SEVENTH INTERNATIONAL CONGRESS
ON SOUND AND VIBRATION
4-7 July 2000, Garmisch-Partenkirchen, Germany

ACCURATE NON-REFLECTING BOUNDARY CONDITIONS FOR TIME-DEPENDENT ACOUSTIC SCATTERING

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Abstract

Accurate radiation boundary conditions for the time-dependent wave equation are formulated in the finite element method as an auxiliary problem for each radial harmonic on a spherical boundary. The method is based on residuals of an asymptotic expansion for the time-dependent radial harmonics. A decomposition into orthogonal transverse modes on the spherical boundary is used so that the residual functions may be computed efficiently and concurrently without altering the local/sparse character of the finite element equations. The method has the ability to vary separately, and up to any desired order, the radial and transverse modal orders of the radiation boundary condition. With the number of equations in the auxiliary Cauchy problem equal to the transverse mode number, the conditions are exact. If fewer equations are used, then the boundary conditions form high-order accurate asymptotic approximations to the exact condition, with corresponding reduction in work and memory. Numerical studies are performed to assess the accuracy and convergence properties of the radiation boundary conditions. The results demonstrate dramatically improved accuracy for time domain simulations compared to standard boundary treatments and improved efficiency over the exact condition.

INTRODUCTION

Hagstrom and Hariharan [1] have derived a sequence of radiation boundary conditions involving first-order differential equations in time and tangential derivatives of auxiliary functions on a circular or spherical boundary. They indicate how these local conditions may be effectively implemented in a finite difference scheme using only local tangential operators, but at the cost of introducing a large number of auxiliary functions at the boundary. In this paper we rederive the sequence of local boundary conditions described in [1] based on the hierarchy of local boundary operators used by Bayliss and Turkel [2] and a recursion relation for the expansion coefficients appearing in a radial asymptotic (multipole) expansion. We then reformulate in terms of spherical harmonics by using a decomposition into orthogonal transverse modes. The resulting procedure then involves a Cauchy problem involving systems of first-order temporal equations, similar to that used in [3,4]. With this reformulation, the auxiliary functions are recognized as residuals of the local boundary operators acting on the asymptotic expansion, and may be implemented efficiently with standard semidiscrete finite element methods without changing the symmetric and sparse structure of the matrix equations.

CONSTRUCTION OF RADIATION BOUNDARY CONDITIONS

We consider time-dependent scattering in an infinite three-dimensional region $\mathcal{R} \subset \mathbb{R}^3$, surrounding an object with surface $\mathcal{S}$. For computation, the unbounded region $\mathcal{R}$ is truncated by an artificial spherical boundary Γ, of radius $\| \boldsymbol{x} \| = R$, see Figure 1.

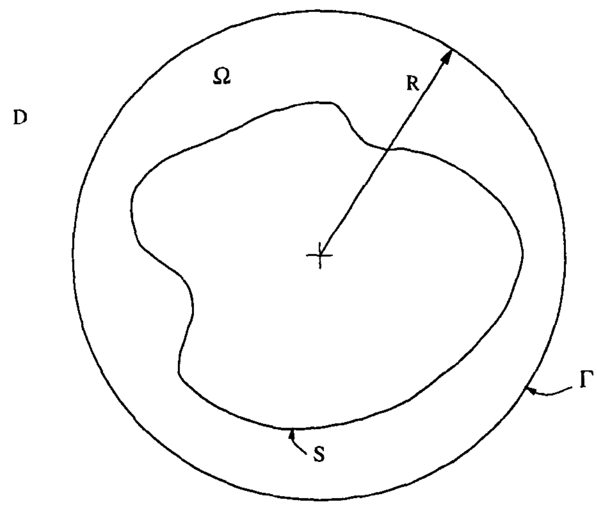


Fig. 1: Illustration of unbounded region $\mathcal{R}$ surrounding a scatterer $\mathcal{S}$. The computational domain $\Omega \subset \mathcal{R}$ is surrounded by a spherical truncation boundary Γ of radius R, with exterior region $\mathcal{D} = \mathcal{R} - \Omega$.

