ME 3070
Foundations of Mechanical Systems

Analytical Position Analysis of Mechanisms and Linkages using Trigonometry

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Approaches for Position Analysis

1. Graphical

- Requires scaled drawings
- CAD system such as SolidWorks or AutoCAD
- By Hand requires Compass, Protractor, Ruler

2. Trigonometry

- Requires Trigonometry of triangular geometry
- Law of Sines and Cosines

3. Position Vector Loop

- Requires Vector Mathematics
- Local attached coordinates and dot products for included angles

Choice of Methods

- **Trig method**: fast for planar mechanisms when a clean triangle exists.

- **Vector-loop method**: scales better, leads naturally to numerical methods and velocity/acceleration analysis.

Approaches to Position Analysis

Graphical Method

This method uses visual diagrams for linkage position analysis, aiding early design and feasibility studies.

Trigonometric Method

Utilizes geometric properties and trigonometric laws to efficiently analyze planar mechanisms with closed-loop triangles.

Vector-Loop Method

Applies vector algebra for position relations, enabling smooth extension to velocity and acceleration analysis.

Graphical Method Tools

Manual Drafting Tools

Ruler, compass, and protractor enable precise scaled drawings and reinforce geometric relationships in mechanisms.

Graphical Method Importance

Graphical methods visually reinforce kinematic principles, intuition, and enhance spatial reasoning for mechanism analysis and design.

Computer-Aided Design (CAD)

CAD offers precision and rapid simulation of mechanisms, enabling parametric design and early error detection.

Vector-Loop Advantages

Structured Mechanism Geometry

Vector-loop method uses vector algebra to represent closed-loop constraints systematically and scalably.

Unified Kinematic Analysis

Differentiating vector-loop equations progresses from position to velocity and acceleration effortlessly.

Dot Product Simplification

Dot products help eliminate unknown vector magnitudes and relate scalar quantities like link lengths and angles.

Compatibility with Computational Tools

Vector-loop equations integrate well with symbolic solvers and numerical methods for advanced engineering applications.

When Trigonometry is Efficient

Triangular Mechanism Geometry

Trigonometry is most efficient when mechanism geometry forms closed triangles within the kinematic loop.

Use of Triangle Laws

Law of Sines, Law of Cosines, and right triangle properties simplify calculation of unknown link angles and lengths.

Computational Efficiency

Single triangle analysis avoids iterative or vector methods, saving time and effort in calculations.

Educational Advantages

Trigonometric solutions enhance intuitive understanding and support learning and rapid engineering checks.

Roadmap

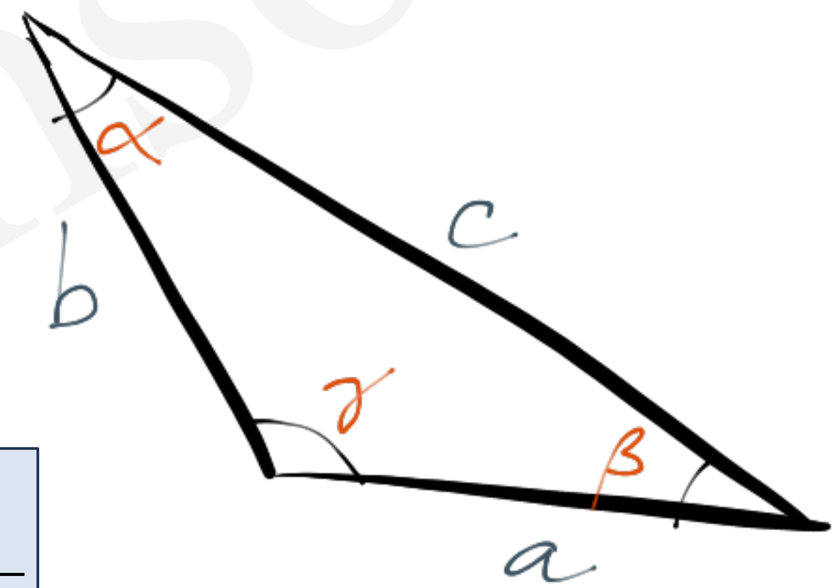
trig laws → worked toggle clamp → four-bar → offset crank-slider.

