# Chapter 4

# Stability and Convergence Analysis for Fluid-Structure Interaction

For the interior problem, where energy is not allowed to flow through domain boundaries, the rate of change of the total energy in the system is constant. In this case the system is referred to as a 'conservative' system and a natural measure of stability is the total energy in the structure and the fluid, see (2.39). In contrast, for the exterior problem a natural measure of stability in the primary variables $\boldsymbol{\mathcal{X}}$, is the total structural acoustic energy in the finite domain $\Omega$ plus the radiation energy absorbed through the artificial boundary $\Gamma_\infty$. It is the radiation energy loss through $\Gamma_\infty$ that renders the exterior problem 'nonconservative'. The purpose of this Chapter is to prove through a rigorous analysis that the proposed space-time variational formulation inherits a stability estimate of the form given by Theorem 2.1 for the continuum problem. It is this estimate that leads to the conclusion that the proposed space-time method for the coupled structural acoustics problem with non-reflecting boundary conditions is unconditionally stable and converges at an optimal rate. The method is termed unconditionally stable if in the absence of external forces, and for arbitrary initial conditions within $\Omega$, the total energy together with the radiation energy absorbed through the artificial boundary is always less than or equal to the initial energy in the system.

## 4.1 Overview

Stability and error estimates for the interior problem, in the context of elastodynamics are given in Hughes and Hulbert [1988]. In this work, the stability and convergence analysis of the time-discontinuous Galerkin/least-squares method for second-order hyperbolic equations is extended to the coupled fluid-structure interaction problem. For the exterior problem, the issue of stability for the non-reflecting boundary conditions must be addressed. In this chapter we prove that the discontinuous Galerkin method, including the non-reflecting boundary conditions, is stable and converges at the optimal rate obtained for the interior problem. In Section 4.3, we prove a series of Lemmas which state the coercivity (positive-definiteness) conditions for the acoustic, radiation, and structural components of the coupled variational equations. Using these results, we then establish an energy decay inequality (stability condition) for the coupled space-time algorithm which is the discrete counterpart to the energy estimate obtained for the continuum problem. In Section 4.4, error estimates are obtained for the uncoupled acoustics problem with non-reflecting boundary conditions. A simple one-dimensional numerical example is performed to verify these energy and error estimates. In order to evaluate the purely temporal behavior of the time-discontinuous Galerkin method, a Fourier transformed second-order ordinary differential equation in time is analyized for stability and convergence. In Section 4.5 we return to the coupled problem, where error estimates are obtained for the coupled fluid-structure system. This chapter concludes in Section 4.6 with an extension to a mixed two-field formulation incorporating independent finite element interpolations for both pressure and velocity potential variables.

## 4.2 Preliminaries

Before proceeding, it is convenient to sum (3.40) over the time slabs and after rearranging terms we have for the fluid,

$$\boxed{B^f_{GLS}(w^h\,,\,\boldsymbol{\chi}^h) \quad = \quad L^f_{GLS}(w^h)} \tag{4.1}$$

with,

$$B^f_{GLS}(w^h\,,\,\boldsymbol{\chi}^h) \;:=\; E_f(w^h\,,\,\phi^h) + A(\boldsymbol{u}^h\,,\,w^h\boldsymbol{n}) + E_r(w^h\,,\,\phi^h) + \sum_{n=0}^{N-1} B^f_{LS}(w^h\,,\,\boldsymbol{\chi}^h)_n \tag{4.2}$$

and where

$$E_f(w^h\,,\,\phi^h) = E_2(w^h\,,\,\phi^h) + E_0(w^h\,,\,\phi^h) \tag{4.3}$$

$$E_r(w^h\,,\,\phi^h) = D_2(w^h\,,\,\phi^h) + D_1(w^h\,,\,\phi^h) + D_0(w^h\,,\,\phi^h) \tag{4.4}$$

and

$$E_2(w^h\,,\,\phi^h) = \sum_{n=0}^{N-1}(\dot{w}^h\,,\,a^2\ddot{\phi}^h)_{Q^f_n} + \sum_{n=1}^{N-1}(\dot{w}^h(t_n^+)\,,\,a^2\,[\![\dot{\phi}^h(t_n)]\!])_{\Omega_f} + (\dot{w}^h(0^+)\,,\,a^2\,\dot{\phi}^h(0^+))_{\Omega_f} \tag{4.5}$$

$$E_0(w^h\,,\,\phi^h) = \sum_{n=0}^{N-1}(\nabla\dot{w}^h\,,\,\nabla\phi^h)_{Q^f_n} + \sum_{n=1}^{N-1}(\nabla w^h(t_n^+)\,,\,[\![\nabla\phi^h(t_n)]\!])_{\Omega_f} + (\nabla w^h(0^+)\,,\,\nabla\phi^h(0^+))_{\Omega_f} \tag{4.6}$$

$$D_2(w^h\,,\,\phi^h) = \sum_{n=0}^{N-1} d_2(\dot{w}^h\,,\,\ddot{\phi}^h)_{(\Upsilon_\infty)_n} + \sum_{n=1}^{N-1} d_2(\dot{w}^h(t_n^+)\,,\,[\![\dot{\phi}^h(t_n)]\!])_{\Gamma_\infty} + d_2(\dot{w}^h(0^+)\,,\,\dot{\phi}^h(0^+))_{\Gamma_\infty} \tag{4.7}$$

$$D_0(w^h\,,\,\phi^h) = \sum_{n=0}^{N-1} d_0(\dot{w}^h\,,\,\phi^h)_{(\Upsilon_\infty)_n} + \sum_{n=1}^{N-1} d_0(w^h(t_n^+)\,,\,[\![\phi^h(t_n)]\!])_{\Gamma_\infty} + d_0(w^h(0^+)\,,\,\phi^h(0^+))_{\Gamma_\infty} \tag{4.8}$$

$$D_1(w^h\,,\,\phi^h) = \sum_{n=0}^{N-1} d_1(\dot{w}^h\,,\,\dot{\phi}^h)_{(\Upsilon_\infty)_n} \tag{4.9}$$

$$A(\boldsymbol{u}^h\,,\,w^h) \;=\; \sum_{n=0}^{N-1}(\dot{\boldsymbol{u}}^h\,,\,\rho_0\,\dot{w}^h\boldsymbol{n})_{(\Upsilon_i)_n} \tag{4.10}$$

with

$$\begin{aligned} L^f_{GLS}(w^h) \;&=\; \sum_{n=0}^{N-1}\Big\{(\dot{w}^h\,,\,f)_{Q^f_n} + L^f_{LS}(w^h)_n\Big\} \\ &+\; (\dot{w}^h(0^+)\,,\,a^2\,\dot{\phi}_0)_{\Omega_f} + (\nabla w^h(0^+)\,,\,\nabla\phi_0)_{\Omega_f} \end{aligned} \tag{4.11}$$

Similarly for the structure, summing (3.39) over the time slabs, we obtain:

$$\boxed{B^s_{GLS}(\boldsymbol{w}^h\,,\,\boldsymbol{\chi}^h) \;=\; L^s_{GLS}(\boldsymbol{w}^h)} \tag{4.12}$$

where

$$B^s_{GLS}(\boldsymbol{w}^h\,,\,\boldsymbol{\chi}^h) \;=\; E_s(\boldsymbol{w}^h\,,\,\boldsymbol{u}^h) - A(\boldsymbol{w}^h\,,\,\phi^h\boldsymbol{n}) + \sum_{n=0}^{N-1} B^s_{LS}(\boldsymbol{w}^h\,,\,\boldsymbol{\chi}^h)_n \tag{4.13}$$

and

$$\begin{aligned} E_s(\boldsymbol{w}^h\,,\,\boldsymbol{u}^h) \;&=\; \sum_{n=0}^{N-1}(\dot{\boldsymbol{w}}^h\,,\,\rho_s\ddot{\boldsymbol{u}}^h)_{Q^s_n} \\ &+\; \sum_{n=1}^{N-1}(\dot{\boldsymbol{w}}^h(t_n^+)\,,\,\rho_s[\![\dot{\boldsymbol{u}}^h(t_n)]\!])_{\Omega_s} + (\dot{\boldsymbol{w}}^h(0^+)\,,\,\rho_s\dot{\boldsymbol{u}}^h(0^+))_{\Omega_s} \\ &+\; \sum_{n=0}^{N-1} a(\dot{\boldsymbol{w}}^h\,,\,\boldsymbol{u}^h)_{Q^s_n} \\ &+\; \sum_{n=1}^{N-1} a(\boldsymbol{w}^h(t_n^+)\,,\,[\![\boldsymbol{u}^h(t_n)]\!])_{\Omega_s} + a(\boldsymbol{w}^h(0^+)\,,\,\boldsymbol{u}^h(0^+))_{\Omega_s} \end{aligned} \tag{4.14}$$

$$\begin{aligned} L^s_{GLS}(\boldsymbol{w}^h) \;&=\; \sum_{n=0}^{N-1}\Big\{(\dot{\boldsymbol{w}}^h\,,\,\bar{\boldsymbol{t}})_{(\Upsilon_\sigma)_n} + L^s_{LS}(\boldsymbol{w}^h)_n\Big\} \\ &+\; (\dot{\boldsymbol{w}}^h(0^+)\,,\,\rho_s\dot{\boldsymbol{u}}_0)_{\Omega_s} + a(\boldsymbol{w}^h(0^+)\,,\,\boldsymbol{u}_0)_{\Omega_s} \end{aligned} \tag{4.15}$$

From (3.60) and (3.61) it follows that for a sufficiently smooth exact solution $\boldsymbol{\mathcal{X}}$, the appropriate consistency conditions for (4.1) and (4.12) are respectively,

$$B^f_{GLS}(w^h\,,\,\boldsymbol{\mathcal{X}}^h - \boldsymbol{\mathcal{X}}) \quad = \quad 0 \tag{4.16}$$

$$B^s_{GLS}(\boldsymbol{w}^h\,,\,\boldsymbol{\mathcal{X}}^h - \boldsymbol{\mathcal{X}}) \quad = \quad 0 \tag{4.17}$$

## 4.3 Stability

In this section we prove the unconditional stability for both the time-discontinuous Galerkin and GLS methods for the coupled structural acoustics problem incorporating non-reflecting boundary conditions. An algorithm is termed unconditional stable if in the absence of forcing terms but for arbitrary initial conditions, the computed total structural acoustic energy plus the radiation energy absorbed through the artificial boundary is always bounded from above by the initial energy in the system. In the subsequent analysis, the notation $(\cdot\,,\,\cdot)_{\Omega_s}$ denotes the $L_2(\Omega_s)$ inner product defined over the structural domain with natural norm $||\cdot||_{\Omega_s}$. For the fluid, $(\cdot\,,\,\cdot)_{\Omega_f}$ denotes the weighted $L_2(\Omega_f)$ inner product with weighting equal to the reference density $\rho_0$. An important inequality that will be used often is,

$$|(u,v)| \leq \frac{1}{2}\,(\frac{1}{\epsilon}||u||^2 + \epsilon||v||^2) \qquad \forall\,\epsilon > 0 \tag{4.18}$$

**Lemma 4.1 (Total Acoustic Energy)**

$$\boxed{E_f(w^h\,,\,w^h) \quad = \quad |||w^h|||_f^2} \tag{4.19}$$

*where*

$$|||w^h|||_f^2 := \quad \mathcal{E}_f(w^h(0^+)) \,+\, \sum_{n=1}^{N-1} \mathcal{E}_f([\![w^h(t_n)]\!]) \,+\, \mathcal{E}_f(w^h(T^-)) \tag{4.20}$$

*and for linear acoustics, the total energy within the domain $\Omega_f$ is defined as in (2.41):*

$$\mathcal{E}_f(w) \quad = \quad \frac{1}{2}\,||a\dot{w}||^2_{\Omega_f} \,+\, \frac{1}{2}\,||\nabla w||^2_{\Omega_f} \tag{4.21}$$

*This is the coercivity condition for the fluid.*

Proof:

$$\begin{aligned}
\sum_{n=0}^{N-1} (\dot{w}^h\,,\,\ddot{w}^h)_{Q_n^f} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{1}{2}\frac{d}{dt} ||a\dot{w}^h||_{\Omega_f}^2\, dt \\
&= \frac{1}{2}\,||a\dot{w}^h(T^-)||_{\Omega_f}^2 - \frac{1}{2}\,||a\dot{w}^h(0^+)||_{\Omega_f}^2 \\
&+ \frac{1}{2}\sum_{n=1}^{N-1} \{||a\dot{w}^h(t_n^-)||_{\Omega_f}^2 - ||a\dot{w}^h(t_n^+)||_{\Omega_f}^2\}
\end{aligned} \tag{4.22}$$