Within Ω, the solution $\phi(\boldsymbol{x}, t) : \Omega \times [0, T] \mapsto \mathbb{R}$, satisfies the scalar wave equation,

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi \qquad \boldsymbol{x} \in \Omega, \; t \in [0, T] \tag{1}$$

$$\phi(\boldsymbol{x}, 0) = \phi_o(\boldsymbol{x}), \qquad \dot{\phi}(\boldsymbol{x}, 0) = \dot{\phi}_o(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \tag{2}$$

and driven by the time-dependent radiation boundary condition on the surface $\mathcal{S}$:

$$\frac{\partial \phi}{\partial n} = g(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \mathcal{S}, \; t \in [0, T] \tag{3}$$

In linear acoustics, the scalar function ϕ may represent the pressure field or a velocity potential. The wave speed is assumed $c > 0$, and real. The initial data ϕ_o and $\dot{\phi}_o$ are assumed to be confined to the computational domain Ω, so that in the exterior region $\mathcal{D} = \mathcal{R} - \Omega$, i.e., the infinite region outside Γ, the scalar field $\phi(\boldsymbol{x}, t)$ satisfies the homogeneous form of the wave equation. In spherical coordinates (r, θ, φ), the external region is defined as, $\mathcal{D} = \{r \geq R\,, \, 0 \leq \theta \leq \pi\,, \, 0 \leq \varphi < 2\pi\}$, and the wave equation takes the form,

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\Delta_\Gamma \phi, \quad \Delta_\Gamma \phi = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2}. \tag{4}$$

The general solution to (4) is given by the spherical harmonic expansion,

$$\phi(r, \theta, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \phi_{nm}(r\,,\, t)\; Y_{nm}(\theta, \varphi) \tag{5}$$

where Y_{nm} are orthogonal spherical harmonics normalized on a unit sphere:

$$Y_{nm}(\theta,\varphi) \;=\; \frac{1}{\sqrt{N_{nm}}}\,P_n^{|m|}(\cos\theta)\,e^{im\varphi}, \quad N_{nm} = \frac{4\pi(n\,+\,|m|)!}{(2n\,+\,1)(n-|m|)!} \tag{6}$$

The time-dependent modes,

$$\phi_{nm}(r,t) = \int_0^{2\pi}\int_0^{\pi} Y^*_{nm}(\theta,\varphi)\,\phi(r,\theta,\varphi,t)\,\sin\theta\,d\theta\,d\varphi, \tag{7}$$

satisfy the radial wave equation,

$$\frac{1}{c^2}\frac{\partial^2\phi_{nm}}{\partial t^2} \;=\; \left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{n(n+1)}{r^2}\right]\phi_{nm}, \qquad r\ge R,\; t\ge 0 \tag{8}$$

For outgoing waves, the solution to (8) may be represented by the radial asymptotic (multipole) expansion:

$$\phi_{nm}(r,t) = \sum_{k=0}^{n} r^{-k-1}\,\phi^k_{nm}(r-ct) \tag{9}$$

Substituting (9) into (8), we obtain the recursion relation for derivatives of the wave functions appearing in the radial harmonic expansion:

$$\left(\phi^k_{nm}\right)' = \frac{k(k-1)-n(n+1)}{2k}\,\phi^{k-1}_{nm}, \qquad k = 1,2,\cdots,n \tag{10}$$

We rederive the local radiation boundary conditions of Hagstrom and Hariharan [1], and then reformulate in terms of spherical harmonics. Here, we use the hierarchy of local operators of Bayliss and Turkel [2],

$$B_p = L_p(L_{p-1}(\cdots(L_2(L_1)))), \quad L_j = \left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\frac{2j-1}{r}\right) \tag{11}$$

