scaling, and two sparse matrix-vector multiplies with $\boldsymbol{M}_\Gamma$. In this form, the storage requirements and number of operations are reduced to $\mathrm{O}(N_{dof} + N_T\, N_\Gamma)$. Since $N_T \ll N_\Gamma$, the storage and cost is considerably lower than a straightforward matrix-vector product. requiring storage and number of operations of $\mathrm{O}(N_{dof} + N_\Gamma^2)$.

To accelerate convergence we use SSOR preconditioning with Eisenstat's trick. Following the ideas in [3], we construct the SSOR preconditioner using the sparse matrix $\boldsymbol{K}_S$, which is stored and split into, $\boldsymbol{K}_S = \boldsymbol{L} + \boldsymbol{\Delta} + \boldsymbol{L}^T$, where $\boldsymbol{\Delta} := \mathrm{diag}\{\boldsymbol{K}_S\}$ is a diagonal matrix and $\boldsymbol{L}$ is a strictly lower triangular matrix. The preconditioner only involves the sparse matrix from the interior finite element mesh, and the local part of the DtN map, yet provides a very good approximation to the complete matrix $\boldsymbol{K} = \boldsymbol{K}_S + \boldsymbol{K}_\Gamma$. The SSOR preconditioner is defined by $\boldsymbol{P} = \mathcal{L}\,\boldsymbol{\Delta}^{-1}\,\mathcal{L}^T$, where $\mathcal{L} = (\boldsymbol{\Delta} + \omega \boldsymbol{L})$. Here $0 < \omega < 2$, is a relaxation parameter. With $\omega \to 0$, the SSOR preconditioner reduces to the diagonal (Jacobi) preconditioner, i.e., $\boldsymbol{P} = \boldsymbol{\Delta}$. Using this sparse SSOR preconditioner, we solve the modified system,

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{A} = \boldsymbol{\Delta}^{1/2}\mathcal{L}^{-1}\,(\boldsymbol{K}_S + \boldsymbol{K}_\Gamma)\mathcal{L}^{-\mathbf{T}}\boldsymbol{\Delta}^{1/2} \tag{23}$$

where $\boldsymbol{x} = \boldsymbol{\Delta}^{1/2}\mathcal{L}^{-\mathbf{T}}\mathbf{d}$, $\boldsymbol{b} = \boldsymbol{\Delta}^{1/2}\mathcal{L}^{-\mathbf{1}}\mathbf{f}$. Iterative solution of the preconditioned system (23) requires a matrix-vector product $\boldsymbol{v} = \boldsymbol{A}\,\boldsymbol{u}$, where $\boldsymbol{A}$ is the complex-symmetric preconditioned matrix and $\boldsymbol{u}$ is an iteration vector. Here the matrix-vector product $\boldsymbol{v} = \boldsymbol{A}\,\boldsymbol{u}$, is computed efficiently using Eisenstat's trick, and the special structure of $\boldsymbol{K}_\Gamma$ written in the matrix outer-product form described earlier,

$$\boldsymbol{A} = \frac{1}{\omega}\boldsymbol{\Delta}^{1/2}\left\{\mathcal{L}^{-T} + \mathcal{L}^{-1}(\boldsymbol{I} + [(\omega - 2)\boldsymbol{\Delta} - \omega \boldsymbol{M}_\Gamma\, \boldsymbol{C}\, \boldsymbol{D}\, \boldsymbol{C}^T \boldsymbol{M}_\Gamma]\mathcal{L}^{-T})\right\}\boldsymbol{\Delta}^{1/2}. \tag{24}$$

Eisenstat's trick with SSOR replaces the matrix-by-vector multiply with $\boldsymbol{K}_S$, with two matrix-vector multiplies with diagonal matrix $\boldsymbol{\Delta}^{1/2}$, and two efficient back-solves with triangular matrix $\mathcal{L}$; a significant reduction in the number of required operations. The storage and cost of computing the matrix-vector multiply $\boldsymbol{K}_\Gamma\,\hat{\boldsymbol{u}}$ is reduced from $\mathrm{O}(N_\Gamma^2)$ to $\mathrm{O}(N_T\, N_\Gamma)$ by exploiting the product decomposition of $\boldsymbol{K}_\Gamma$.

