Chapter 2

Coupled Problem Statement

In this work, the exterior structural acoustics problem involving solution of the coupled wave equation defined over an infinite domain is transformed to a coupled fluid-structure interaction problem defined over a finite computational domain through the introduction of an artificial fluid truncation boundary. For uniqueness, radiation boundary conditions are prescribed on the fluid truncation boundary in the form of a linear operator which approximates the asymptotic behavior of the solution at infinity. The coupled system for the exterior structural acoustics problem is composed of four distinct components:

(i) the structure,
(ii) the acoustic fluid,
(iii) structure-fluid interface conditions, and
(iv) non-reflecting boundary conditions.

In the following, the governing differential equations for each of the four components are described. After collecting these equations, the strong (local) form of the fluid-structure initial/boundary-value problem is stated for structural acoustics. In the remainder of this chapter, a number of important energy estimates for the coupled problem are derived. These results play a central role in the analysis of stability for both the continuum and discrete problems and form the basis for the design of the space-time finite element methods presented in this thesis.

The coupled fluid-structure system is illustrated in Figure 2.1, and consists of

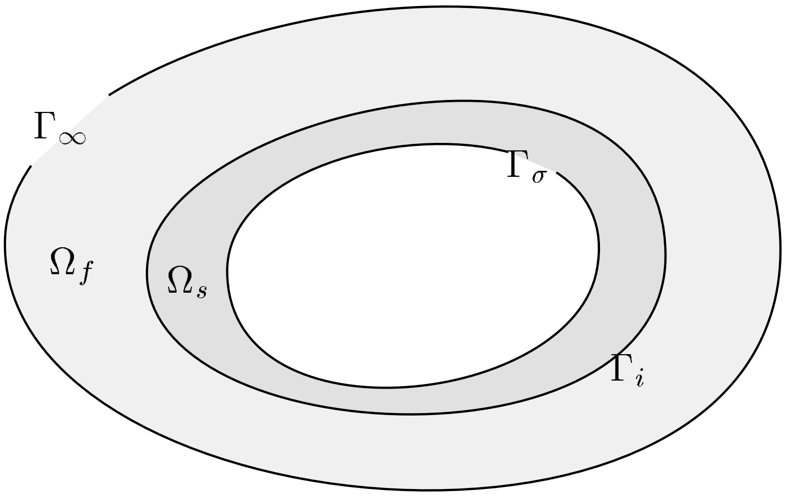

Figure 2.1: Coupled system for the exterior fluid-structure interaction problem, with artificial boundary Γ_∞ enclosing the finite computational domain $\Omega = \Omega_f \cup \Omega_s$.

an artificial truncation boundary Γ_∞ enclosing a finite computational domain Ω. The finite computational domain is composed of the fluid domain Ω_f, which in turn surrounds the structural domain Ω_s such that $\Omega = \Omega_f \cup \Omega_s$. The fluid boundary $\partial\Omega_f$, is divided into a fluid-structure interface boundary Γ_i, and an artificial boundary Γ_∞. The structural boundary $\partial\Omega_s$, is composed of the shared fluid-structure interface boundary Γ_i and a traction boundary Γ_σ. The infinite domain outside the artificial boundary and extending to infinity is denoted by Ω_∞. In this thesis, the domain interior to the elastic body is assumed to be in vacuo, although the inclusion of an interior fluid can be accommodated with no special difficulty. In the following, the equations governing the structure, the acoustic fluid, and their interaction are summarized.

2.1 Governing Equations for Structural Acoustics

The governing equations for the structure are stated for a linear solid continuum without the explicit imposition of kinematic constraints common to structural models such as plates and shells. In this way, full generality of the formulation is maintained in order to accommodate a wide class of shell models into the formulation. The acoustic fluid is modeled under the usual assumptions governing sound wave propagation; a compressible and inviscid fluid linearized about an equilibrium state with constant density and velocity. The fluid is assumed to be homogeneous outside the computational domain Ω_∞, but may be inhomogeneous within Ω_f. In this work the density of the fluid is assumed comparable to the inertial and elastic forces in the structure so that significant interaction exists.

2.2 The Structure

Consider an elastic body occupying the bounded domain $\Omega_s \in \mathbb{R}^d$, where $d = 1$, 2 or 3 is the number of space dimensions. The structural displacement vector is denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{x} \in \overline{\Omega}_s := \Omega_s \cup \partial\Omega_s$, and $t \in I =]0, T[$, is the time interval of interest of length $T > 0$. The equations of motion for the structure are governed by the linear momentum balance, which can be stated as,

$$\boxed{\nabla \cdot \boldsymbol{\sigma} \quad = \quad \rho_s \ddot{\boldsymbol{u}} \qquad \text{in} \quad Q_s := \Omega_s \times]0\,,\, T[} \tag{2.1}$$

where $\rho_s = \rho_s(\boldsymbol{x}) > 0$ is the structural density, a superposed dot indicates partial differentiation with respect to time t, and $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor which is related to the displacements through the constitutive equation,

$$\boldsymbol{\sigma} \quad = \quad \boldsymbol{C} : \boldsymbol{\varepsilon}[\boldsymbol{u}] \tag{2.2}$$