We interpret the residuals of the operators (11) acting on the asymptotic expansion (9) as a sequence of functions with reduced radial order. We apply $B_1 = L_1$ to the radial expansion (9), with the result,

$$B_1\,\phi_{nm} = \left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\frac{1}{r}\right)\phi_{nm} = w^1_{nm} \tag{12}$$

$$w^1_{nm}(r,t) = \sum_{k=1}^{n} -k\,r^{-k-2}\phi^k_{nm} \tag{13}$$

The function w^1_{nm} defines the remainder of the radial expansion. ith error $O(R^{-5})$ for $n \ge 2$. In general, applying B_{j+1} to (9), we have by induction,

$$B_{j+1}\,\phi_{nm} = L_{j+1}(w^j_{nm}) = \left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\frac{2j+1}{r}\right)w^j_{nm} = w^{j+1}_{nm} \tag{14}$$

where the residual of B_j is defined as,

$$w^j_{nm}(r,t) = \sum_{k=j}^{n} a^j_k \, r^{-k-j-1} \phi^k_{nm}, \quad a^j_k = (-1)^j \frac{k!}{(k-j)!}. \tag{15}$$

The order of the residuals are reduced, $w^{j+1}_{nm} = O(r^{-2}) w^j_{nm}$, $w^j_{nm}(r,t) = O(r^{-2j})\phi_{nm} = O(r^{-2j-1})$, and $B_j \phi_{nm} = 0$ is exact for harmonics $n < j$. For $j = 1, 2, \cdots, p_n$, using the recursion relation for $(\phi^k_{nm})'$ given in (10), and the definition for a^j_k, we eliminate radial derivatives in (14) in favor of a recursive sequence for w^j_{nm}, with the result,

$$\frac{r}{c}\frac{\partial y^j_{nm}}{\partial t} = \frac{1}{4}[j(j-1) - n(n+1)]\, y^{j-1}_{nm} - j y^j_{nm} + y^{j+1}_{nm} \tag{16}$$

where $y^j_{nm}(r,t) = 2^{1-j} r^j w^j_{nm}$, and $y^0_{nm} = \phi_{nm}$. Applying the spherical harmonic expansion to (12) and (16), and making use of the eigenvalues for the spherical harmonics Y_{nm}, we obtain,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\right)\phi = r\, y_1 \tag{17}$$

$$\left(\frac{r}{c}\frac{\partial}{\partial t} + j\right) y_j = \frac{1}{4}[j(j-1) + \Delta_\Gamma] y_{j-1} + y_{j+1} \tag{18}$$

where

$$y_j(r,\theta,\varphi,t) = 2^{1-j} r^j \sum_{n\geq 0} \sum_{|m|\leq n} w^j_{nm}(r,t)\, Y_{nm}(\theta,\varphi) \tag{19}$$

for j = 1,2,..., p, and $y_0 = 2\phi, y_{p+1} = 0$. Applied to the radiation boundary $r = R$, (17), together with the sequence of p equations (18), is a scaled version of the sequence of local conditions involving tangential derivatives derived by Hagstrom and Hariharan [1]. If the solution ϕ contains only N harmonics, then with $p = N$ auxiliary functions, the radiation boundary condition is exact.

Here, instead of solving the sequence of boundary conditions for the auxiliary functions $y_j(R,\theta,\varphi,t)$ at points (θ,φ), on Γ, we opt to reformulate them as harmonics. First, we recognize that when evaluated on the artificial boundary at $r = R$, the sequence (16) forms a system of first-order ordinary differential equations in time for the auxiliary functions, $v^j_{nm}(t) = y^j_{nm}(R,t) = 2^{1-j} R^j\, w^j_{nm}(R,t)$. Let $\boldsymbol{v}_{nm}(t) = \{2^{1-j} R^j\, w^j_{nm}(R,t)\}$, $j = 1, 2, \cdots p_n$, and define a time-dependent vector function of order p_n,

$$\boldsymbol{v}_{nm}(t) = [v^1_{nm}(t)\,,\, v^2_{nm}(t)\,,\, \cdots\,,\, v^{p_n}_{nm}(t)]^T \tag{20}$$

then the first-order system of equations may be written as a matrix differential equation for each spherical harmonic similar to the Cauchy problem appearing in [3,4]:

$$\frac{d}{dt}\boldsymbol{v}_{nm}(t) = \boldsymbol{A}_n\, \boldsymbol{v}_{nm}(t) \;+\; \boldsymbol{b}_n\, \phi_{nm}(R,t) \tag{21}$$