Therefore,

$$\begin{aligned}
E_2(w^h\,,\,w^h) &= \frac{1}{2}\,||a\dot{w}^h(T^-)||_{\Omega_f}^2 + \frac{1}{2}\,||a\dot{w}^h(0^+)||_{\Omega_f}^2 \\
&+ \frac{1}{2}\sum_{n=1}^{N-1} \left\{||a\dot{w}^h(t_n^+)||_{\Omega_f}^2 - 2(\dot{w}^h(t_n^+)\,,\,a^2\,\dot{w}^h(t_n^-))_{\Omega_f} + ||a\dot{w}^h(t_n^-)||_{\Omega_f}^2\right\} \\
&= \frac{1}{2}\,||a\dot{w}^h(T^-)||_{\Omega_f}^2 + \frac{1}{2}\,||a\dot{w}^h(0^+)||_{\Omega_f}^2 + \frac{1}{2}\sum_{n=1}^{N-1} ||a[\![\dot{w}^h(t_n)]\!]||_{\Omega_f}^2
\end{aligned} \tag{4.23}$$

Similarly,

$$E_0(w^h\,,\,w^h) = \frac{1}{2}\,||\nabla w^h(T^-)||_{\Omega_f}^2 + \frac{1}{2}\,||\nabla w^h(0^+)||_{\Omega_f}^2 + \frac{1}{2}\sum_{n=1}^{N-1} ||[\![\nabla w^h(t_n)]\!]||_{\Omega_f}^2 \tag{4.24}$$

Combining (4.23) and (4.24) and using (4.3) completes the proof. ∎

**Lemma 4.2 (Radiation Energy)**
*For the exterior problem, assuming the non-reflecting boundary conditions satisfy condition (3.35), then*

$$\boxed{E_r(w^h\,,\,w^h) = |||w^h|||_r^2} \tag{4.25}$$

*where*

$$
\begin{aligned}
|||w^h|||_r^2 \;\; := \;\; & \sum_{n=0}^{N-1} d_1(\dot{w}^h \,,\, \dot{w}^h)_{(\Upsilon_\infty)_n} \\
+ \;\; & \mathcal{E}_r(w^h(0^+)) + \sum_{n=1}^{N-1} \mathcal{E}_r([\![w^h(t_n)]\!]) + \mathcal{E}_r(w^h(T^-))
\end{aligned}
\tag{4.26}
$$

*and the operator $\mathcal{E}_r(\cdot)$ is defined as,*

$$
\mathcal{E}_r(w) := \frac{1}{2}\, d_2(\dot{w} \,,\, \dot{w})_{\Gamma_\infty} + \frac{1}{2}\, d_0(w \,,\, w)_{\Gamma_\infty} \tag{4.27}
$$

*This is the coercivity condition for the non-reflecting boundary operators.*

Equation (4.26) represents a norm that is stronger than the radiation energy absorbed through the truncation boundary $\Gamma_\infty$ given by,

$$
\begin{aligned}
\mathcal{R}(w) &= \int_0^T \int_{\Gamma_\infty} \rho_0 \dot{w} \mathbb{S}_m w \; d\Gamma \; dt \\
&= \mathcal{E}_r(w) + d_1(\dot{w}^h \,,\, \dot{w}^h)_{\Upsilon_\infty}
\end{aligned}
\tag{4.28}
$$

Proof: Proceeding as in Lemma 4.1,

$$
\begin{aligned}
\sum_{n=0}^{N-1} d_2(\dot{w}^h \,,\, \ddot{w}^h)_{(\Upsilon_\infty)_n} \;\; = \;\; & \frac{1}{2}\, d_2(\dot{w}^h(T^-) \,,\, \dot{w}^h(T^-))_{\Gamma_\infty} - \frac{1}{2}\, d_2(\dot{w}^h(0^+) \,,\, \dot{w}^h(0^+))_{\Gamma_\infty} \\
+ \;\; & \frac{1}{2} \sum_{n=1}^{N-1} \Big\{ d_2(\dot{w}^h(t_n^-) \,,\, \dot{w}^h(t_n^-))_{\Gamma_\infty} - d_2(\dot{w}^h(t_n^+) \,,\, \dot{w}^h(t_n^+))_{\Gamma_\infty} \Big\}
\end{aligned}
\tag{4.29}
$$

Therefore,

$$
\begin{aligned}
D_2(w^h \,,\, w^h) \;\; = \;\; & \frac{1}{2}\, d_2(\dot{w}^h(T^-) \,,\, \dot{w}^h(T^-))_{\Gamma_\infty} + \frac{1}{2}\, d_1(\dot{w}^h(0^+) \,,\, \dot{w}^h(0^+))_{\Gamma_\infty} \\
+ \;\; & \frac{1}{2} \sum_{n=1}^{N-1} d_2([\![\dot{w}^h(t_n)]\!] \,,\, [\![\dot{w}^h(t_n)]\!])_{\Gamma_\infty}
\end{aligned}
\tag{4.30}
$$

Similarly,

$$
\begin{aligned}
D_0(w^h\,,\,w^h) \quad &= \quad \frac{1}{2}\,d_0(w^h(T^-)\,,\,w^h(T^-))_{\Gamma_\infty} \;+\; \frac{1}{2}\,d_0(w^h(0^+)\,,\,w^h(0^+))_{\Gamma_\infty} \\
&+ \quad \frac{1}{2}\sum_{n=1}^{N-1} d_0([\![w^h(t_n)]\!]\,,\,[\![w^h(t_n)]\!])_{\Gamma_\infty}
\end{aligned} \tag{4.31}
$$

Combining (4.30), (4.31) and (4.4) completes the proof. □

**Lemma 4.3 (Total Structural Energy)**

$$
\boxed{E_s(\boldsymbol{w}^h\,,\,\boldsymbol{w}^h) \quad = \quad |||\boldsymbol{w}^h|||_s^2} \tag{4.32}
$$

*where*

$$
|||\boldsymbol{w}^h|||_s^2 := \quad \mathcal{E}_s(\boldsymbol{w}^h(0^+)) \;+\; \sum_{n=1}^{N-1} \mathcal{E}_s([\![\boldsymbol{w}^h(t_n)]\!]) \;+\; \mathcal{E}_s(\boldsymbol{w}^h(T^-)) \tag{4.33}
$$

*and for elastodynamics, the total energy within the domain $\Omega_s$ is defined as in (2.40):*

$$
\mathcal{E}_s(\boldsymbol{w}) \quad = \quad \frac{1}{2}\,(\dot{\boldsymbol{w}}\,,\,\rho_s\dot{\boldsymbol{w}})_{\Omega_s} \;+\; \frac{1}{2}\,a(\boldsymbol{w}\,,\,\boldsymbol{w})_{\Omega_s} \tag{4.34}
$$

*This is the coercivity condition for the structure.*

The proof of Lemma 4.3 follows the same steps taken to prove Lemma 4.1 and so is omitted.

The norm associated with the coupled fluid-structure system (4.1) and (4.12) is defined by:

$$
\begin{aligned}
|||\boldsymbol{\psi}^h|||^2 \quad &= \quad |||\boldsymbol{w}^h|||_s^2 \;+\; |||w^h|||_f^2 \;+\; |||w^h|||_r^2 \\
&+ \quad \sum_{n=0}^{N-1} \Big\{ (\mathcal{L}_s\boldsymbol{w}^h\,,\,\rho_s^{-1}\boldsymbol{\tau}\mathcal{L}_s\boldsymbol{w}^h)_{\tilde{Q}_n^s} \\
&+ \quad ([\![\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)(\boldsymbol{x})]\!]\cdot\boldsymbol{n}\,,\,\rho_s^{-1}\boldsymbol{s}[\![\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)(\boldsymbol{x})]\!]\cdot\boldsymbol{n})_{(\tilde{\Upsilon}_s)_n} \\
&+ \quad (\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)\cdot\boldsymbol{n}\,,\,\rho_s^{-1}\boldsymbol{s}\boldsymbol{\sigma}(\nabla\boldsymbol{w}^h)\cdot\boldsymbol{n})_{(\Upsilon_\sigma)_n} \\
&+ \quad (\mathcal{L}_1\boldsymbol{\psi}^h\,,\,\rho_s^{-1}\boldsymbol{s}\mathcal{L}_1\boldsymbol{\psi}^h)_{(\Upsilon_i)_n}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad ||c\tau^{1/2}\mathcal{L}_f w^h||^2_{\tilde{Q}^f_n} + ||cs^{1/2}\mathcal{L}_r w^h||^2_{(\tilde{\Upsilon}_\infty)_n} \\
&+ \quad ||cs^{1/2}[\![ w^h_{,n}(\boldsymbol{x})]\!]||^2_{(\tilde{\Upsilon}_f)_n} + (\mathcal{L}_2\boldsymbol{\psi}^h \,,\, c^2 s\mathcal{L}_2\boldsymbol{\psi}^h)_{(\Upsilon_i)_n}\Big\}
\end{aligned} \tag{4.35}
$$

**Lemma 4.4 (Stability condition for the coupled system)**

$$
\boxed{|||\boldsymbol{\psi}^h|||^2 \quad = \quad B^s_{GLS}(\boldsymbol{w}^h \,,\, \boldsymbol{\psi}^h) + B^f_{GLS}(w^h \,,\, \boldsymbol{\psi}^h)} \tag{4.36}
$$

Proof: The proof follows directly from the superposition of (4.1) and (4.12),

$$
\begin{aligned}
B^s_{GLS}(\boldsymbol{w}^h \,,\, \boldsymbol{\psi}^h) \quad &+ \quad B^f_{GLS}(w^h \,,\, \boldsymbol{\psi}^h) \\
&= \quad E_s(\boldsymbol{w}^h \,,\, \boldsymbol{w}^h) - A(\boldsymbol{w}^h \,,\, w^h\boldsymbol{n}) \\
&+ \quad E_f(w^h \,,\, w^h) + A(\boldsymbol{w}^h \,,\, w^h\boldsymbol{n}) + E_r(w^h \,,\, w^h) \\
&+ \quad \sum_{n=0}^{N-1}\Big\{B^s_{GLS}(\boldsymbol{w}^h \,,\, \boldsymbol{\psi})_n + B^f_{GLS}(w^h \,,\, \boldsymbol{\psi})_n\Big\}
\end{aligned} \tag{4.37}
$$

Using the definitions in Lemmas 4.1 , 4.2 and 4.3, and canceling the coupling terms $A(\cdot\,,\,\cdot)$ completes the proof. The discontinuous Galerkin method ($\boldsymbol{\tau} = 0\,,\boldsymbol{s} = 0$) and/or ($\tau = 0\,,\, s = 0$), also satisfies a coercivity condition. □

**Remark**

- By the choice of displacement/velocity potential variables $\boldsymbol{\chi}^h$, weighting $\dot{\boldsymbol{\psi}}^h$, and weakly enforcing the traction and kinematic interface boundary conditions across $\Gamma_\infty \times I_n$, then the coupling terms $A(\cdot\,,\,\cdot)$, cancel when the structure and the fluid variational equations are added together. Recall that an analogous result occurred in the construction of Lemma 2.1 for the continuum problem. This is the key result which allows for the establishment of a well-defined norm (4.35) for the coupled problem.

In the following, we establish the discrete counterpart to the energy identity obtained in Theorem 2.1 for the continuum problem.