In this expression, the strain tensor is denoted by $\boldsymbol{\varepsilon}[\boldsymbol{u}] = \nabla^s \boldsymbol{u}$, where $\nabla^s \boldsymbol{u}$ is the symmetric gradient of the displacement vector. The elastic coefficients $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{x})$ are assumed to satisfy the usual positive-definiteness (pointwise stability) and major and minor symmetry assumptions for an elastic solid continuum. Pointwise stability implies that,

$$C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \;\geq\; \gamma||\boldsymbol{\varepsilon}||^2_{\Omega_s} \tag{2.3}$$

for some constant $\gamma > 0$. By the minor symmetries, $\boldsymbol{\sigma}$ is symmetric and depends only upon the symmetric gradient $\nabla^s \boldsymbol{u}$ of the displacement. Under the usual assumption of isotropic materials, the constitutive relation specializes to the generalized Hooke's law:

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \tag{2.4}$$

where λ and μ are the Lame constants, $\mathrm{tr}(\boldsymbol{\varepsilon})$ is the trace of the strain tensor and $\mathbf{1}$ is the second-order identity tensor. Other constitutive relationships can also be used, for example the visco-elastic relationships described in ABBOUD [1990].

With Γ_σ and Γ_i denoting non-overlapping subregions of $\partial\Omega_s$ such that $\partial\Omega_s = \overline{\Gamma_\sigma \cup \Gamma_i}$, prescribed tractions $\bar{\boldsymbol{t}}$, are enforced through the boundary condition,

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} \quad = \quad \bar{\boldsymbol{t}} \qquad \text{on} \;\; \Upsilon_\sigma := \Gamma_\sigma \times]0\,,\, T[\tag{2.5}$$

The specification of coupling conditions on the structure-fluid interface Γ_i are discussed later in Section 2.3.3. Here we note that two interface conditions are required: (i) a kinematic interface condition, and (ii) a traction interface condition. Prescribed displacements (i.e. essential boundary conditions) are set to zero for clarity of presentation, although the inclusion of non-homogeneous conditions is straightforward.

2.3 The Acoustic Fluid

The acoustic fluid is modeled under the usual assumptions governing the linearized theory of sound. Under the standard acoustic approximation, the motion of an inviscid, irrotational, and compressible fluid is regarded as a small perturbation from a uniform reference state. Linearizing the conservation equations of compressible inviscid flow around a reference state in which the fluid has a uniform density ρ_0 and is at rest, i.e.

$$\rho_0 = \text{const}\,,\; p_o = \text{const}\,,\; \boldsymbol{v}_0 = 0 \tag{2.6}$$

with products of small quantities neglected, the equations of momentum and of continuity for the acoustic fluid take the form (see e.g. ZAUDERER [1989]):

$$\rho_0 \dot{\boldsymbol{v}} + \nabla p \quad = \quad 0 \tag{2.7}$$

$$\dot{\rho} + \rho_0 \nabla \cdot \boldsymbol{v} \quad = \quad 0 \tag{2.8}$$

where p, ρ, and $\boldsymbol{v}$ denote perturbations of pressure, density and velocity from the reference state. These equations relate the local rates of change of velocity and density to the pressure gradient and to the velocity divergence, respectively. The momentum equation (2.7) expresses the inertial nature of the fluid, while the continuity equation (2.8) expresses the compressibility of the fluid. In this work, the relationship between changes in pressure and density are given by the linearized equation of state, $\dot{p} = c^2\dot{\rho}$, where $c > 0$ is the speed of sound (velocity of acoustic wave propagation).

2.3.1 First-order hyperbolic form

Combining (2.8) with the equation of state $\dot{p} = c^2\dot{\rho}$, results in a system of first-order hyperbolic equations in terms of pressure and velocity variables:

$$\rho_0 \dot{\boldsymbol{v}} + \nabla p = 0 \tag{2.9}$$

$$\dot{p} + \rho_0 c^2 \nabla \cdot \boldsymbol{v} = 0 \tag{2.10}$$

Defining the state vector as,

$$\boldsymbol{U} = (p\,,\ v_1\,,\ v_2\,,\ v_3)^T \tag{2.11}$$

where v_i are the velocity components in $\mathbb{R}^3$, these equations can be arranged in standard first-order hyperbolic form as,