Here, the constant $p_n \times p_n$, tri-diagonal matrix $\boldsymbol{A}_n = \{A^{ij}_n\}$, is defined with band:

$$\boldsymbol{A}_n = \frac{c}{R}\, \mathrm{B}\left[\frac{1}{4}(n+i)(i-n-1)\,,\, -i\,,\, 1\right] \tag{22}$$

The constant vector $\boldsymbol{b}_n = \{b_n^j\}$ is defined by

$$\boldsymbol{b}_n = -\frac{n(n+1)c}{2R}[1\,,\,0\,,\,\dots\,,\,0]^T \tag{23}$$

Taking the spherical harmonic expansion of (12), i.e., multiplying by Y_{nm}, summing over n and m, and evaluating on the truncation boundary at $r = R$, gives the new boundary condition,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\right)\phi = R\sum_{n=1}^{\infty}\sum_{m=-n}^{n} v_{nm}^1(t)\,Y_{nm}(\theta,\varphi), \qquad \text{on } \Gamma \tag{24}$$

where the component $v_{nm}^1(t)$ satisfies the Cauchy problem for each harmonic defined by the first-order matrix system (21), with initial condition $\boldsymbol{v}_{nm}(0) = 0$, and driven by the radial modes $\phi_{nm}(R,t)$, defined by the spherical harmonic transform (7), evaluated at $r = R$.

The auxiliary functions in (21) satisfy the property, $v_{nm}^{j+1} = O(R^{-2})v_{nm}^j$, so that $v_{nm}^{j+1} < v_{nm}^j$. Thus for accurate solutions, it is sufficient to use a radial modal order which is less than the angular modal order, i.e. $p_n < n$. In this case the boundary condition (24) forms a *uniform asymptotic approximation* to the exact condition. Computation of eigenvalues for the first-order system of residual functions verified solutions are stable [5]. We denote the truncated boundary condition (24) by RBC1(N,P), where N defines the number of terms included in the truncated series, and $P \leq N$ defines the maximum number of equations, included in the Cauchy problem (21). In general, do to the rapid convergence of the functions v_{nm}^j, then $P \ll N$ is sufficient for accurate solutions, with corresponding reduction in storage and work. When $P = N$, the boundary condition RBC1(N,N) is exact for modes $n \leq N$, with $O(R^{-3})$ error for modes $n > N$.

In [5], a linear transformation is used to established the equivalence of our exact version of the RBC to the non-reflecting boundary conditions (NRBC) derived in [3,4]. The analysis in [5], provides a straightforward derivation of the NRBC with a clear physical interpretation of the auxiliary functions, interpreted as wave functions appearing in the multipole expansion. Several improvements of our radiation boundary condition over the NRBC derived in [3,4] have been identified including a banded tri-diagonal coefficient matrix for the auxiliary variables, reduced memory and computational work needed to store and solve the auxiliary functions for each harmonic, and the ability to to vary separately the radial and transverse modal orders of the radiation boundary condition. Furthermore, using asymptotic radial wave expansions and Fourier modes similar to that used in this paper for the three-dimensional wave equation, we have developed an efficient and accurate formulation on a circle in two-dimensions [6] – a result not possible with the approach used in [3,4].