NUMERICAL EXAMPLE

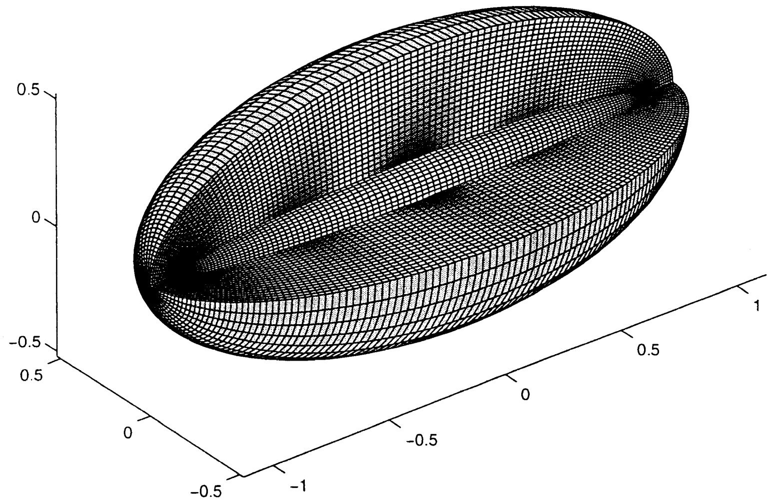

Fig. 2: Scattering of a plane wave from a prolate spheroid defined by aspect ratio of major to minor axes of 10:1. The wave travels from right-to-left along the axis of symmetry $\phi^{(i)} = e^{-ikz}$. The radiation boundary is positioned at $a_0/a_1 = 1.2$. Solution contours for the real-part of the finite element solution using the modified spheroidal DtN condition MDtN(B2) with $N = 15$, and normalized wavenumber $c = kf = 2\pi$.

Fig. 2 shows a representative finite element mesh used to model scattering from a 'soft' prolate spheroid. The computational domain Ω is discretized with a uniform mesh of standard 8-node trilinear 'brick' elements. For the example problem, we use a finer mesh of $[40 \times 480 \times 80]$ evenly spaced elements in $[0.1 \leq \mu \leq 0.5,\ 0 \leq \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi]$, respectively. A spiral of wedge elements is used at the poles. The model has over 1.5 million total nodes with $N_\Gamma = 38,242$ nodes on Γ. Solutions obtained using the baseline DtN is denoted 'DtN', while the first and second modified conditions are denoted 'MDtN(B1)' and 'MDtN(B2)', respectively.