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{i=1}^{3} \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0} \tag{2.12}$$

and $A_i \in \mathbb{R}^{4\times 4}$, $i = 1, \cdots 3$ are sparse unsymmetric matrices for the system in $\mathbb{R}^3$. Considerable progress has been made in the development of stabilized time-discontinuous Galerkin finite elements methods for first-order hyperbolic and parabolic systems; see for example Johnson et al. [1984,1986], Hughes [1987b], Shakib et al. [1991b] and references therein. These methods are immediately applicable to the classical linear acoustics equations written in the first-order form (2.12). However, in this approach, the state vector consists of both pressure and velocity components, requiring a large number of solution variables, which may be computationally uneconomical. Another difficulty with this approach is the coupling between the natural second-order form of the momentum equations governing the structure, and a first-order form for the system of equations describing the acoustic fluid. As a result of the mismatch between the second-order and first-order systems for the structure and fluid respectively, rigorous stability and convergence proofs for the coupled fluid-structure system are difficult (if not impossible) to obtain. In addition, for the exterior problem, incorporation of high-order accurate non-reflecting boundary conditions involving high-order temporal derivatives in a stable and efficient formulation for first-order systems is difficult to achieve.

2.3.2 Second-order hyperbolic form

For these reasons, a more natural approach for the coupled problem is to recast the first-order system of acoustic equations given in (2.12) as a second-order hyperbolic equation in terms of a single scalar variable. The most common choice is to eliminate the velocity in terms of the scalar pressure variable. Taking the divergence of (2.9) and the time derivative of (2.10), and after eliminating the velocity in terms of pressure we obtain the second-order scalar wave equation,

$$\nabla^2 p = a^2 \ddot{p} \tag{2.13}$$

where $a = c^{-1} > 0$ is the slowness and $\nabla^2 = \mathrm{Div}(\mathrm{Grad})$ is the Laplacian operator. Traditional variational (weak) formulations for the coupled structural acoustics problem are based on this second-order hyperbolic wave equation in terms of the scalar acoustic pressure p, together with the equations of elastodynamics in terms of the structural displacement $\boldsymbol{u}$, see e.g. ZIENKIEWICZ [1978] for this approach in the context of the semidiscrete finite element method. While this approach reduces the number of solution variables in the formulation, the use of $\{\boldsymbol{u}\,,\,p\}$ variables leads to an unsymmetric coupled system of equations for the fluid-structure problem. As a result of this lack of symmetry, stability and energy estimates for both the continuum and discrete structural-acoustic problems are difficult to obtain *a priori*.

To eliminate this difficulty for the coupled problem, scalar velocity potential ϕ, is chosen as the solution variable for the acoustic fluid, in conjunction with the displacement vector $\boldsymbol{u}$, for the structure. As a consequence of this choice of variables, the coupled variational equations associated with the fluid-structure interaction problem give rise to a globally symmetric form, which in the context of the space-time finite element formulation, allows for the prove of stability and convergence of the method.

The use of the velocity potential follows directly from the assumption of an irrotational acoustic fluid. Taking the curl of (2.9) and using the fact that the curl of ∇p vanishes, implies,

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{v}) = 0 \tag{2.14}$$

This result indicates that the acoustic part of the velocity field is indeed irrotational. Only the irrotational part exhibits the fluctuations associated with sound propagation so that the acoustic velocity $\boldsymbol{v}$, can be written as the gradient of the velocity potential ϕ:

$$\boxed{\boldsymbol{v} = \nabla \phi} \tag{2.15}$$

Substituting (2.15) into (2.9) and (2.10) results in a form involving the Laplacian differential operator,

$$p + \rho_0 \dot{\phi} \;=\; 0 \tag{2.16}$$

$$\dot{p} + \rho_0 c^2 \nabla^2 \phi \;=\; 0 \tag{2.17}$$

Eliminating the pressure in terms of the velocity potential $\phi(\boldsymbol{x}\,,\,t)$ with $\boldsymbol{x} \in \Omega_f$, gives the desired result,

$$\boxed{\nabla^2 \phi = a^2 \ddot{\phi} \qquad \text{in} \quad Q_f := \Omega_f \times\,]0\,,\,T[} \tag{2.18}$$

This second-order hyperbolic wave equation in terms of the acoustic velocity potential ϕ, is used as the basis for the fluid representation in the variational formulations developed in this thesis. In the presentation to follow, we will show how this choice of variable for the acoustic fluid leads directly to a globally symmetric coupled system from which *a priori* stability and energy estimates are obtained for both the continuum and discrete fluid-structure interaction problem.