FINITE ELEMENT FORMULATION

The statement of the weak form for the initial-boundary value problem in the computational domain Ω may be stated as: Given g , c, find $\phi(\boldsymbol{x},t)$ in $\Omega \cup \partial\Omega$, such that for all admissible weighting functions $\delta\phi$, the following variational equation is satisfied,

$$(\delta\phi\,,\,\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2})_\Omega + (\delta\phi\,,\,\frac{1}{c}\frac{\partial\phi}{\partial t})_\Gamma + K(\delta\phi\,,\,\phi) = F(\delta\phi) \tag{25}$$

$$K(\delta\phi\,,\,\phi) \;:=\; \int_{\Omega} \nabla\delta\phi \cdot \nabla\phi \; d\Omega \,+\, \frac{1}{R}\int_{\Gamma} \delta\phi\,\phi\; d\Gamma \tag{26}$$

$$F(\delta\phi) \;:=\; \int_{S} \delta\phi\, g\; dS \,+\, R\sum_{n=1}^{\infty}\sum_{m=-n}^{n} v_{nm}^{1} \int_{\Gamma} \delta\phi\, Y_{nm}\; d\Gamma \tag{27}$$

Using a standard Galerkin semi-discrete approximation, $\phi(\boldsymbol{x},t) \approx \phi^h(\boldsymbol{x},t) = \boldsymbol{N}(x)\,\boldsymbol{\phi}(t)$, leads to the system of second-order ordinary differential equations in time:

$$\boldsymbol{M}\ddot{\boldsymbol{\phi}}(t) \,+\, \boldsymbol{C}\dot{\boldsymbol{\phi}}(t) \,+\, \boldsymbol{K}\boldsymbol{\phi}(t) = \boldsymbol{F}(t), \qquad t > 0 \tag{28}$$

In the above, $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ are standard arrays associated with the discretization of the wave equation and the local B_1 operator; and $\boldsymbol{F}(t)$ is the discrete force vector composed of a standard load vector and a part associated with the auxiliary functions appearing in the radiation boundary condition. Here, we approximate the harmonics $Y_{nm}(\theta,\varphi)$ by a projection onto the finite-dimensional basis, $P_n^m(\cos\theta)\,\cos m\varphi \approx \boldsymbol{N}(\theta,\varphi)\,\boldsymbol{\psi}_{nm}^{c}$, where $\boldsymbol{\psi}_{nm} = \{Y_{nm,l}\}$, $l = 1\,,\,2\,,\,\cdots\,,\,N_\Gamma$, is a vector containing the nodal values of the harmonic defined by (n,m) on Γ, i.e., $\psi_{nm,l} = Y_{nm}(\theta_l\,,\,\varphi_l)$. A similar expression is used for the odd functions. Using this expansion in (27) we have,

$$\boldsymbol{F}_\Gamma(t) = R\sum_{n=1}^{N}\sum_{m=0}^{n}{}'\; \boldsymbol{M}_\Gamma\,[\,\boldsymbol{\psi}_{nm}^{c}\,v_{nm,1}^{c}(t) \,+\, \boldsymbol{\psi}_{nm}^{s}\,v_{nm,1}^{s}(t)\,] \tag{29}$$

The functions $v_{nm,1}^{c}$ and $v_{nm,1}^{s}$ are the first element of the vector arrays $\boldsymbol{v}_{nm}^{c} = \{v_{nm,j}^{c}\}$, and $\boldsymbol{v}_{nm}^{s} = \{v_{nm,j}^{s}\}$, which satisfy the system of first-order differential equations (21) driven by the even and odd radial modes at $r = R$. For the even modes,

$$\phi_{nm}^{c}(R,t) = \frac{2}{RN_{nm}}\;\boldsymbol{\psi}_{nm}^{cT}\,\boldsymbol{M}_\Gamma\,\boldsymbol{\phi}_\Gamma(t) \tag{30}$$