**Theorem 4.1 (Energy Estimate)**
*In the absence of external loading, i.e.* $f = 0$ *and* $\bar{\boldsymbol{t}} = 0$*, and assuming the radiation boundary operators satisfy (3.35), then at the end of a time interval the total structural energy, i.e.* $\mathbb{E}(\boldsymbol{\chi}^h(t_n^-))$*, within the computational domain* $\Omega = \Omega_s \cup \Omega_f$*, plus the radiation energy absorbed through the artificial boundary* $\Gamma_\infty$*, i.e.* $\mathcal{R}(\phi^h(t_n^-))$*, is bounded above by the initial energy in the system, i.e.,* $\mathbb{E}(\boldsymbol{\chi}_0)$*.*

Proof: With $f = 0$ and $\bar{\boldsymbol{t}} = 0$,

$$
\begin{aligned}
L^s_{GLS}(\boldsymbol{u}^h) + L^f_{GLS}(\phi^h) \quad &= \quad (\dot{\boldsymbol{u}}^h(0^+)\,,\, \rho_s \dot{\boldsymbol{u}}_0)_{\Omega_s} + a(\boldsymbol{u}^h(0^+)\,,\, \boldsymbol{u}_0)_{\Omega_s} \\
&+ \quad (\dot{\phi}^h(0^+)\,,\, a^2\, \dot{\phi}_0)_{\Omega_f} + (\nabla\phi^h(0^+)\,,\, \nabla\phi_0)_{\Omega_f} \\
&\leq \quad |(\dot{\boldsymbol{u}}^h(0^+)\,,\, \rho_s \dot{\boldsymbol{u}}_0)_{\Omega_s} + a(\boldsymbol{u}^h(0^+)\,,\, \boldsymbol{u}_0)_{\Omega_s} \\
&+ \quad (\dot{\phi}^h(0^+)\,,\, a^2\, \dot{\phi}_0)_{\Omega_f} + (\nabla\phi^h(0^+)\,,\, \nabla\phi_0)_{\Omega_f}| \\
&\leq \quad \frac{1}{2}\, (\dot{\boldsymbol{u}}^h(0^+)\,,\, \rho_s \dot{\boldsymbol{u}}^h(0^+))_{\Omega_s} + \frac{1}{2}\, (\dot{\boldsymbol{u}}_0\,,\, \rho_s \dot{\boldsymbol{u}}_0)_{\Omega_s} \\
&+ \quad \frac{1}{2}\, a(\boldsymbol{u}^h(0^+)\,,\, \boldsymbol{u}^h(0^+))_{\Omega_s} + \frac{1}{2}\, a(\boldsymbol{u}_0\,,\, \boldsymbol{u}_0)_{\Omega_s} \\
&+ \quad \frac{1}{2}\, ||a\dot{\phi}^h(0^+)||^2_{\Omega_f} + \frac{1}{2}\, ||a\dot{\phi}_0||^2_{\Omega_f} \\
&+ \quad \frac{1}{2}\, ||\nabla\phi^h(0^+)||^2_{\Omega_f} + \frac{1}{2}\, ||\nabla\phi_0||^2_{\Omega_f} \\
&= \quad \mathcal{E}_s(\boldsymbol{u}^h(0^+)) + \mathcal{E}_s(\boldsymbol{u}_0) \\
&+ \quad \mathcal{E}_f(\phi^h(0^+)) + \mathcal{E}_f(\phi_0)
\end{aligned} \tag{4.38}
$$

where $\mathcal{E}_s(\boldsymbol{u}_0)$ and $\mathcal{E}_f(\phi_0)$ denote the initial total energy in the domains $\Omega_s$ and $\Omega_f$ respectively. Adding (4.1) and (4.12), and using Lemma 4.4 and the definition of $|||\boldsymbol{\psi}^h|||^2$ results in the energy estimate,

$$
\mathcal{E}_s(\boldsymbol{u}^h(T^-)) + \mathcal{E}_f(\phi^h(T^-)) + \mathcal{R}(\phi^h(T^-)) \leq \mathcal{E}_s(\boldsymbol{u}_0) + \mathcal{E}_f(\phi_0) \tag{4.39}
$$

Since $T$ is arbitrary,

$$\boxed{\mathbb{E}(\boldsymbol{\mathcal{X}}^h(t_n^-)) + \mathcal{R}(\phi^h(t_n^-)) \leq \mathbb{E}(\boldsymbol{\mathcal{X}}_0), \quad n = 0, 1, 2, \ldots, N-1} \tag{4.40}$$

where,

$$\mathbb{E}(\boldsymbol{\mathcal{X}}^h(t_n^-)) := \mathcal{E}_s(\boldsymbol{u}^h(t_n^-)) + \mathcal{E}_f(\phi^h(t_n^-)) \tag{4.41}$$

$$\mathbb{E}(\boldsymbol{\mathcal{X}}_0) := \mathcal{E}_s(\boldsymbol{u}_0) + \mathcal{E}_f(\phi_0) \tag{4.42}$$

and $\mathcal{R}(\phi^h(t_n^-))$ is defined as in (4.28). Equation (4.40) also holds for the time-discontinuous Galerkin method where ($\boldsymbol{\tau} = 0\,, \boldsymbol{s} = 0$) and/or ($\tau = s = 0$). ∎

This result is the algorithmic counterpart to the energy identity derived for the continuum problem and stated in Theorem 2.1. The energy decay inequality in (4.40) states that the total energy in the fluid-structure system, plus the energy transported through the radiation boundary, is always less than, or equal to the initial energy in the system, which implies that *the space-time formulation is unconditionally stable.*

**Corollary 4.1 (Energy decay inequality for the interior problem)**
*For the interior (bounded) problem, energy transport across boundaries is not allowed so that $\mathcal{R} = 0$, and (4.40) reduces to,*

$$\boxed{\mathbb{E}(\boldsymbol{\mathcal{X}}^h(t_n^-)) \leq \mathbb{E}(\boldsymbol{\mathcal{X}}_0), \quad n = 0, 1, 2, \ldots, N-1} \tag{4.43}$$

The result implies that the algorithm is dissipative. As discussed earlier, it is this numerical dissipation which helps control (damp) unwanted oscillations that may occur near discontinuities in the solution.

**Corollary 4.2 (Energy decay inequality for acoustics)**
*For the uncoupled problem of linear acoustics, with zero sources, $f = 0$, and 'rigid' boundary conditions, i.e. $\dot{\boldsymbol{u}}^h = \boldsymbol{0}$, then (4.40) reduces to,*

$$\boxed{\mathcal{E}_f(\phi^h(t_n^-)) + \mathcal{R}(\phi^h(t_n^-)) \leq \mathcal{E}_f(\phi_0), \quad n = 0, 1, 2, \ldots, N-1} \tag{4.44}$$

As a specific example of the energy estimate (4.40), consider the first-order $\mathbb{S}_1$, non-reflecting boundary condition defined in (2.26). In this case the radiation energy (4.28) reduces to,

$$\begin{aligned}\mathcal{R}(\phi^h) &= \frac{1}{2} d_0(\phi^h\,,\,\phi^h)_{\Gamma_\infty} + d_1(\dot{\phi}^h\,,\,\dot{\phi}^h)_{\Upsilon_\infty} \\ &= \frac{1}{2}\, c_0 ||\phi^h||^2_{\Gamma_\infty} + c_1 ||\dot{\phi}^h||^2_{\Upsilon_\infty} \qquad (4.45)\end{aligned}$$

where $c_0 = 1/R$ and $c_1 = 1/c$ are positive constants. As a result (4.40) specializes to,

$$\mathbb{E}(\boldsymbol{\mathcal{X}}^h(t_n^-)) + \frac{1}{2}\, c_0 ||\phi^h||^2_{\Gamma_\infty} + c_1 ||\dot{\phi}^h||^2_{\Upsilon_\infty} \le \mathbb{E}(\boldsymbol{\mathcal{X}}_0) \qquad (4.46)$$

This result is the discrete counterpart to (2.60).

## 4.4 Error Estimates for Acoustics

Before stating the convergence properties for the coupled fluid-structure system, error estimates for the simplified uncoupled problem of linear acoustics with 'rigid' boundary conditions are derived. To further simplify the analysis, non-reflecting boundary operators are restricted to the first-order $\mathbb{S}_1$ condition defined in (2.26). In this case, second-order time-derivatives do not appear in the radiation boundary condition and the boundary term $d_2(\cdot\,,\,\cdot)$ is omitted. The proof of convergence follows along the same lines as in Hughes and Hulbert [1988], with the additional complication of including the non-reflecting boundary operators for the exterior problem.

For the acoustics problem with 'rigid' boundary conditions, (i.e. $\dot{\boldsymbol{u}}^h \cdot \boldsymbol{n} = 0$ on $\Gamma_\infty$), the coupling operator $A(\boldsymbol{u}^h\,,\,w^h\boldsymbol{n})$ vanishes and the norm associated with (4.1) reduces to,

$$\begin{aligned}|||w^h|||^2 &= |||w^h|||^2_f + |||w^h|||^2_r \\ &+ \sum_{n=0}^{N-1} \Big\{ ||c\tau^{1/2}\mathcal{L}_f w^h||^2_{\tilde{Q}^f_n} + ||cs^{1/2}\mathcal{L}_r w^h||^2_{(\tilde{\Upsilon}_\infty)_n} \\ &+ ||cs^{1/2}[\![w^h_{,n}(\boldsymbol{x})]\!]||^2_{(\tilde{\Upsilon}_f)_n} + ||cs^{1/2}w^h_{,n}||^2_{(\Upsilon_i)_n} \Big\} \qquad (4.47)\end{aligned}$$

The coercivity condition for the uncoupled acoustics problem is,

$$|||w^h|||^2 \quad = \quad B^f_{GLS}(w^h\,,\, w^h) \tag{4.48}$$

where the right-hand-side is defined by (4.1) with $A_n = 0$. The appropriate consistency condition is,

$$B^f_{GLS}(w^h\,,\, e) \quad = \quad 0 \tag{4.49}$$

where $e = \phi^h - \phi$.

Let $\tilde{\phi}^h \in \mathcal{T}^h$ denote an interpolant of $\phi$, and let $\eta = \tilde{\phi}^h - \phi$ denote the interpolation error. Then the solution error, $e = \phi^h - \phi$, can be split into the sum

$$e = e^h + \eta \tag{4.50}$$

where

$$e^h = \phi^h - \tilde{\phi}^h \in \mathcal{W}^h \tag{4.51}$$

is the difference between the finite element solution and the interpolant. An appropriate space-time mesh parameter is given by,

$$\boxed{h = \max\{c\Delta t\,,\, \Delta x\}} \tag{4.52}$$

where $c$ is the acoustic wave speed and $\Delta x$ and $\Delta t$ are maximum element diameters in space and time, respectively.

**Theorem 4.2 (Error Estimates)**
*Assume $\tau$ and $s$ satisfy,*

$$c_1 h \quad \leq \quad \tau \quad \leq \quad c_2 h \tag{4.53}$$

$$c_3 \quad \leq \quad s \quad \leq \quad c_4 \tag{4.54}$$

*where* $c_1\,, c_2\,, c_3$ and $c_4$ *are positive constants* ($c_2 > c_1\,, c_4 > c_3$). *Assuming the exact solution $\phi$ to the local acoustics problem with $\mathbb{S}_1$, is smooth in the sense that, $\phi \in H^{p+1}(Q_f)$, then the interpolation error, $\eta$, satisfies the following estimates:*

$$\sum_{n=0}^{N-1} d_1(\dot{\eta}\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.55}$$

$$\sum_{n=0}^{N-1} ||a\tau^{-1/2}\dot{\eta}||^2_{Q_n^f} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.56}$$

$$\sum_{n=0}^{N-1} ||c\tau^{1/2}\mathcal{L}_f\eta||^2_{\tilde{Q}_n^f} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.57}$$

$$\sum_{n=0}^{N-1} ||as^{-1/2}\dot{\eta}||^2_{(\Upsilon_\infty)_n} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.58}$$

$$\sum_{n=0}^{N-1} ||cs^{1/2}\mathcal{L}_r\eta||^2_{(\Upsilon_\infty)_n} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.59}$$

$$\sum_{n=0}^{N-1} ||as^{-1/2}\dot{\eta}||^2_{\tilde{\Upsilon}_n\cup(\Upsilon_i)_n} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.60}$$

$$\sum_{n=0}^{N-1} \left\{||cs^{1/2}[\![\eta_{,n}(\boldsymbol{x})]\!]||^2_{\tilde{\Upsilon}_n} + ||cs^{1/2}\eta_{,n}||_{(\Upsilon_i)_n}\right\} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.61}$$

$$\mathcal{E}_f(\eta(T^-)) + \mathcal{E}_f(\eta(0^+)) + \sum_{n=1}^{N-1}\left\{\mathcal{E}_f(\eta(t_n^-)) + \mathcal{E}_f(\eta(t_n^+))\right\} \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.62}$$

$$\begin{aligned} & d_0(\eta(T^-)\,,\,\eta(T^-))_{\Gamma_\infty} + d_0(\eta(0^+)\,,\,\eta(0^+))_{\Gamma_\infty} \\ + & \sum_{n=1}^{N-1}\left\{d_0(\eta(t_n^-)\,,\,\eta(t_n^-))_{\Gamma_\infty} + d_0(\eta(t_n^+)\,,\,\eta(t_n^+))_{\Gamma_\infty}\right\} \quad \leq \quad c(\phi)\, h^{2p-1} \end{aligned} \tag{4.63}$$

*where $c(\phi)$ is independent of the mesh parameter $h$. Then,*

$$\boxed{|||e|||^2 \quad \leq \quad c(\phi)\, h^{2p-1}} \tag{4.64}$$

The following Lemmas aid in the proof of Theorem 4.2.