2.3.3 Fluid-Structure Coupling

The unique solution to the coupled problem requires two interface conditions between the structure and fluid: (i) a traction boundary condition, and (ii) a kinematic compatibility condition. The traction boundary condition represents the acoustic fluid pressure acting on the structure and is given by,

$$\boxed{\boldsymbol{\sigma} \cdot \boldsymbol{n} \quad = \quad \rho_0 \dot{\phi} \boldsymbol{n} \qquad \text{on} \quad \Upsilon_i := \Gamma_i \times \,]0\,,\, T[} \tag{2.19}$$

where $p = -\rho_0 \dot{\phi}$ and $\boldsymbol{n}$ is the unit outward normal to Ω_s, and inward normal to Ω_f, on Γ_i. The kinematic boundary condition provides for the compatibility of the normal velocity across the fluid-structure interface and is given by,

$$\boxed{\nabla \phi \cdot \boldsymbol{n} \quad = \quad \dot{\boldsymbol{u}} \cdot \boldsymbol{n} \qquad \text{on} \quad \Upsilon_i := \Gamma_i \times \,]0\,,\, T[} \tag{2.20}$$

where $\boldsymbol{v} = \nabla \phi$.

Remark

- When pressure p, is used in place of the velocity potential ϕ, as the basic solution variable for the fluid, the kinematic boundary condition enforcing compatibility of normal velocity is usually replaced with a kinematic condition enforcing compatibility of normal accelerations, i.e.,

$$\nabla p \cdot \boldsymbol{n} \quad = \quad -\rho_0 \ddot{\boldsymbol{u}} \cdot \boldsymbol{n} \qquad \text{on} \quad \Upsilon_i := \Gamma_i \times \,]0\,,\, T[\tag{2.21}$$

  This condition is in effect the time derivative of (2.20) and is required for a displacement-pressure $\{\boldsymbol{u}\,,\, p\}$ variational formulation where it is incorporated as a Neumann ('natural') boundary condition.

2.3.4 Non-Reflecting Boundary Conditions

The exterior structural acoustics problem typically involves the the radiation and scattering of sound waves emanating from a vibrating structure submerged in an infinite acoustic fluid. When a domain based computational method such as the finite element method is used, the fluid domain must be truncated at a finite distance from the structure. In order to eliminate or minimize reflection of outgoing waves, radiation (non-reflecting) boundary conditions are applied on the fluid truncation boundary. If accurate non-reflecting boundary conditions are used then the fluid truncation boundary can be moved closer to the structure and the finite element model will be able to use fewer elements than otherwise would be possible and considerable cost savings may result.

In this thesis, non-reflecting boundary conditions are imposed on the artificial boundary Γ_∞ in the form of a linear operator $\mathbb{S}_m$, relating Dirichlet data ϕ, to the normal derivative $\nabla\phi \cdot \boldsymbol{n}$, through the mapping,

$$\boxed{\nabla\phi \cdot \boldsymbol{n} \quad = \quad -\mathbb{S}_m\phi \qquad \text{on} \quad \Upsilon_\infty := \Gamma_\infty \times]0, T[} \tag{2.22}$$

This boundary condition approximates the asymptotic behavior of the solution at infinity as described by the Sommerfeld condition, i.e.

$$\lim_{r\to\infty} r^{(d-1)/2}\left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right)\phi = 0 \qquad r + ct = \text{constant} \tag{2.23}$$

where r is the radial distance from the source. This condition asserts that at infinity all waves are outgoing. The non-reflecting boundary condition (2.22) can take several different forms depending on the local (differential) or nonlocal (integral) operators appearing in $\mathbb{S}_m$. The specific definition of the operator $\mathbb{S}_m$ relating Dirichlet to Neumann data is given in Chapters 5 and 6. Here we note that the index m, is related to the order of temporal derivatives appearing in the operator.

For an artificial boundary taken to be a circle in two-dimensions ($d = 2$) or a sphere in three-dimensions ($d = 3$), the obvious choice for $\mathbb{S}_m$ is to use a simple Sommerfeld-like boundary condition on the truncation boundary Γ_∞ of the form,

$$\nabla\phi \cdot \boldsymbol{n} \quad = \quad -\frac{1}{c}\dot{\phi} \qquad \text{on} \quad \Upsilon_\infty := \Gamma_\infty \times]0, T[\tag{2.24}$$

where $\nabla\phi \cdot \boldsymbol{n} = \partial\phi/\partial n$ is the normal derivative on Γ_∞. Restated in terms of the physical variables of pressure, $p = -\rho_0\dot{\phi}$, and velocity $\boldsymbol{v} = \nabla\phi$, this boundary condition becomes,

$$\boldsymbol{v} \cdot \boldsymbol{n} = p/z_o \tag{2.25}$$

where $z_o = \rho_0 c$, is the characteristic impedance for plane-wave propagation. In one-dimension ($d = 1$), this condition is exact not only at infinity, but also at any finite distance r. However, in multi-dimensions ($d = 2, 3$), this boundary condition may produce large spurious reflections, resulting in large errors in the numerical solution, except at very high frequencies or large r where the operator acts as a plane-wave (PW) absorber.