In the above, $\boldsymbol{\phi}_\Gamma(t) = \{\phi_I(t)\}$, $I = 1,2,\cdots,N_\Gamma$, is a vector of nodal solutions on the artificial boundary Γ with N_Γ nodes. $\boldsymbol{M}_\Gamma$ is a symmetric $N_\Gamma \times N_\Gamma$ matrix defined by the inner-product, $(\boldsymbol{N}\,,\,\boldsymbol{N})_\Gamma$. This matrix may be diagonalized using nodal (Lobatto) quadrature to reduce cost/memory. This implementation does not disturb in any way the symmetric, and sparse structure of the finite element matrix equations (28). As discussed in [4], one time-integration approach is to apply the central difference method directly to (28). This explicit method requires only the forcing term $\boldsymbol{F}^k = \boldsymbol{F}(t_k)$ at time step $t_k = k\Delta t$. Therefore, to update the solution $\boldsymbol{d}^{k+1} = \boldsymbol{\phi}(t_{k+1})$, only the evaluation of $\boldsymbol{v}_{nm}^{k} = \boldsymbol{v}_{nm}(t_k)$ is needed. To numerically solve (21), either the explicit second-order Adams-Bashforth method or the the implicit second-order Adams-Moulton method (trapezoidal rule) may be used. The computational work required in solving is negligible, since the matrices $\boldsymbol{A}_n$, are tridiagonal, relatively small (usually $N \leq 25$), and remain constant. When $p_n < n$, the work is further reduced. Complete algorithms for computing the solution concurrently with auxiliary functions on Γ, using either implicit or explicit time-integrators, are given in [4].

NUMERICAL EXAMPLES

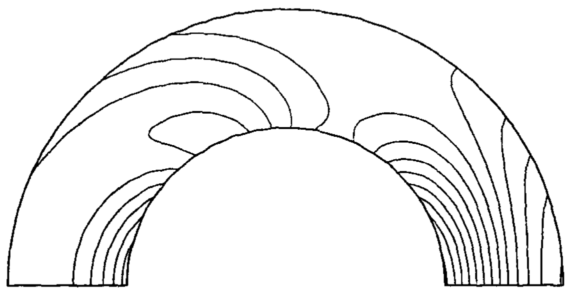


Fig. 2: Scattering from a sphere with wave incident from the ($\theta = \pi$) direction, and normalized frequency $\omega a/c = \pi$. Solution contours at steady-state (t = 15), using RBC1(10,10) and $R/a = 1.75$