**Lemma 4.5**

$$
\begin{aligned}
E_2(e^h\,,\,\eta) \quad &= \quad -\sum_{n=0}^{N-1}(\ddot{e}^h\,,\,a^2\,\dot{\eta})_{Q_n^f} \\
&- \quad \sum_{n=1}^{N-1}(\llbracket\dot{e}^h(t_n)\rrbracket\,,\,a^2\,\dot{\eta}(t_n^-))_{\Omega_f}\;+\;(\dot{e}^h(T^-)\,,\,a^2\,\dot{\eta}(T^-))_{\Omega_f}
\end{aligned} \tag{4.65}
$$

$$
\begin{aligned}
E_0(e^h\,,\,\eta) \quad &= \quad -\sum_{n=0}^{N-1}(\nabla e^h\,,\,\nabla\dot{\eta})_{Q_n^f} \\
&- \quad \sum_{n=1}^{N-1}(\llbracket\nabla e^h(t_n)\rrbracket\,,\,\nabla\eta(t_n^-))_{\Omega_f}\;+\;(\nabla e^h(T^-)\,,\,\nabla\eta(T^-))_{\Omega_f}
\end{aligned} \tag{4.66}
$$

$$
\begin{aligned}
D_0(e^h\,,\,\eta) \quad &= \quad -\sum_{n=0}^{N-1}d_0(e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \\
&- \quad \sum_{n=1}^{N-1}d_0(\llbracket e^h(t_n)\rrbracket\,,\,\eta(t_n))_{\Gamma_\infty}\;+\;d_0(e^h(T^-)\,,\,\eta(T^-))_{\Gamma_\infty}
\end{aligned} \tag{4.67}
$$

Results (4.65),(4.66), and (4.67) follow directly using definitions (4.5), (4.6), and (4.8), and from integration-by-parts in time. □

**Lemma 4.6**

$$
\begin{aligned}
&(\ddot{e}^h\,,\,a^2\,\dot{\eta})_{Q_n^f}\;+\;(\nabla e^h\,,\,\nabla\dot{\eta})_{Q_n^f}\;+\;d_0(e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n}\;+\;d_1(\dot{e}^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \\
= \quad &(\boldsymbol{\mathcal{L}}_f e^h\,,\,\dot{\eta})_{\tilde{Q}_n^f}\;+\;(\boldsymbol{\mathcal{L}}_r e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n}\;+\;(\llbracket e^h_{,n}(\boldsymbol{x})\rrbracket\,,\,\dot{\eta})_{\tilde{\Upsilon}_n^f}+\;(e^h_{,n}\,,\,\dot{\eta})_{(\Upsilon_i)_n}
\end{aligned} \tag{4.68}
$$

Proof:
By integration-by-parts and the divergence theorem,

$$
\begin{aligned}
(\nabla e^h\,,\,\nabla\dot{\eta})_{Q_n^f} \quad &= \quad -(\nabla^2 e^h\,,\,\dot{\eta})_{\tilde{Q}_n^f}\;+\;(e^h_{,n}\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \\
&+ \quad (\llbracket e^h_{,n}(\boldsymbol{x})\rrbracket\,,\,\dot{\eta})_{\tilde{\Upsilon}_n^f}\;+\;(e^h_{,n}\,,\,\dot{\eta})_{(\Upsilon_i)_n}
\end{aligned} \tag{4.69}
$$

Thus

$$(\ddot{e}^h\,,\,a^2\,\dot{\eta})_{Q_n^f}\;+\;(\nabla e^h\,,\,\nabla\dot{\eta})_{Q_n^f}\;+\;d_0(e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n}\;+\;d_1(\dot{e}^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n}$$

$$\begin{aligned} &= \;(a^2\ddot{e}^h - \nabla^2 e^h\,,\,\dot{\eta})_{\tilde{Q}_n^f}\;+\;(e^h_{,n} + \mathbb{S}_1 e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \\ &+\;([\![e^h_{,n}(\boldsymbol{x})]\!]\,,\,\dot{\eta})_{\tilde{\Upsilon}_n^f}\;+\;(e^h_{,n}\,,\,\dot{\eta})_{(\Upsilon_i)_n} \end{aligned} \tag{4.70}$$

where use is made of (3.29),(3.30), and (3.31) with the first-order $m = 1$, non-reflecting boundary operator defined as in (2.26), i.e.

$$\mathbb{S}_1 e^h = \frac{1}{R}e^h + \frac{1}{c}\dot{e}^h \tag{4.71}$$

Employing the definitions (3.52) and (3.53) completes the proof. □

The following two inequalities are also needed.

$$\begin{aligned} &\sum_{n=1}^{N-1}\Big\{-\,([\![\dot{e}^h(t_n)]\!]\,,\,a^2\,\dot{\eta}(t_n^-))_{\Omega_f}\;-\;([\![\nabla e^h(t_n)]\!]\,,\,\nabla\eta(t_n^-))_{\Omega_f}\Big\} \\ &\quad +(\dot{e}^h(T^-)\,,\,a^2\,\dot{\eta}(T^-))_{\Omega_f} + (\nabla e^h(T^-)\,,\,\nabla\eta(T^-))_{\Omega_f} \\ &\leq \;\frac{1}{2}\Big[\sum_{n=1}^{N-1}\{\mathcal{E}_f([\![e^h(t_n)]\!])\;+\;4\mathcal{E}_f(\eta(t_n^-))\} \\ &+\;\mathcal{E}_f(e^h(T^-)) + 4\mathcal{E}_f(\eta(T^-))\Big] \end{aligned} \tag{4.72}$$

Similarly,

$$\begin{aligned} &-\sum_{n=1}^{N-1} d_0([\![e^h(t_n)]\!]\,,\,\eta(t_n^-))_{\Gamma_\infty}\;+\;d_0(e^h(T^-)\,,\,\eta(T^-))_{\Gamma_\infty} \\ &\leq\;\frac{1}{2}\Big[\sum_{n=1}^{N-1}\Big\{\frac{1}{2}d_0([\![e^h(t_n)]\!]\,,\,[\![e^h(t_n)]\!])_{\Gamma_\infty}\;+\;2\,d_0(\eta(t_n^-)\,,\,\eta(t_n^-))_{\Gamma_\infty}\Big\} \\ &+\;\frac{1}{2}d_0(e^h(T^-)\,,\,e^h(T^-))_{\Gamma_\infty}\;+\;2\,d_0(\eta(T^-)\,,\,\eta(T^-))_{\Gamma_\infty}\Big] \end{aligned} \tag{4.73}$$

Results (4.72) and (4.73) follow directly from repeated use of inequality (4.18). The proof of Theorem 4.2 proceeds as follows,

$$
\begin{aligned}
|||e^h|||^2 &= B^f_{GLS}(e^h\,,\,e^h) \qquad \text{(by the coercivity condition, (4.48))}\\
&= B^f_{GLS}(e^h\,,\,e-\eta) \qquad \text{(by (4.50))}\\
&= -B^f_{GLS}(e^h\,,\,\eta) \qquad \text{(by the consistency condition, (4.49))}\\
&\le \left|B^f_{GLS}(e^h\,,\,\eta)\right|\\
&= \Big| E_2(e^h\,,\,\eta) + E_0(e^h\,,\,\eta) \;+\; D_1(e^h\,,\,\eta) + D_0(e^h\,,\,\eta)\\
&+ \sum_{n=0}^{N-1}\Big\{(\mathcal{L}_f e^h\,,\,c^2\tau\mathcal{L}_f\eta)_{\tilde{Q}^f_n} \;+\; (\mathcal{L}_r e^h\,,\,c^2 s\mathcal{L}_r\eta)_{(\tilde{\Upsilon}_\infty)_n}\\
&+ ([\![e^h_{,n}(\boldsymbol{x})]\!]\,,\,c^2 s[\![\eta_{,n}(\boldsymbol{x})]\!])_{\tilde{\Upsilon}^f_n} + (e^h_{,n}\,,\,c^2 s\eta_{,n})_{(\Upsilon_i)_n}\Big\}\Big|\\
&\qquad \text{(by definition of } B^f_{GLS}, (4.3)\ -(4.9))\\
&= \Big|\sum_{n=0}^{N-1}\Big\{-(\mathcal{L}_f e^h\,,\,\dot{\eta})_{\tilde{Q}^f_n} + (\mathcal{L}_f e^h\,,\,c^2\tau\mathcal{L}_f\eta)_{\tilde{Q}^f_n}\\
&- (\mathcal{L}_r e^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} + (\mathcal{L}_r e^h\,,\,c^2 s\mathcal{L}_r\eta)_{(\Upsilon_\infty)_n} \;+\; 2d_1(\dot{e}^h\,,\,\dot{\eta})_{(\Upsilon_\infty)_n}\\
&- ([\![e^h_{,n}(\boldsymbol{x})]\!]\,,\,\dot{\eta})_{\tilde{\Upsilon}^f_n} \;-\; (e^h_{,n}\,,\,\dot{\eta})_{(\Upsilon_i)_n}\\
&+ ([\![e^h_{,n}(\boldsymbol{x})]\!]\,,\,c^2 s[\![\eta_{,n}(\boldsymbol{x})]\!])_{\tilde{\Upsilon}^f_n} \;+\; (e^h_{,n}\,,\,c^2 s\eta_{,n})_{(\Upsilon_i)_n}\Big\}\\
&+ \sum_{n=1}^{N-1}\Big\{-([\![\dot{e}^h(t_n)]\!]\,,\,a^2\,\dot{\eta}(t_n^-))_{\Omega_f} \;-\; ([\![\nabla e^h(t_n)]\!]\,,\,\nabla\eta(t_n^-))_{\Omega_f}\Big\}\\
&+ (\dot{e}^h(T^-)\,,\,a^2\,\dot{\eta}(T^-))_{\Omega_f} + (\nabla e^h(T^-)\,,\,\nabla\eta(T^-))_{\Omega_f}\\
&- \sum_{n=1}^{N-1} d_0([\![e^h(t_n)]\!]\,,\,\eta(t_n^-))_{\Gamma_\infty} \;+\; d_0(e^h(T^-)\,,\,\eta(T^-))_{\Gamma_\infty}\Big|\\
&\qquad \text{(by Lemmas 4.5 and 4.6)}\\
&\le \sum_{n=0}^{N-1}\Big\{\frac{1}{4}\,||c\tau^{1/2}\mathcal{L}_f e^h||^2_{\tilde{Q}^f_n} \;+\; ||a\tau^{-1/2}\dot{\eta}||^2_{\tilde{Q}^f_n}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{4}\,||c\tau^{1/2}\mathcal{L}_f e^h||^2_{\tilde{Q}^f_n} + ||c\tau^{1/2}\mathcal{L}_f \eta||^2_{\tilde{Q}^f_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}\mathcal{L}_r e^h||^2_{(\Upsilon_\infty)_n} + ||as^{-1/2}\dot{\eta}||^2_{(\Upsilon_\infty)_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}\mathcal{L}_r e^h||^2_{(\Upsilon_\infty)_n} + ||cs^{1/2}\mathcal{L}_r \eta||^2_{(\Upsilon_\infty)_n} \\
&+ \quad \frac{1}{2}\,d_1(\dot{e}^h\,,\,\dot{e}^h)_{(\Upsilon_\infty)_n} + 2d_1(\dot{\eta}\,,\,\dot{\eta})_{(\Upsilon_\infty)_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}[\![e^h_{,n}(\boldsymbol{x})]\!]||^2_{\tilde{\Upsilon}^f_n} + ||as^{-1/2}\dot{\eta}||^2_{\tilde{\Upsilon}^f_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}e^h_{,n}||^2_{(\Upsilon_i)_n} + ||as^{-1/2}\dot{\eta}||^2_{(\Upsilon_i)_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}[\![e^h_{,n}(\boldsymbol{x})]\!]||^2_{\tilde{\Upsilon}^f_n} + ||cs^{1/2}[\![\eta_{,n}(\boldsymbol{x})]\!]||^2_{\tilde{\Upsilon}^f_n} \\
&+ \quad \frac{1}{4}\,||cs^{1/2}e^h_{,n}||^2_{(\Upsilon_i)_n} + ||cs^{1/2}\eta_{,n}||^2_{(\Upsilon_i)_n}\Big\} \\
&+ \quad \frac{1}{2}\sum_{n=1}^{N-1}\Big\{\mathcal{E}_f([\![e^h(t_n)]\!]) + 4\mathcal{E}_f(\eta(t_n^-))\Big\} \\
&+ \quad \frac{1}{2}\,\mathcal{E}_f(e^h(T^-)) + 2\mathcal{E}_f(\eta(T^-)) \\
&+ \quad \frac{1}{4}\sum_{n=1}^{N-1}\Big\{d_0([\![e^h(t_n)]\!]\,,\,[\![e^h(t_n)]\!])_{\Gamma_\infty} + 4d_0(\eta(t_n^-)\,,\,\eta(t_n^-))_{\Gamma_\infty}\Big\} \\
&+ \quad \frac{1}{4}\,d_0(e^h(T^-)\,,\,e^h(T^-))_{\Gamma_\infty} + 4d_0(\eta(T^-)\,,\,\eta(T^-))_{\Gamma_\infty} \\
&\qquad \text{(by (4.18) and results (4.72) and (4.73))}
\end{aligned}
$$

The terms involving $e^h$ may be subsumed by the left-hand side. The interpolation estimates (4.55)–(4.63) then yield

$$
|||e^h|||^2 \quad \leq \quad c(\phi)\,h^{2p-1} \tag{4.74}
$$

Likewise,

$$
|||\eta|||^2 \quad \leq \quad c(\phi)\,h^{2p-1} \tag{4.75}
$$

By the triangle inequality,

$$|||e|||^2 \quad \leq \quad 2|||e^h|||^2 + 2|||\eta|||^2 \quad \leq \quad c(\phi)\, h^{2p-1} \tag{4.76}$$

which completes the proof. □

This result indicates that the error as measured in the natural norm (4.48) for the discontinuous GLS solution to the exterior acoustics problem converges at the rate $(2p-1)/2$.