Because of the potential loss of accuracy resulting from the use of (2.24), an improved non-reflecting boundary is needed. A number of high-order accurate non-reflecting boundary conditions are available and will be discussed in detail in Chapters 5 and 6. An example is given here for the first-order accurate boundary condition,

$$\nabla \phi \cdot \boldsymbol{n} \quad = \quad -\frac{1}{R}\phi - \frac{1}{c}\dot{\phi} \qquad \text{on} \quad \Upsilon_\infty := \Gamma_\infty \times]0, T[\tag{2.26}$$

This non-reflecting boundary condition is exact for axially symmetric waves in $\mathbb{R}^3$ and will henceforth be denoted as $\mathbb{S}_1$.

2.4 The Strong Form in the Time-Domain

After collecting the governing equations defined in the preceding sections, the strong (local) form of the fluid-structure initial/boundary-value problem may be stated as:

Given a prescribed traction $\bar{\boldsymbol{t}} : \Gamma_\sigma \times I \mapsto \mathbb{R}^d$ and a source $f : \Omega_f \times I \mapsto \mathbb{R}$,
Find $\boldsymbol{u} : \bar{\Omega}_s \times I \mapsto \mathbb{R}^d$, and $\phi : \bar{\Omega}_f \times I \mapsto \mathbb{R}$, such that

$$\nabla \cdot \boldsymbol{\sigma} - \rho_s \ddot{\boldsymbol{u}} \quad = \quad \boldsymbol{0} \qquad \text{in } Q_s := \Omega_s \times I \tag{2.27}$$

$$\boldsymbol{\sigma} \quad = \quad \boldsymbol{C} : \nabla^s \boldsymbol{u} \qquad \text{in } Q_s := \Omega_s \times I \tag{2.28}$$

$$\nabla^2 \phi - a^2 \ddot{\phi} \quad = \quad f \qquad \text{in } Q_f := \Omega_f \times I \tag{2.29}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} \quad = \quad \bar{\boldsymbol{t}} \qquad \text{on } \Upsilon_\sigma := \Gamma_\sigma \times I \tag{2.30}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} \quad = \quad \rho_0 \dot{\phi} \boldsymbol{n} \qquad \text{on } \Upsilon_i := \Gamma_i \times I \tag{2.31}$$

$$\nabla\phi \cdot \boldsymbol{n} \quad = \quad \dot{\boldsymbol{u}} \cdot \boldsymbol{n} \qquad \text{on} \quad \Upsilon_i := \Gamma_i \times I \tag{2.32}$$

$$\nabla\phi \cdot \boldsymbol{n} \quad = \quad -\mathbb{S}_m\phi \qquad \text{on} \quad \Upsilon_\infty := \Gamma_\infty \times I \tag{2.33}$$

with initial conditions,

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_o(\boldsymbol{x}) \qquad \text{for} \quad \boldsymbol{x} \in \Omega_s \tag{2.34}$$

$$\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{u}}_o(\boldsymbol{x}) \qquad \text{for} \quad \boldsymbol{x} \in \Omega_s \tag{2.35}$$

$$\phi(\boldsymbol{x}, 0) = \phi_o(\boldsymbol{x}) \qquad \text{for} \quad \boldsymbol{x} \in \Omega_f \tag{2.36}$$

$$\dot{\phi}(\boldsymbol{x}, 0) = \dot{\phi}_o(\boldsymbol{x}) \qquad \text{for} \quad \boldsymbol{x} \in \Omega_f \tag{2.37}$$

Equations (2.27) – (2.33) together with the initial boundary conditions define a linear coupled system of second-order hyperbolic type due to the positive definite condition on $\boldsymbol{C}$ and $c > 0$. Thus the exterior structural acoustics problem, in its local form, consists of solving the equations of elastodynamics for the displacement field within Ω_s and the scalar wave equation in Ω_f for the acoustic velocity potential field, due to a traction loading, acoustic sources, and initial conditions. In addition, the traction and normal velocity must be continuous across the fluid-structure interface Γ_i, and the velocity potential must satisfy the radiation boundary condition (2.33) on Γ_∞.

2.5 Energy Estimates for the Continuum Problem

In this section we establish some *a priori* estimates for the continuum problem in terms of the energy for the hyperbolic structure-fluid coupled system using structural displacement $\boldsymbol{u}$, and acoustic velocity potential ϕ variables. The study of these important mathematical properties provides a general framework for the design and stability analysis of the proposed space-time finite element algorithm proposed in this thesis.