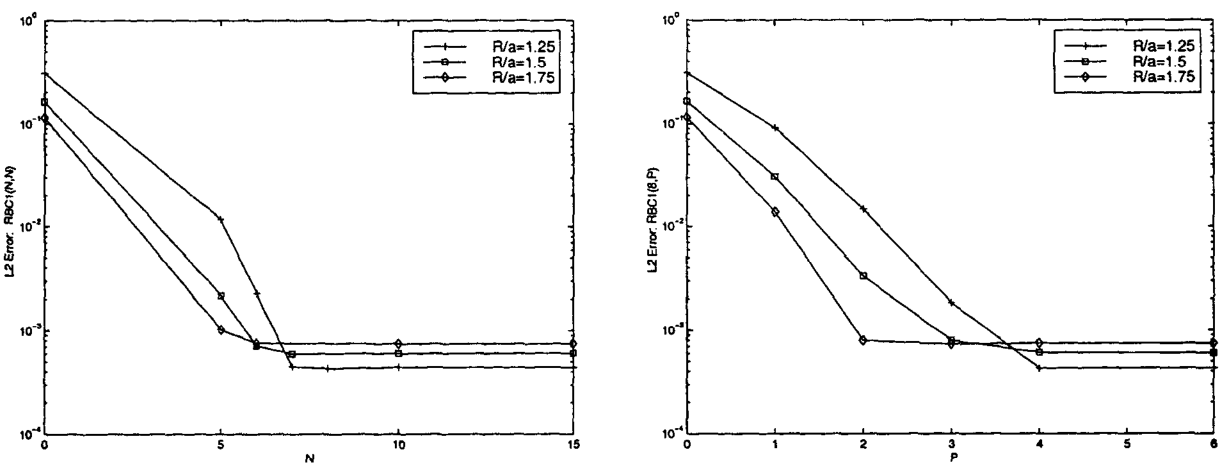


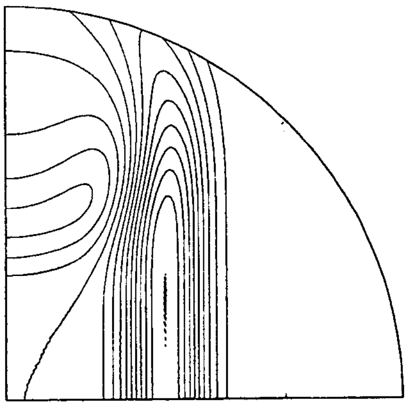
Fig. 3: Maximum L_2 error during steady-state measured at $r/a = 1.25$. Radiation boundary condition applied at truncation boundary Γ postioned at $R/a = 1.25$, 1.5, 1.75. Numerical solutions using (Left): RBC1(N, N), (Right): RBC1($8, P$). As the radiation boundary is moved further away from the source, the number of modes N required to obtain a fixed level of accuracy is reduced. The uniform approximation to the exact condition is sufficiently accurate with $N/3 \leq P \leq N$.

In the first example, we study plane-wave scattering from a rigid sphere of radius $r = a$, The computation is driven from rest to steady-state with a time step $\Delta t = 0.01$. Contours for the scattered solution are shown in Figure 2. Figure 3 shows the maximum L_2 error during steady-state.

In the second example, we solve for transient radiation in a semi-infinite region defined by circular transducer of radius $a = 1$ mounted in an infinite rigid planar baffle. The normal velocity on the transducer surface is a Gaussian pulse. Figure 4 shows the pressure field solution.

CONCLUSIONS

Asymptotic and exact local radiation boundary conditions first derived by Hagstrom and Hariharan for the time-dependent wave equation, are rederived based on the hierarchy of local boundary operators used by Bayliss and Turkel and a recursion relation for the expansion coefficients appearing in the asymptotic (multipole) expansion for radial wave harmonics.



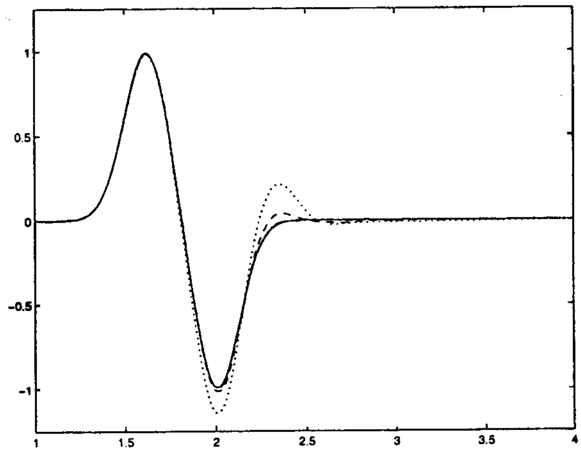


Fig. 4: (Left) Solution contours of pressure field using RBC1(20,4), $t = 0.9$. The solution obtained using RBC1(20,4) is nearly identical to the analytical solution at all observation points. (Right) Time-history on-axis at $\theta = 0$, $R = 1.125$. Solid line denotes analytic solution; Dotted line denotes RBC1(20,0); Dashed line denotes RBC(20,1); Dashed-Dotted line denotes RBC1(20,3). The solution for RBC1(20,0) and RBC1(20,1) exhibits large errors; both overshooting and undershooting the exact solution. The RBC1(20,3) condition matches the analytical solution well. Accurate solutions are obtained with the asymptotic form of RBC1(N,P). In this case it is sufficient to use a small number ($P = 3$) of auxiliary functions.

By introducing a decomposition into spherical harmonics we reformulate the sequence of local boundary conditions as a Cauchy problem involving systems of first-order temporal equations, similar to that used in [3,4]. The use of spherical harmonics allows the boundary conditions to be implemented efficiently and concurrently without altering the local character of the finite element equations. The accuracy rapidly converges with the number of residual functions included in the Cauchy problem.

Support for this work was provided by the National Science Foundation under Grant CMS-9702082 in conjunction with a Presidential Early Career Award for Scientists and Engineers (PECASE).

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