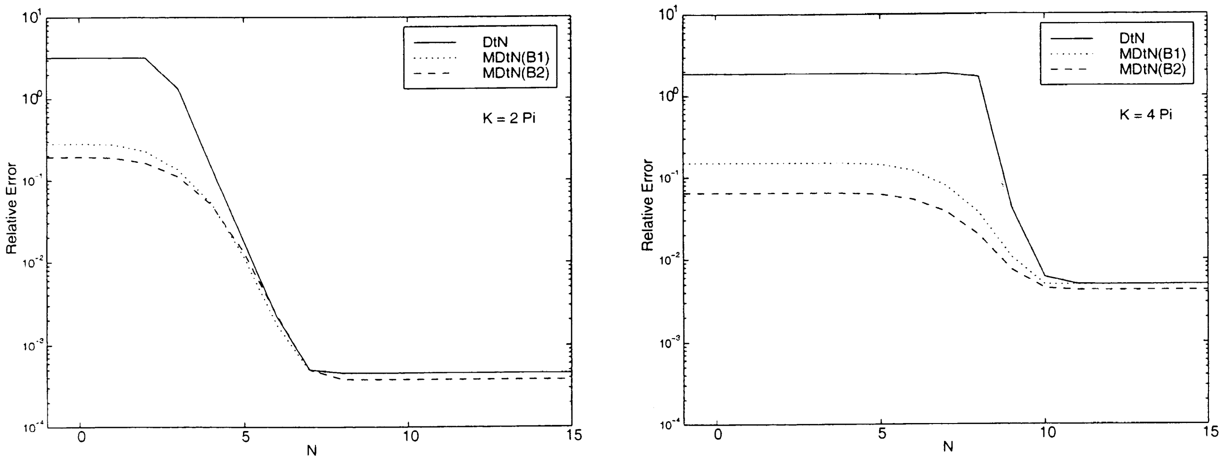

Fig. 3: L_2 error vs. number of harmonics $N-1$, included in the DtN map. Modified DtN with B_2 is more accurate. Number of modes N needed for accurate solution increases with frequency.

CONCLUSIONS

Modified DtN boundary conditions based on first B_1, and second B_2, local boundary operators are derived for spheroidal boundaries in a form suitable for finite element implementation. The conditions are exact for N spheroidal harmonics on the radiation boundary.

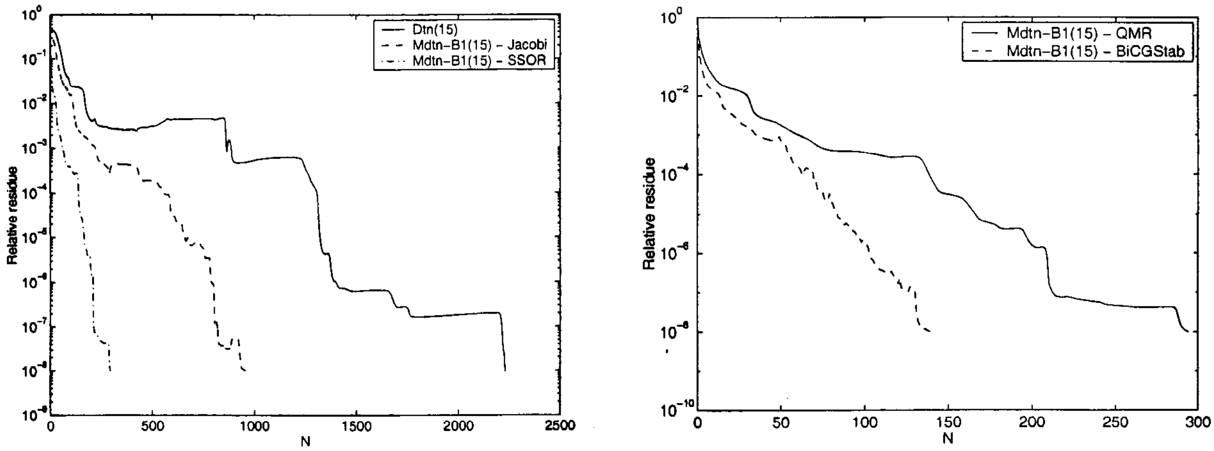

Fig. 4: Iterative performance ($c = 2\pi$). (Left): Comparison of DtN, and MDtN conditions with Jacobi and SSOR preconditioning using quasi-minimal residual (QMR) solver [4]. Results show significant speedup with SSOR preconditioning. (Right): Convergence is accelerated with biconjugate gradient stabilized (BiCG-Stab) solver [5], compared to QMR. Both methods show almost monotonically decreasing residuals. Wall clock times less than 45 minutes using 300 Mhz UltraSparc cpu.

Modifications are more accurate than the baseline DtN map, yet require no extra memory and little extra cost. The number of harmonics included in the modified DtN map can be increased as needed to accurately model high-frequency scattering. The special structure of the discrete DtN map as a rank-m update allows for efficient matrix-vector products in block form using Level-2 BLAS, without assembling the full DtN matrix. SSOR preconditioning based on the local operators for spheroidal boundaries, together with Eisenstat's trick, provide an efficient means to accelerate convergence of Krylov-subspace iterative solution methods for the resulting complex-symmetric equation system. Iterative results show significant speedup with SSOR preconditioned BiCG-Stab compared to QMR.

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