2.5.1 Energy Identities

In this section an energy identity for the continuum problem is established which relates the total structural and acoustic energies to the radiation energy absorbed through the fluid truncation boundary. Let the solution vector be defined in abstract form as,

$$\boxed{\boldsymbol{\mathcal{X}} := \{\boldsymbol{u}\,,\,\phi\}} \tag{2.38}$$

Then define the total energy for the coupled domain $\Omega = \Omega_s \cup \Omega_f$ by the expression,

$$\boxed{\mathbb{E}(\boldsymbol{\mathcal{X}}) := \mathcal{E}_s(\boldsymbol{u}) + \mathcal{E}_f(\phi)} \tag{2.39}$$

where the total structural energy in terms of the structural displacements is,

$$\mathcal{E}_s(\boldsymbol{u}) := \quad \frac{1}{2}\int_{\Omega_s} [\rho_s \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})] \; d\Omega \tag{2.40}$$

and the acoustic energy or 'wave energy' in terms of the velocity potential is,

$$\begin{aligned} \mathcal{E}_f(\phi) \quad &:= \quad \frac{1}{2}\int_{\Omega_f} \rho_0 [(\frac{p}{\rho_0 c})^2 + \boldsymbol{v} \cdot \boldsymbol{v}] \; d\Omega \\ &= \quad \frac{1}{2}\int_{\Omega_f} \rho_0 [a^2(\dot{\phi})^2 + \nabla\phi \cdot \nabla\phi] \; d\Omega \end{aligned} \tag{2.41}$$

The first and second terms in (2.40) are the kinetic and strain energy for the elastic structure respectively. The form of the acoustic energy written in terms of velocity potential is in reverse order: the first term (time derivative squared) and second term (gradient squared) in (2.41) are the potential and kinetic energies respectively. The reverse order of the kinetic and potential energies for $\mathcal{E}_s(\boldsymbol{u})$ and $\mathcal{E}_f(\phi)$ reflect the reciprocal nature of the structure and fluid in an energy sense.

The total energy norm for the coupled system:

An important property of (2.39) is that in the Hilbert space

$$\mathcal{V} = [H^1(\Omega_s)]^d \times H^1(\Omega_f) \tag{2.42}$$

the total energy for the coupled system $\mathbb{E}(\boldsymbol{\mathcal{X}})$ with abstract solution vector $\boldsymbol{\mathcal{X}} \in \mathcal{V}$ defines a natural energy norm on the space $\mathcal{V}$,

$$\boxed{\mathbb{E}(\boldsymbol{\mathcal{X}}) := ||\boldsymbol{\mathcal{X}}||^2_{\mathcal{V}}} \tag{2.43}$$

This result follows directly from the positive definite assumption on $\boldsymbol{C}$ and $c > 0$.

The acoustic intensity:

The acoustic intensity for sound waves is defined as the vector, $\boldsymbol{I} := p\boldsymbol{v}$, where p and $\boldsymbol{v}$ are the acoustic pressure and velocity respectively, see Lighthill [1978]. In terms of the velocity potential the acoustic intensity vector is given by,

$$\boldsymbol{I} = -\rho_0 \dot{\phi} \nabla \phi \tag{2.44}$$

whose component $\boldsymbol{I} \cdot \boldsymbol{n}$ is the rate per unit area at which energy is transmitted in the direction of the unit normal $\boldsymbol{n}$. For the exterior problem, the rate of energy (power) absorbed through the artificial boundary Γ_∞ is then,

$$\mathcal{D}(\phi) := -\int_{\Gamma_\infty} \rho_0 \dot{\phi} \nabla \phi \cdot \boldsymbol{n} \, d\Gamma \tag{2.45}$$

In terms of the absorbing boundary conditions defined through the linear map $\mathbb{S}_m$ we have,

$$\boxed{\mathcal{D}(\phi) = \int_{\Gamma_\infty} \rho_0 \dot{\phi} \mathbb{S}_m \phi \, d\Gamma} \tag{2.46}$$

With these definitions in place, we are in position to prove the following energy estimate for the coupled structural acoustics problem in exterior domains.

Lemma 2.1 (Energy estimate)
Under the conditions of zero sources $f = 0$ in Ω_f, and homogeneous traction boundary conditions $\bar{\boldsymbol{t}} = 0$ on Γ_σ, then the time rate of change of structural acoustic energy in $\Omega = \Omega_s \cup \Omega_f$ is equal to the total rate of transport through the artificial boundary Γ_∞,

i.e.

$$\boxed{\frac{d}{dt}\mathbb{E}(\boldsymbol{\mathcal{X}}) = -\mathcal{D}(\phi), \qquad \forall\ t \geq 0} \tag{2.47}$$

where $\boldsymbol{\mathcal{X}} = \{\boldsymbol{u}\,,\,\phi\}$.