### 4.4.1 Confirmation of Energy and Error Estimates

To verify the convergence and energy properties of the space-time finite element formulation, the response of a one-dimensional model problem in the semi-infinite interval $0 \leq x < \infty$ was calculated by introducing a truncation boundary $\Gamma_\infty = L$, and imposing an exact non-reflecting boundary condition. Consider the following initial/boundary-value problem,

$$\ddot{\phi} \;=\; c^2\, \phi_{,xx} \qquad \text{for } x \in \Omega\,,\; t > 0 \tag{4.77}$$

$$\phi(0,t) \;=\; 0, \qquad \phi_{,x}(L,t) = -(1/c)\dot{\phi}(L,t) \tag{4.78}$$

$$\phi(x,0) \;=\; \phi_0(x), \qquad \dot{\phi}(x,0) = 0 \tag{4.79}$$

where $\Omega = ]0, L[$ and the condition at $x = L$ in (4.78) is the exact 'plane-wave' non-reflecting boundary condition for this one-dimension problem. The initial pulse,

$$\phi_0(x) = \frac{1}{4}\left(1 - \cos\frac{2\pi}{\lambda}(x - x_0)\right)^2 \tag{4.80}$$

is positioned at the distance $x_0 = 2.4$ from the fixed end, with wavelength $\lambda = .8$, and the computational domain is taken as $\Omega = ]0, 4[$, i.e. $L = 4$. The domain $\Omega$ is discretized uniformly with standard biquadratic interpolations, $Q2$, in space-time, with $3 \times 3$ Gaussian quadrature used throughout.

The exact solution is given by D'Alembert. Half of the initial pulse propagates with velocity $c = 1$, in each direction, i.e.

$$\phi(x,t) = \frac{1}{2}[\phi_0(\xi_+) + \phi_0(\xi_-)] \tag{4.81}$$

for $0 \leq \xi_+ \leq \lambda$ and $0 \leq \xi_- \leq \lambda$ where $\xi_+ = x - x_0 - ct$, and $\xi_- = x - x_0 + ct$. At time $t = 0.8$, the wave propagating to the right, $\phi_0(\xi_+)$, reaches the truncation boundary $\Gamma_\infty = L$, and at $t = 1.6$, this wave leaves $\Omega$ entirely. At time $t = 2.4$ the wave propagating to the left, $\phi_0(\xi_-)$, reaches the fixed end and is reflected back. At time $t = 7.2$, the reflected wave leaves $\Omega$, resulting in a quiescent solution. The form of $\phi_0$ is chosen such that the solution $\phi(x,t)$ has sufficient smoothness (continuity) for the interpolation estimates given in (4.55) - (4.63) to be valid. For $\phi_0$ defined in (4.80), and quadratic space-time elements, $p = 2$, the exact solution satisfies the requirement $\phi \in H^3$.

Figure 4.1 shows the numerical solution using the time-discontinuous Galerkin/least-squares algorithm at three representative times. Each space-time slab was discretized with a uniform mesh of 160 $Q2$ elements. The intrinsic time scale used for all computations is,

$$\tau = \frac{\Delta t}{8\sqrt{1 + C^2}} \tag{4.82}$$

where $C = c\Delta t/\Delta x$ is the Courant number. This definition for $\tau$ is a generalization of the $\tau$ given in Shakib et.al. [1991b]. The effect of the slowness parameter $s$, has little consequence on the solution to this problem, and as pointed out by Hulbert and Hughes [1990], due to the complications involved with its assembly, this term is omitted in all the computations.

The energy properties of the exact solution are summarized in the following. The total energy within $\Omega$ is the sum of the kinetic and potential energies, i.e.

$$\mathcal{E}(\phi(t)) = \frac{1}{2}||a\dot{\phi}||^2_\Omega + \frac{1}{2}||\phi_{,x}||^2_\Omega \tag{4.83}$$

where

$$\begin{aligned} ||a\dot{\phi}||^2_\Omega &= \int_0^L (a\dot{\phi}(x,t))^2 \, dx \\ ||\phi_{,x}||^2_\Omega &= \int_0^L (\phi_{,x}(x,t))^2 \, dx \end{aligned}$$

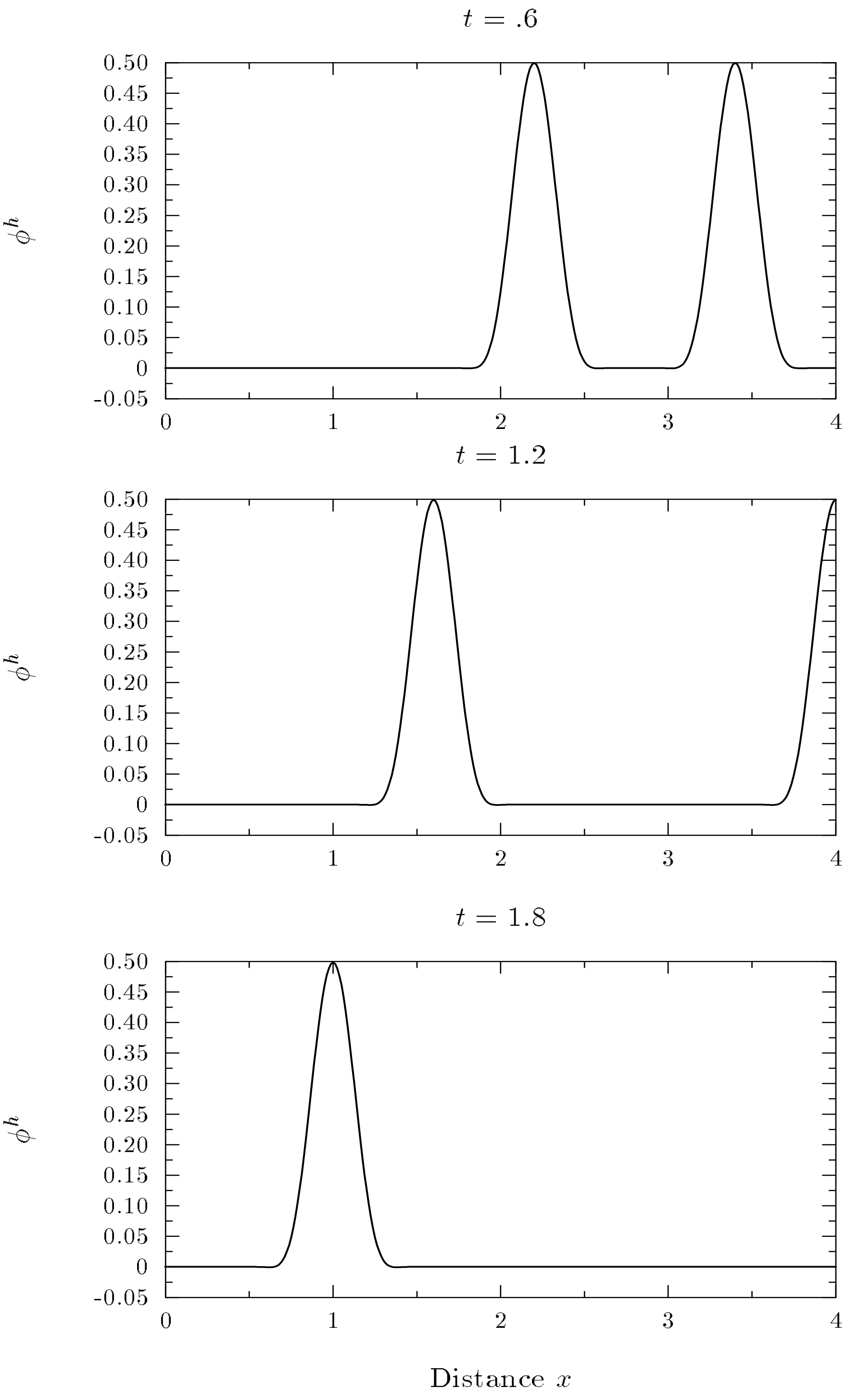


Figure 4.1: One-dimension semi-infinite problem: Numerical solution using time-discontinuous GLS algorithm with 160 $Q2$ elements and a Courant number of $C = 1$. At time $t = .6$ two waves travel in opposite directions. At $t = 1.2$ the wave propagating to the right is absorbed by the non-reflecting boundary located at $x = 4$. At $t = 1.8$ only the left-going wave remains.

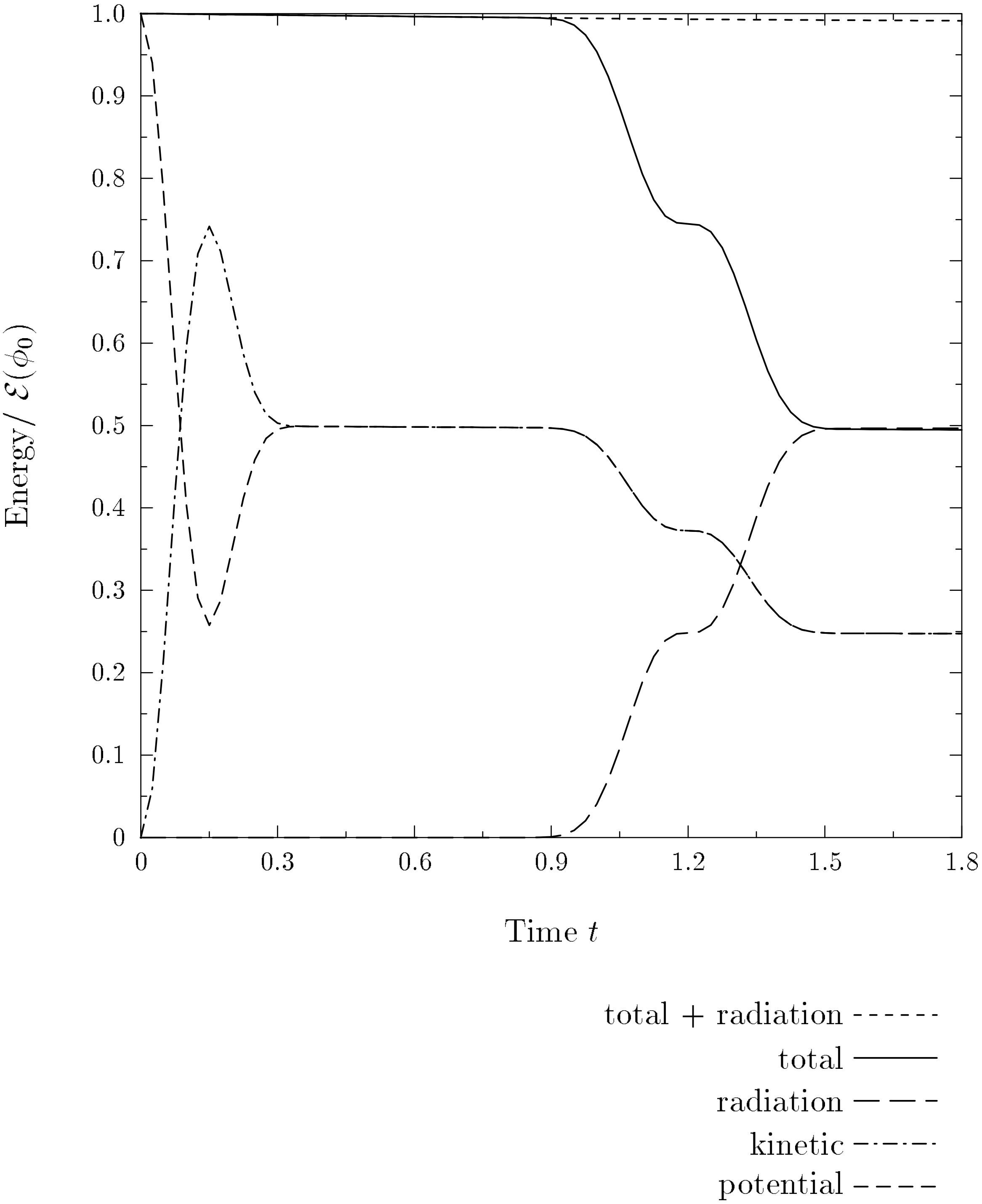


Figure 4.2: Energy calculations using time-discontinuous Galerkin/least-squares algorithm with 160 $Q2$ elements and Courant number $C = 1$. (Total energy = Kinetic energy + Potential energy)

For the initial condition $\phi_0$ given in (4.80), $||a\dot{\phi}||^2_\Omega = ||\phi_{,x}||^2_\Omega$ for $t \geq \lambda/2c$. In addition to the energy in $\Omega$, the power flow, or rate of transfer of energy past the truncation boundary $\Gamma_\infty = L$, is

$$\frac{1}{c}(\dot{\phi}(L,t))^2 \geq 0 \tag{4.84}$$

and it can be shown that the total energy in $\Omega$, plus the energy entering the exterior domain $x > L$, is conserved, i.e.

$$\mathcal{E}(\phi(t)) + \mathcal{R}(\phi(t)) = \mathcal{E}(\phi_0) \tag{4.85}$$

where the radiation energy defined in (4.28) is specialized in one-dimension to,

$$\mathcal{R}(\phi) = \frac{1}{c}\int_0^t (\dot{\phi}(L,\eta))^2 \; d\eta \tag{4.86}$$

Figure 4.2 shows the energy calculations for the time-discontinuous Galerkin/least-squares formulation, with the Courant number set at $C = 1$. At $t = 0$, all the energy is potential, and as the initial pulse begins to propagate, energy is transformed from potential to kinetic. For $t \geq \lambda/2c = 0.4$, the initial pulse has completely separated into a left and right wave, and the space-time finite element solution preserves the equality of the potential and kinetic energies, i.e. $||a\dot{\phi}^h||^2_\Omega = ||\phi^h_{,x}||^2_\Omega$. At $t = 0.8$, the right going wave reaches $\Gamma_\infty = L$, and the loss in total energy $\mathcal{E}(\phi^h)$ within $\Omega$, is absorbed by the non-reflecting boundary condition as measured by the radiation energy $\mathcal{R}(\phi^h)$. Results confirm that the Galerkin/least-squares method is indeed dissipative, and verifies the energy-decay inequality given in Theorem 4.1, i.e. the total energy plus the radiation energy is bounded above by the initial energy in the computational domain, i.e.

$$\mathcal{E}(\phi^h(t_n^-)) + \mathcal{R}(\phi^h(t_n^-)) \leq \mathcal{E}(\phi_0) \tag{4.87}$$

To study the numerical convergence, the response was calculated for the time interval $0 \leq t \leq T = 1.8$. The convergence rates for the time-discontinuous Galerkin and Galerkin/least-squares formulations are shown in Figure 4.3. Errors were measured in two different norms: $|||\cdot|||_\tau$ denotes the norm, (4.47), with $s = 0$, and $|||\cdot|||_0$ denotes the Galerkin norm, with $\tau = 0$, $s = 0$. Note that the convergence rates are identical for all formulations measured in both norms, and are the same as those predicted by Theorem 4.2. Results for this example show that the Galerkin formulation converges at the same rate as the GLS formulation, with slightly larger errors.

### 4.4.2 Temporal Stability and Convergence

In order to evaluate the purely temporal behavior of the time-discontinuous Galerkin method, it suffices to consider the Fourier transform of the scalar wave equation from physical space to wavenumber space. Assuming an infinite spatial domain with zero sources, a Fourier synthesis of the wave equation leads to the following second-order ordinary differential equation (ODE).

$$\boxed{\ddot{\phi}(t) + \omega^2 \phi(t) = 0 \qquad t \in ]0, T[} \tag{4.88}$$

with initial conditions, $\phi(0) = \phi_0$ and $\dot{\phi}(0) = \dot{\phi}_0$. In the above, $\omega = ck$ is the frequency, $c$ is the wave speed, and $k$ is the wavenumber. The time-discontinuous Galerkin method applied to this ODE is,

$$B_n(w^h\,,\,\phi^h) \quad = \quad L_n(w^h)\,, \qquad n \,=\, 0, 1, ..., N-1 \tag{4.89}$$

where

$$\begin{aligned} B_n(w^h\,,\,\phi^h) \;&:=\; \int_{t_n}^{t_{n+1}} (\dot{w}^h \ddot{\phi}^h + \omega^2 \dot{w}^h \phi^h)\, dt \\ &\quad +\; \dot{w}^h(t_n^+)\dot{\phi}^h(t_n^+) + \omega^2 w^h(t_n^+)\phi^h(t_n^+) \end{aligned} \tag{4.90}$$

$$L_n(w^h) \;:=\; \dot{w}^h(t_n^+)\dot{\phi}^h(t_n^-) + \omega^2 w^h(t_n^+)\phi^h(t_n^-) \tag{4.91}$$

Summing (4.89) over the time slabs, we obtain,

$$B(w^h\,,\,\phi^h) \;=\; \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (\dot{w}^h \ddot{\phi}^h + \omega^2 \dot{w}^h \phi^h)\, dt$$

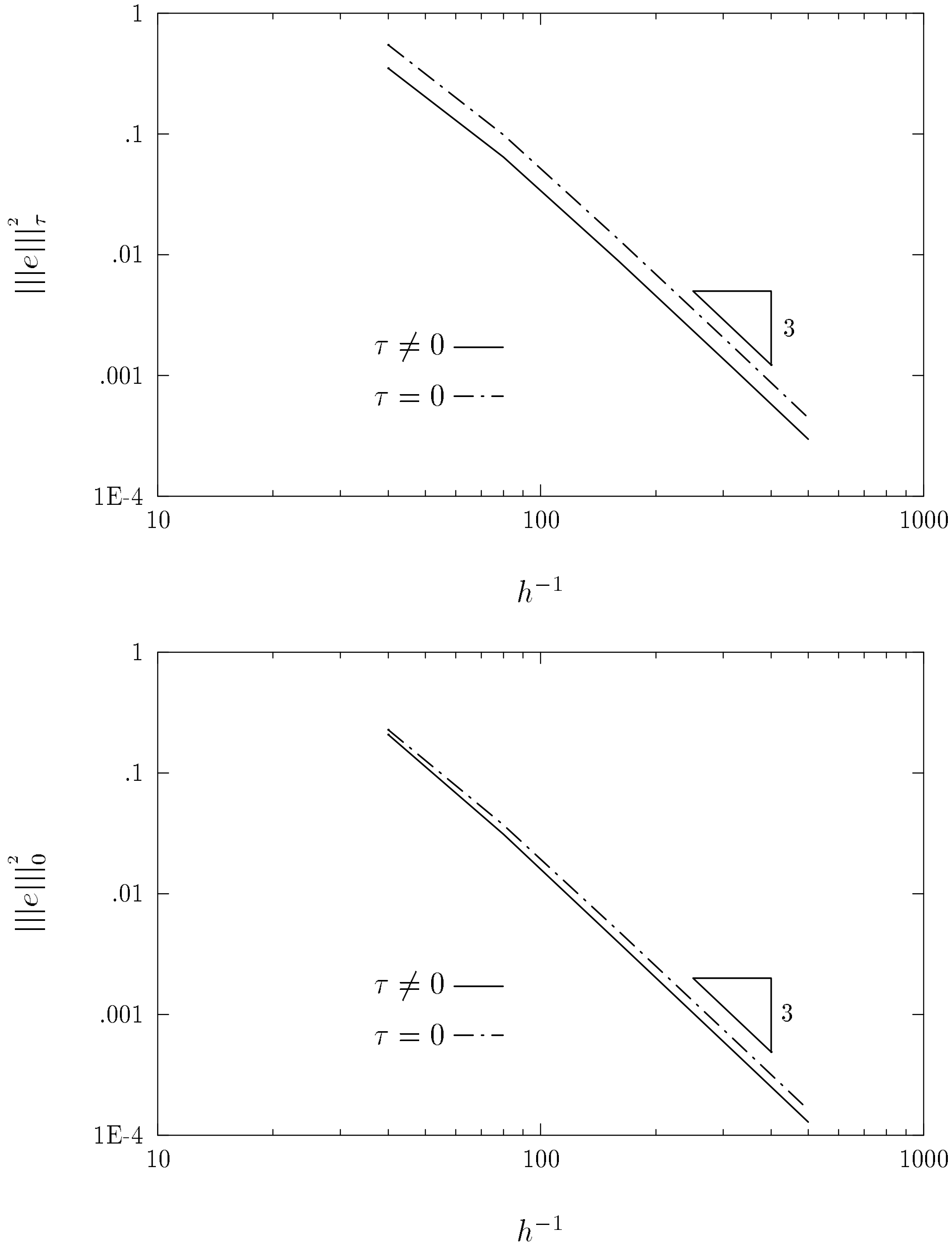


Figure 4.3: Convergence of the numerical error for different formulations employing the $Q2$ element and $C = 1$; $h$ is the length of the element.

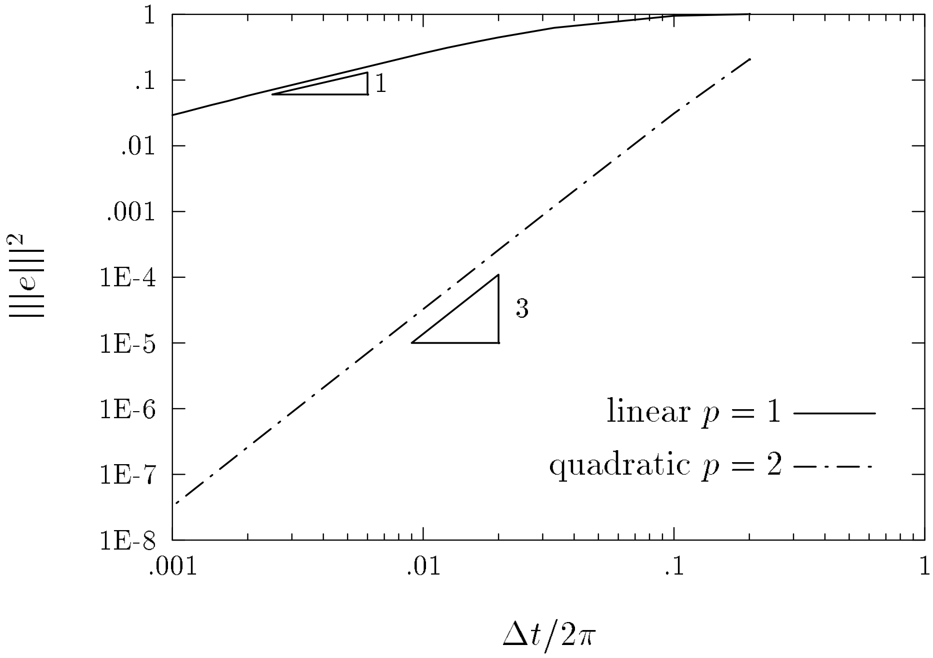


Figure 4.4: Temporal convergence for the time-discontinuous Galerkin formulation applied to a second-order ordinary differential equation in time: Convergence rates employing linear and quadratic interpolation functions.

$$
\begin{aligned}
&+ \sum_{n=1}^{N-1} \left\{ \dot{w}^h(t_n^+)[\![\dot{\phi}^h(t_n)]\!] + \omega^2 w^h(t_n^+)[\![\phi^h(t_n)]\!] \right\} \\
&+ \dot{w}^h(0^+)\dot{\phi}^h(0^+) + \omega^2 w^h(0^+)\phi^h(0^+)
\end{aligned}
\tag{4.92}
$$

The norm associated with (4.92) is defined by:

$$
\begin{aligned}
|||w^h|||^2 \;:=\;& \frac{1}{2}\,\dot{w}^h(T^-)^2 + \frac{1}{2}\,\dot{w}^h(0^+)^2 + \frac{1}{2}\sum_{n=1}^{N-1}[\![\dot{w}^h(t_n)]\!]^2 \\
+\;& \frac{1}{2}\,w^h(T^-)^2 + \frac{1}{2}\,w^h(0^+)^2 + \frac{1}{2}\sum_{n=1}^{N-1}[\![w^h(t_n)]\!]^2
\end{aligned}
\tag{4.93}
$$

with the coercivity condition:

$$
B(w^h\,,\,w^h) = |||w^h|||^2 \tag{4.94}
$$

Consider the time discretization over the interval $I_n = ]t_n\,,\,t_{n+1}[$, with interpolation order $p$:

$$\phi^h(t) = \sum_{a=1}^{p+1} N_a(t)\phi_a \tag{4.95}$$

$$w^h(t) = \sum_{a=1}^{p+1} N_a(t)w_a \tag{4.96}$$

In this expression, $\phi_1 = \phi^h(t_n^+)$ and $\phi_{p+1} = \phi^h(t_{n+1}^-)$ denote unknown discrete values associated with the end point nodes, while the values $\phi_2 \cdots \phi_p$ define a set of internal variables. Substituting these interpolations into (4.89) yields a system of coupled $(p+1)$ equations in $(p+1)$ unknowns $\{\phi_1\,,\, \phi_2 \cdots \phi_{p+1}\}$, which all depend on the solution at the previous time interval, i.e. $\phi^h(t_n^-)$ and $\dot{\phi}^h(t_n^-)$. The solution is obtained in sequence from one time interval to the next, data from the end of the previous time step are employed as initial conditions for the current time interval.