Proof:
The proof of this important result employs integration by parts and provides the key motivation for the space-time finite element formulation proposed in this thesis. Computing the time derivative we have,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_s(\boldsymbol{u}) &= \int_{\Omega_s} [\rho_s \dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}} \,+\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})]\ d\Omega \\
&= \int_{\Omega_s} [\dot{\boldsymbol{u}} \cdot (\nabla \cdot \boldsymbol{\sigma}) \,+\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})]\ d\Omega \\
&= \int_{\Gamma_i} \rho_0 \dot{\phi}\ \dot{\boldsymbol{u}} \cdot \boldsymbol{n}\ d\Gamma
\end{aligned} \tag{2.48}$$

Similarly,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_f(\phi) &= \int_{\Omega_f} \rho_0 [a^2 \dot{\phi}\,\ddot{\phi} \,+\, \nabla\dot{\phi} \cdot \nabla\phi]\ d\Omega \\
&= \int_{\Omega_f} \rho_0 [\dot{\phi}\,\nabla^2\phi \,+\, \nabla\dot{\phi} \cdot \nabla\phi]\ d\Omega \\
&= -\int_{\Gamma_i} \rho_0 \dot{\phi}\ \dot{\boldsymbol{u}} \cdot \boldsymbol{n}\ d\Gamma - \int_{\Gamma_\infty} \rho_0 \dot{\phi}\ \mathbb{S}_m \phi\ d\Gamma
\end{aligned} \tag{2.49}$$

Combining (2.48) and (2.49) gives the desired energy estimate (2.47). $\square$

Remarks

1. It is important to note that by choosing the velocity potential ϕ for the acoustic variable and enforcing the compatibility of normal velocity (2.20) across Γ_i, then the coupling integrals defined on the fluid-structure interface boundary Γ_i, cancel when (2.48) and (2.49) are added together to form (2.47). This is the key property associated with the choice of $\boldsymbol{\mathcal{X}}$ variables and plays an important role in the stability (energy) estimates for the space-time formulation.

If the acoustic pressure is used as the fluid variable, together with the compatibility of normal accelerations (2.21) as required in a coupled variational equation for structural acoustics, then the coupling integrals defined on Γ_i, *will not cancel*. It is this property resulting from the use of $\{\boldsymbol{u}\,,\,p\}$ variables that destroys the inherent symmetry for the coupled structural acoustics system and consequently invalidates the energy estimate given in Lemma 2.1,

2. For the interior (bounded) problem with zero internal sources and homogeneous boundary conditions, energy transport across boundaries is not allowed ($\mathbb{S}_m = 0$), so that $\mathcal{D}(\phi) = 0$, and (2.47) reduces to the familiar result:

$$\frac{d}{dt}\mathbb{E}(\boldsymbol{\mathcal{X}}) = 0 \qquad \forall\, t \geq 0 \tag{2.50}$$

which implies that the total energy within the interior domain Ω is conserved, i.e.

$$\mathbb{E}(\boldsymbol{\mathcal{X}}(t)) = \mathbb{E}(\boldsymbol{\mathcal{X}}(0)) \qquad \forall\, t \geq 0 \tag{2.51}$$

Theorem 2.1 (Energy identity)
At any given time $t > 0$, the energy in $\Omega = \Omega_s \cup \Omega_f$ plus the energy absorbed through the artificial boundary Γ_∞ is equal to the initial energy in the coupled system. i.e.

$$\boxed{\mathcal{E}_s(\boldsymbol{u}(t)) + \mathcal{E}_f(\phi(t)) + \mathcal{R}(\phi(t)) = \mathcal{E}_s(\boldsymbol{u}_0) + \mathcal{E}_f(\phi_0), \qquad \forall\, t \geq 0} \tag{2.52}$$

where

$$\mathcal{R}(\phi(t)) := \quad \int_0^t \mathcal{D}(\phi(\tau))\; d\tau \tag{2.53}$$

is the radiation energy.

Proof: Integrating (2.47) over time,

$$\mathbb{E}(\boldsymbol{\mathcal{X}})\big|_0^t + \int_0^t \mathcal{D}(\phi(\tau))\; d\tau = 0 \tag{2.54}$$

and using definitions (2.40), (2.41) and (2.53) gives,

$$\mathcal{E}_s(\boldsymbol{u}(t)) + \mathcal{E}_f(\phi(t)) + \mathcal{R}(\phi(t)) = \mathcal{E}_s(\boldsymbol{u}(0)) + \mathcal{E}_f(\phi(0)), \qquad \forall\, t \geq 0 \tag{2.55}$$

Using the initial conditions, $\boldsymbol{u}(\boldsymbol{x}\,,\,0) = \boldsymbol{u}_0$ and $\phi(\boldsymbol{x}\,,\,0) = \phi_0$ gives the desired energy identity (2.52). □

This result provides a natural measure of stability for the exterior structural acoustics problem, and will be used to guide the design of the space-time finite element formulation described in the next Chapter.