To study the temporal convergence, a numerical solution was obtained using a frequency of $\omega = 1$ and initial data $\phi_0 = 1$ and $\dot{\phi}_0 = 0$. The response was calculated using both linear $p = 1$, and quadratic $p = 2$, interpolation functions. The convergence rates are shown in Figure 4.4 for the time interval $t \in ]0\,,\, 6\pi[$. Numerical errors, $e = \phi^h - \phi$, are measured in the natural norm emanating from the time-discontinuous variational equations, i.e. $|||e|||^2$. The slope of the curves indicate a convergence rate of the order $\mathcal{O}(\Delta t^{2p-1})$ as expected.

## 4.5 Error Estimates for the Coupled Problem

The convergence proof for the coupled fluid-structure problem follows the same basic steps as the uncoupled acoustics problem. As before, only the first-order non-reflecting boundary condition $\mathbb{S}_1$ will be considered. Let $\P\boldsymbol{\mathcal{X}}^h \in \mathcal{S}^h \times \mathcal{T}^h$ denote an interpolant of $\boldsymbol{\mathcal{X}} := \{\boldsymbol{u}\,,\, \phi\}$. Then the error $\boldsymbol{E} = \boldsymbol{\mathcal{X}}^h - \boldsymbol{\mathcal{X}}$, can be written as:

$$\boldsymbol{E} = \boldsymbol{E}^h + \boldsymbol{H} \tag{4.97}$$

where

$$\boldsymbol{E}^h = (\boldsymbol{\mathcal{X}}^h - \P\boldsymbol{\mathcal{X}}^h) \in \mathcal{V}^h \times \mathcal{W}^h \tag{4.98}$$

$$\boldsymbol{H} \;=\; \P\boldsymbol{\chi}^h - \boldsymbol{\chi} \quad \text{(interpolation error)} \tag{4.99}$$

In components,

$$\boldsymbol{E} \;=\; \{\boldsymbol{e}\,,\, e\} \tag{4.100}$$

$$\boldsymbol{E}^h \;=\; \{\boldsymbol{e}^h\,,\, e^h\} \tag{4.101}$$

$$\boldsymbol{H} \;=\; \{\boldsymbol{\eta}\,,\, \eta\} \tag{4.102}$$

where $\boldsymbol{e} = \boldsymbol{u}^h - \boldsymbol{u}$ and $e = \phi^h - \phi$. An appropriate space-time mesh parameter for the structural domain $\Omega_s$ is given by,

$$h_s = \max\{c_L \Delta t\,,\, \Delta x\} \tag{4.103}$$

where $c_L$ is the dilatational wave speed and $\Delta x$ and $\Delta t$ are maximum element diameters in space and time, respectively. For the fluid domain $\Omega_f$,

$$h_f = \max\{c\Delta t\,,\, \Delta x\} \tag{4.104}$$

where $c$ is the acoustic wave speed.

**Theorem 4.3 (Error Estimates for the Coupled Problem)**
*Assume $\tau$ and $s$ satisfy the conditions given previously in (4.53) and (4.54). Similarly, assume $\boldsymbol{\tau}$ and $\boldsymbol{s}$ satisfy,*

$$c_1 h_s \quad \leq \quad ||\boldsymbol{\tau}|| \quad \leq \quad c_2 h_s \tag{4.105}$$

$$c_3 \quad \leq \quad ||\boldsymbol{s}|| \quad \leq \quad c_4 \tag{4.106}$$

*where $c_1$, $c_2$, $c_3$ and $c_4$ are positive constants ($c_2 > c_1$, $c_4 > c_3$). Assuming $\boldsymbol{\chi} \in (H^{k+1}(Q_s))^d \times H^{p+1}(Q_f)$, together with standard interpolation estimates for $\boldsymbol{H}$, then*

$$|||\boldsymbol{E}|||^2 \quad \leq \quad c(\boldsymbol{u})\, h_s^{2k-1} \,+\, c(\phi)\, h_f^{2p-1} \tag{4.107}$$

*where $c(\boldsymbol{u})$ and $c(\phi)$ are values that are independent of the mesh parameters.*

Proof: The proof follows the same basic steps as Theorem 4.2:

$$
\begin{aligned}
|||\boldsymbol{E}^h|||^2 &= B^s_{GLS}(\boldsymbol{e}^h\,,\,\boldsymbol{E}^h) + B^f_{GLS}(e^h\,,\,\boldsymbol{E}^h) && \text{(by Lemma 4.4)}\\
&= B^s_{GLS}(\boldsymbol{e}^h\,,\,\boldsymbol{E}-\boldsymbol{H}) + B^f_{GLS}(e^h\,,\,\boldsymbol{E}-\boldsymbol{H}) && \text{(by (4.97))}\\
&= -B^s_{GLS}(\boldsymbol{e}^h\,,\,\boldsymbol{H}) - B^f_{GLS}(e^h\,,\,\boldsymbol{H}) && \text{(by (4.16) and (4.17))}\\
&\leq \Big|B^s_{GLS}(\boldsymbol{e}^h\,,\,\boldsymbol{H}) + B^f_{GLS}(e^h\,,\,\boldsymbol{H})\Big|\\
&= \Big|E_s(\boldsymbol{e}^h\,,\,\boldsymbol{\eta}) - A(\boldsymbol{e}^h\,,\,\eta\boldsymbol{n})\\
&\quad + E_f(e^h\,,\,\eta) + A(\boldsymbol{\eta}\,,\,e^h\boldsymbol{n}) + E_r(e^h\,,\,\eta)\\
&\quad + \sum_{n=0}^{N-1}\Big\{(\mathcal{L}_s\boldsymbol{e}^h\,,\,\rho_s^{-1}\boldsymbol{\tau}\mathcal{L}_s\boldsymbol{\eta})_{\tilde{Q}^s_n}\\
&\quad + ([\![\boldsymbol{\sigma}(\nabla\boldsymbol{e}^h)(\boldsymbol{x})]\!]\cdot\boldsymbol{n}\,,\,\rho_s^{-1}\boldsymbol{s}[\![\boldsymbol{\sigma}(\nabla\boldsymbol{\eta})(\boldsymbol{x})]\!]\cdot\boldsymbol{n})_{(\tilde{\Upsilon}_s)_n}\\
&\quad + (\boldsymbol{\sigma}(\nabla\boldsymbol{e}^h)\cdot\boldsymbol{n}\,,\,\rho_s^{-1}\boldsymbol{s}\boldsymbol{\sigma}(\nabla\boldsymbol{\eta})\cdot\boldsymbol{n})_{(\Upsilon_\sigma)_n}\\
&\quad + (\mathcal{L}_1\boldsymbol{E}^h\,,\,\rho_s^{-1}\boldsymbol{s}\mathcal{L}_1\boldsymbol{H})_{(\Upsilon_i)_n}\\
&\quad + (\mathcal{L}_f e^h\,,\,c^2\tau\mathcal{L}_f\eta)_{\tilde{Q}^f_n} + (\mathcal{L}_r e^h\,,\,c^2 s\mathcal{L}_r\eta)_{(\tilde{\Upsilon}_\infty)_n}\\
&\quad + ([\![e^h_{,n}(\boldsymbol{x})]\!]\,,\,c^2 s[\![\eta_{,n}(\boldsymbol{x})]\!])_{\tilde{\Upsilon}^f_n} + (\mathcal{L}_2\boldsymbol{E}^h\,,\,c^2 s\mathcal{L}_2\boldsymbol{H})_{(\Upsilon_i)_n}\Big|\\
&\qquad\qquad \text{(by definition of } B^s_{GLS} \text{ and } B^f_{GLS})\\
&= \Big|\sum_{n=0}^{N-1}\Big\{-(\mathcal{L}_s\boldsymbol{e}^h\,,\,\dot{\boldsymbol{\eta}})_{\tilde{Q}^s_n} + (\mathcal{L}_s\boldsymbol{e}^h\,,\,\rho_s^{-1}\boldsymbol{\tau}\mathcal{L}_s\boldsymbol{\eta})_{\tilde{Q}^s_n}\\
&\quad - ([\![\boldsymbol{\sigma}(\nabla\boldsymbol{e}^h)(\boldsymbol{x})]\!]\cdot\boldsymbol{n}\,,\,\dot{\boldsymbol{\eta}})_{\tilde{\Upsilon}^s_n}\\
&\quad - (\boldsymbol{\sigma}(\nabla\boldsymbol{e}^h)\cdot\boldsymbol{n}\,,\,\dot{\boldsymbol{\eta}})_{(\Upsilon_\sigma)_n}\\
&\quad - (\mathcal{L}_1\boldsymbol{E}^h\,,\,\dot{\boldsymbol{\eta}})_{(\Upsilon_i)_n}\\
&\quad + ([\![\boldsymbol{\sigma}(\nabla\boldsymbol{e}^h)(\boldsymbol{x})]\!]\cdot\boldsymbol{n}\,,\,\rho_s^{-1}\boldsymbol{s}[\![\boldsymbol{\sigma}(\nabla\boldsymbol{\eta})(\boldsymbol{x})]\!]\cdot\boldsymbol{n})_{(\tilde{\Upsilon}_s)_n}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (\boldsymbol{\sigma}(\nabla \boldsymbol{e}^h) \cdot \boldsymbol{n} \,,\, \rho_s^{-1} \boldsymbol{s} \boldsymbol{\sigma}(\nabla \boldsymbol{\eta}) \cdot \boldsymbol{n})_{(\Upsilon_\sigma)_n} \\
&+ \quad (\mathcal{L}_1 \boldsymbol{E}^h \,,\, \rho_s^{-1} \boldsymbol{s} \mathcal{L}_1 \boldsymbol{H})_{(\Upsilon_i)_n} \Big\} \\
&\quad \sum_{n=0}^{N-1} \Big\{ -(\mathcal{L}_f e^h \,,\, \dot{\eta})_{\tilde{Q}_n^f} + (\mathcal{L}_f e^h \,,\, c^2 \tau \mathcal{L}_f \eta)_{\tilde{Q}_n^f} \\
&- \quad (\mathcal{L}_r e^h \,,\, \dot{\eta})_{(\Upsilon_\infty)_n} + (\mathcal{L}_r e^h \,,\, c^2 s \mathcal{L}_r \eta)_{(\Upsilon_\infty)_n} + 2 d_1 (\dot{e}^h \,,\, \dot{\eta})_{(\Upsilon_\infty)_n} \\
&- \quad ([\![ e^h_{,n}(\boldsymbol{x}) ]\!] \,,\, \dot{\eta})_{\tilde{\Upsilon}_n^f} - (\mathcal{L}_2 \boldsymbol{E}^h \,,\, \dot{\eta})_{(\Upsilon_i)_n} \\
&+ \quad ([\![ e^h_{,n}(\boldsymbol{x}) ]\!] \,,\, c^2 s [\![ \eta_{,n}(\boldsymbol{x}) ]\!])_{\tilde{\Upsilon}_n^f} + (\mathcal{L}_2 \boldsymbol{E}^h \,,\, c^2 s \mathcal{L}_2 \boldsymbol{H})_{(\Upsilon_i)_n} \Big\} \\
&+ \quad \sum_{n=1}^{N-1} \Big\{ -([\![ \dot{\boldsymbol{e}}^h(t_n) ]\!] \,,\, \rho_s \dot{\boldsymbol{\eta}}(t_n^-))_{\Omega_s} - a([\![ \boldsymbol{e}^h(t_n) ]\!] \,,\, \boldsymbol{\eta}(t_n^-))_{\Omega_s} \Big\} \\
&+ \quad (\dot{\boldsymbol{e}}^h(T^-) \,,\, \rho_s \dot{\boldsymbol{\eta}}(T^-))_{\Omega_s} + a(\boldsymbol{e}^h(T^-) \,,\, \boldsymbol{\eta}(T^-))_{\Omega_s} \\
&+ \quad \sum_{n=1}^{N-1} \Big\{ -([\![ \dot{e}^h(t_n) ]\!] \,,\, a^2 \, \dot{\eta}(t_n^-))_{\Omega_f} - ([\![ \nabla e^h(t_n) ]\!] \,,\, \nabla \eta(t_n^-))_{\Omega_f} \Big\} \\
&+ \quad (\dot{e}^h(T^-) \,,\, a^2 \, \dot{\eta}(T^-))_{\Omega_f} + (\nabla e^h(T^-) \,,\, \nabla \eta(T^-))_{\Omega_f} \\
&- \quad \sum_{n=1}^{N-1} d_0([\![ e^h(t_n) ]\!] \,,\, \eta(t_n^-))_{\Gamma_\infty} + d_0(e^h(T^-) \,,\, \eta(T^-))_{\Gamma_\infty} \Big|
\end{aligned}
$$

(by integration by parts)

$$
\leq \quad \frac{1}{2} |||\boldsymbol{E}^h|||^2 + c(\boldsymbol{u}) h_s^{2k-1} + c(\phi) h_f^{2p-1}
$$

The last lines follows from repeated use of (4.18) and standard interpolation estimates. Subsuming the terms involving $\boldsymbol{E}^h$ to the left-hand-side and from the triangle inequality and the interpolation estimates we have,

$$
|||\boldsymbol{E}|||^2 \quad \leq \quad 2|||\boldsymbol{E}^h|||^2 + 2|||\boldsymbol{H}|||^2 \quad \leq \quad c(\boldsymbol{u}) \, h_s^{2k-1} + c(\phi) \, h_f^{2p-1} \tag{4.108}
$$

which completes the proof. □

This result indicates that the error for the coupled system is controlled by the convergence rates in both the structure and the fluid. In other words, for an accurate solution to the coupled fluid-structure problem, discretizations for both the structural domain and the fluid domain must be adequately resolved.

## 4.6 Two-Field Formulation

In Hughes and Hulbert [1988] an extension of the time-discontinuous Galerkin space-time formulation is given where independent interpolation functions are used for the solution and its time derivative, e.g. for elastodynamics, independent interpolations are used for the displacements and velocities. In this section the single-field formulation described earlier is extended to a two-field formulation incorporating second-order non-reflecting boundary conditions with independent velocity potential and pressure variables. In the following development, only the uncoupled acoustics problem is considered and the superscript indicating fluid terms is omitted for clarity, e.g. $Q_n^f = Q_n$.

The finite element interpolation functions for the two-field formulation are stated in terms of independent trial velocity potential $\phi^h = \phi_1^h$, and trial pressure $p^h = -\rho_0 \phi_2^h$, variables:

*Trial velocity potential*

$$\mathcal{T}_1^h = \left\{ \phi_1^h \middle| \phi_1^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n\Big),\ \phi_1^h\Big|_{Q_n^e} \in \mathcal{P}^k(Q_n^e) \right\}$$

*Trial pressure*

$$\mathcal{T}_2^h = \left\{ \phi_2^h \middle| \phi_2^h \in C^0\Big(\bigcup_{n=0}^{N-1} Q_n\Big),\ \phi_2^h\Big|_{Q_n^e} \in \mathcal{P}^l(Q_n^e) \right\}$$

For the case of pure Neumann boundary conditions, the trial spaces and weighting spaces are the same, i.e. $\mathcal{T}_1^h = \mathcal{W}_1^h$ and $\mathcal{T}_2^h = \mathcal{W}_2^h$. The statement of the time-discontinuous Galerkn method for the two-field formulation is:

Find $\boldsymbol{U}^h := \{\phi_1^h, \phi_2^h\} \in \mathcal{T}_1^h \times \mathcal{T}_2^h$, such that $\forall\ \boldsymbol{W}^h := \{w_1^h, w_2^h\} \in \mathcal{W}_1^h \times \mathcal{W}_2^h$,

$$\boxed{B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) = L_n(\boldsymbol{W}^h), \qquad n = 0, 1, ..., N-1} \tag{4.109}$$

with the following definitions,

$$B_n(\boldsymbol{W}^h, \boldsymbol{U}^h) \;:=\; E_f(\boldsymbol{W}^h, \boldsymbol{U}^h)_n + E_r(\boldsymbol{W}^h, \boldsymbol{U}^h)_n \tag{4.110}$$

$$\begin{aligned} E_f(\boldsymbol{W}^h, \boldsymbol{U}^h)_n \;:=\;& (w_2^h, a^2 \dot{\phi}_2^h)_{Q_n} + (\nabla w_2^h, \nabla \phi_1^h)_{Q_n} \\ +\;& (w_2^h(t_n^+), a^2\, \phi_2^h(t_n^+))_\Omega + (\nabla w_1^h(t_n^+), \nabla \phi_1^h(t_n^+))_\Omega \\ +\;& (\nabla w_1^h, \nabla \mathcal{L}\boldsymbol{U}^h)_{Q_n} \end{aligned} \tag{4.111}$$

$$\begin{aligned} E_r(\boldsymbol{W}^h, \boldsymbol{U}^h)_n \;:=\;& d_2(w_2^h, \dot{\phi}_2^h)_{(\Upsilon_\infty)_n} + d_1(w_2^h, \phi_2^h)_{(\Upsilon_\infty)_n} + d_0(w_2^h, \phi_1^h)_{(\Upsilon_\infty)_n} \\ +\;& d_2(w_2^h(t_n^+), \phi_2^h(t_n^+))_{\Gamma_\infty} + d_0(w_1^h(t_n^+), \phi_1^h(t_n^+))_{\Gamma_\infty} \\ +\;& d_0(w_1^h, \mathcal{L}\boldsymbol{U}^h)_{(\Upsilon_\infty)_n} \end{aligned} \tag{4.112}$$

$$\begin{aligned} L_n(\boldsymbol{W}^h) \;:=\;& (w_2^h, f)_{Q_n} \\ +\;& (w_2^h(t_n^+), a^2\, \phi_2^h(t_n^-))_\Omega + (\nabla w_1^h(t_n^+), \nabla \phi_1^h(t_n^-))_\Omega \\ +\;& d_2(w_2^h(t_n^+), \phi_2^h(t_n^-))_{\Gamma_\infty} + d_0(w_1^h(t_n^+), \phi_1^h(t_n^-))_{\Gamma_\infty} \end{aligned} \tag{4.113}$$

in which,

$$\boxed{\mathcal{L}\boldsymbol{U}^h \;=\; \dot{\phi}_1^h - \phi_2^h} \tag{4.114}$$

In this formulation the pressures, i.e. $p^h = -\rho_0 \phi_2^h$, and velocity potentials, i.e. $\phi^h = \phi_1^h$ are governed by independent interpolations. If we select the pressure space such that, $\mathcal{L}\boldsymbol{U}^h = \dot{\phi}_1^h - \phi_2 = 0$, then the formulation specializes to the single-field formulation presented in Chapter 3.

Since the proofs for stability of the two-field formulation follow nearly identical steps as those used for the single-field formulation, results are presented without proof.

Summing (4.109) over the time slabs we obtain,

$$\boxed{B(\boldsymbol{W}^h\,,\boldsymbol{U}^h) \quad = \quad L(\boldsymbol{W}^h)} \tag{4.115}$$

where

$$\begin{aligned}
B(\boldsymbol{W}^h\,,\boldsymbol{U}^h) \;=\; & \sum_{n=0}^{N-1}(w_2^h\,,\,a^2\,\dot{\phi}_2^h)_{Q_n} \\
+ & \sum_{n=1}^{N-1}(w_2^h(t_n^+)\,,\,a^2\,[\![\phi_2^h(t_n)]\!])_\Omega + (w_2^h(0^+)\,,\,a^2\,\phi_2^h(0^+))_\Omega \\
+ & \sum_{n=0}^{N-1}\Big\{(\nabla w_2^h\,,\,\nabla\phi_1^h)_{Q_n} + (\nabla w_1^h\,,\,\nabla\boldsymbol{\mathcal{L}}\boldsymbol{U}^h)_{Q_n}\Big\} \\
+ & \sum_{n=1}^{N-1}(\nabla w_1^h(t_n^+)\,,\,[\![\nabla\phi_1^h(t_n)]\!])_\Omega + (\nabla w_1^h(0^+)\,,\,\nabla\phi_1^h(0^+))_\Omega \\
+ & \sum_{n=0}^{N-1}d_2(w_2^h\,,\,\dot{\phi}_2^h)_{(\Upsilon_\infty)_n} \\
+ & \sum_{n=1}^{N-1}d_2(w_2^h(t_n^+)\,,\,[\![\phi_2^h(t_n)]\!])_{\Gamma_\infty} + d_2(w_2^h(0^+)\,,\,\phi_2^h(0^+))_{\Gamma_\infty} \qquad (4.116) \\
+ & \sum_{n=0}^{N-1}\Big\{d_0(w_2^h\,,\,\phi_1^h)_{(\Upsilon_\infty)_n} + d_0(w_1^h\,,\,\boldsymbol{\mathcal{L}}\boldsymbol{U}^h)_{(\Upsilon_\infty)_n}\Big\} \\
+ & \sum_{n=1}^{N-1}d_0(w_1^h(t_n^+)\,,\,[\![\phi_1^h(t_n)]\!])_{\Gamma_\infty} + d_0(w_1^h(0^+)\,,\,\phi_1^h(0^+))_{\Gamma_\infty} \\
+ & \sum_{n=0}^{N-1}d_1(w_2^h\,,\,\phi_2^h)_{(\Upsilon_\infty)_n} \qquad (4.117)
\end{aligned}$$

with

$$\begin{aligned}
L(\boldsymbol{W}^h) \;=\; & \sum_{n=0}^{N-1}(w_2^h\,,\,f)_{Q_n} \\
+ & (w_2^h(0^+)\,,\,a^2\,\dot{\phi}_0)_\Omega + (\nabla w_1^h(0^+)\,,\,\nabla\phi_0)_\Omega \qquad (4.118)
\end{aligned}$$

The norm associated with (4.115) is defined by:

$$
\begin{aligned}
|||\boldsymbol{W}^h|||^2 \quad := \quad & \mathcal{E}_f(\boldsymbol{W}^h(0^+)) + \sum_{n=1}^{N-1} \mathcal{E}_f([\![\boldsymbol{W}^h(t_n)]\!]) + \mathcal{E}_f(\boldsymbol{W}^h(T^-)) \\
+ \quad & \mathcal{E}_r(\boldsymbol{W}^h(0^+)) + \sum_{n=1}^{N-1} \mathcal{E}_r([\![\boldsymbol{W}^h(t_n)]\!]) + \mathcal{E}_r(\boldsymbol{W}^h(T^-)) \\
+ \quad & \sum_{n=0}^{N-1} d_1(w_2^h\,,\, w_2^h)_{(\Upsilon_\infty)_n}
\end{aligned}
\tag{4.119}
$$

where

$$
\begin{aligned}
\mathcal{E}_f(\boldsymbol{W}^h) \quad &= \quad \frac{1}{2}\,||a w_2^h||_\Omega^2 + \frac{1}{2}\,||\nabla w_1^h||_\Omega^2 \\
\mathcal{E}_r(\boldsymbol{W}^h) \quad &= \quad \frac{1}{2}\,d_2(w_2^h\,,\, w_2^h)_{\Gamma_\infty} + \frac{1}{2}\,d_0(w_1^h\,,\, w_1^h)_{\Gamma_\infty}
\end{aligned}
\tag{4.120}
$$

The coercivity condition is:

$$
B(\boldsymbol{W}^h\,,\, \boldsymbol{W}^h) = |||\boldsymbol{W}^h|||^2 \tag{4.121}
$$

For additional stability GLS terms can be appended to (4.109). Convergence proofs for the two-field formulation follow closely the same steps as the proofs for the the single-field formulation. Detailed proofs for the interior problem using the two-field formulation can be found in Hughes and Hulbert [1988]. A two-field formulation for the coupled fluid-structure problem is obtained from a combination of the exterior acoustics variational equation (4.109) and the structural variational equation in Hughes and Hulbert [1988], together with consistent coupling of the interface boundary conditions on $\Gamma_i$. Details will be given in a follow-up article to Thompson and Pinsky [1993e].