As a specific example for the energy estimates given in (2.47) and (2.52), consider the first-order non-reflecting boundary condition defined in (2.26). In this case the rate of energy absorbed through the artificial boundary Γ_∞ is given by the surface integral (2.46) with the normal derivative replaced with,

$$-\,\mathbb{S}_1\phi = -\frac{1}{R}\phi - \frac{1}{c}\dot{\phi} \tag{2.56}$$

Defining the following weighted L_2 norms on Γ_∞ as,

$$||\phi||^2_{\Gamma_\infty} \;:=\; \int_{\Gamma_\infty} \rho_0(\phi)^2 \; d\Gamma \tag{2.57}$$

$$||\dot{\phi}||^2_{\Upsilon_\infty} \;:=\; \int_0^t ||\dot{\phi}||^2_{\Gamma_\infty} \; d\tau \tag{2.58}$$

then with the linear operator $\mathbb{S}_1$ defined in (2.26), the rate of energy transport through the artificial boundary specializes to,

$$\begin{aligned}\mathcal{D}(\phi) \;&=\; \frac{1}{R}\int_{\Gamma_\infty} \rho_0 \;\dot{\phi}\phi \; d\Gamma \;+\; \frac{1}{c}\int_{\Gamma_\infty} \rho_0 \;(\dot{\phi})^2 \; d\Gamma \\ &=\; \frac{1}{2}\frac{d}{dt}c_0||\phi||^2_{\Gamma_\infty} + c_1||\dot{\phi}||^2_{\Gamma_\infty}\end{aligned} \tag{2.59}$$

where $c_0 = 1/R$ and $c_1 = 1/c$ are positive constants. An energy identity of the form (2.52) is obtained by integrating (2.59) over time with the result,

$$\boxed{\mathcal{E}_s(\boldsymbol{u}) + \mathcal{E}_f(\phi) = \mathcal{E}_s(\boldsymbol{u}_0) + \mathcal{E}_f(\phi_0) - \frac{1}{2}c_0||\phi||^2_{\Gamma_\infty} - c_1||\dot{\phi}||^2_{\Upsilon_\infty}} \tag{2.60}$$

This is the stability (energy) estimate for the continuum problem with $\mathbb{S}_1$. It states that the total energy for the structure $\mathcal{E}_s$ and the fluid $\mathcal{E}_f$ at any time $t > 0$ is equal to the total initial energy in the system minus the radiation energy absorbed through the artificial boundary Γ_∞. An analogous result will be derived for the proposed space-time finite element formulation incorporating $\mathbb{S}_1$, see Theorem 4.1.

2.5.2 Uniqueness

A fundamental question pertaining to the exterior structural acoustics problem is whether a solution that satisfies the governing differential equations including the approximate non-reflecting boundary is unique. In this section we show how energy estimates of the form (2.60) can be used to help establish uniqueness of the solution. The question of uniqueness is approached by supposing that two solutions exist and then showing that this assumption leads to a contradiction, see e.g. ZAUDERER [1989] Chapter 6 and ACHENBACH [1987] Chapter 3. To prove that the initial/boundary-value problem for the coupled fluid-structure system written in terms of $\boldsymbol{\mathcal{X}} = \{\boldsymbol{u}\,,\,\phi\}$ variables and incorporating the first-order boundary operator $\mathbb{S}_1$, has a unique solution, we show that by assuming two unique solutions exist, given by $\boldsymbol{\mathcal{X}}_1$ and $\boldsymbol{\mathcal{X}}_2$, then it must be concluded that they are identical. Since the problem is linear, by superposition $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_1 - \boldsymbol{\mathcal{X}}_2$ must be a solution of the homogeneous problem with zero sources, $f = 0$, zero boundary data, $\bar{\boldsymbol{t}} = \boldsymbol{0}$, and zero initial data $\boldsymbol{\mathcal{X}}(0) = \boldsymbol{0}$. As a result, for the homogeneous problem, the energy estimate (2.60) reduces to,

$$\mathcal{E}_s(\boldsymbol{u}) + \mathcal{E}_f(\phi) + \frac{1}{2}c_0||\phi||^2_{\Gamma_\infty} + c_1||\dot{\phi}||^2_{\Upsilon_\infty} = 0 \tag{2.61}$$

Furthermore, both $\mathcal{E}_s(\boldsymbol{u})$ and $\mathcal{E}_f(\phi)$ are positive definite, and since c_0 and c_1 are positive on Γ_∞, the boundary integrals are non-negative as well, hence for (2.61) to hold we conclude,

$$\mathcal{E}_s(\boldsymbol{u}) = \mathcal{E}_f(\phi) = \frac{1}{2}c_0||\phi||^2_{\Gamma_\infty} = c_1||\dot{\phi}||^2_{\Upsilon_\infty} = 0 \tag{2.62}$$

and it follows that, $\boldsymbol{u} = \boldsymbol{0}$ and $\phi = 0$, which implies that the two supposedly different solutions must, in fact, be identical $\boldsymbol{\mathcal{X}}_1 = \boldsymbol{\mathcal{X}}_2$, so that only one solution exists. A remark here is that a crucial property for this proof was the non-negative integrands appearing in the the boundary integrals. Uniqueness proofs for high-order non-reflecting boundary operators $\mathbb{S}_m$ follow similar lines, assuming the non-negative property of the boundary integrals is maintained.