Trigonometric Laws for Oblique Triangles

- Law of Sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$

- Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

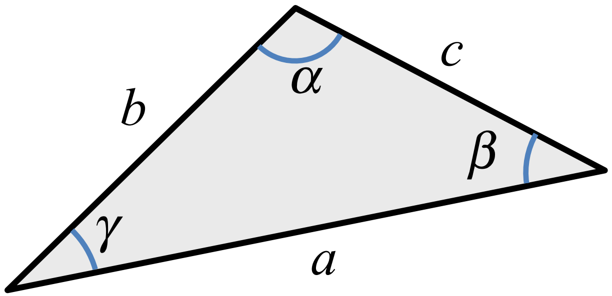


Specialized to right triangle

$$\gamma = 90^\circ, \quad \cos 90^\circ = 0, \qquad c = \sqrt{a^2 + b^2}$$

Analytical Position Analysis:

Use triangles & trigonometry for side lengths and interior angles.



$$c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

$$b = \sqrt{a^2 + c^2 - 2ac\cos\beta}$$

$$a = \sqrt{b^2 + c^2 - 2bc\cos\alpha}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Position Analysis (Trig) Recipe

1. Draw a kinematic diagram + label link lengths and angles.

2. Pick the triangle that closes the vector loop.

3. Choose Law of Cosines to get the *first* unknown angle (or diagonal).

4. Use Law of Sines/Cosines to get the remaining angles.

5. Take differences in fixed distances or fixed angles for displacement.

6. Sanity checks:
   - triangle inequality,
   - angle ranges,
   - “open vs crossed” configuration.

Law of Sines Question

Law of Sines Formula

The Law of Sines states that the ratio of a triangle's side length to the sine of its opposite angle is constant: a/sin(A) = b/sin(B) = c/sin(C).

Use in Mechanism Analysis

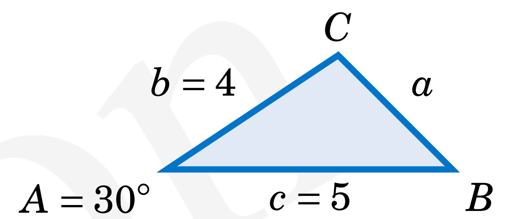
Useful in mechanisms when two angles and one side are known, aiding in computing unknown angles or sides in linkage triangles.

Ambiguity in Solutions

The sine function's ambiguity means both acute and obtuse angles can satisfy the law, requiring careful interpretation in applications.

Example 3.4

Case 3: Two sides and the angle between them.
Solve the triangle $\triangle ABC$ given $A = 30°$, $b = 4$, and $c = 5$.



Solution: We will use the Law of Cosines to find a, use it again to find B, then use $C = 180° - A - B$. First, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 4^2 + 5^2 - 2(4)(5)\cos 30° = 6.36 \quad \Rightarrow \quad \boxed{a = 2.52}. \end{aligned}$$

Now we use the formula for b^2 to find B:

$$b^2 = c^2 + a^2 - 2ca\cos B \quad \Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\Rightarrow \quad \cos B = \frac{5^2 + (2.52)^2 - 4^2}{2(5)(2.52)} = 0.6091$$

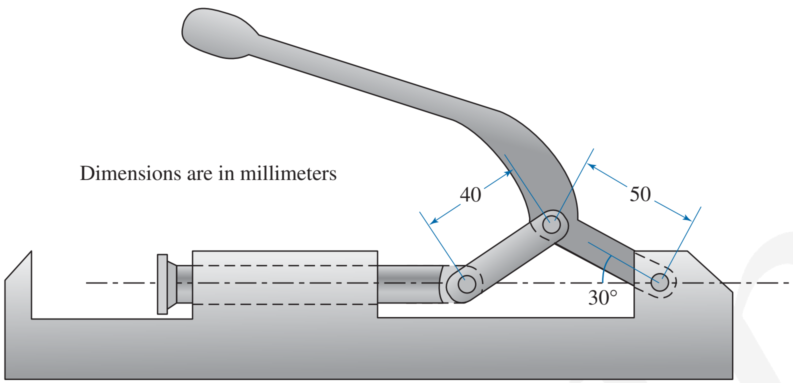
$$\Rightarrow \quad \boxed{B = 52.5°}$$

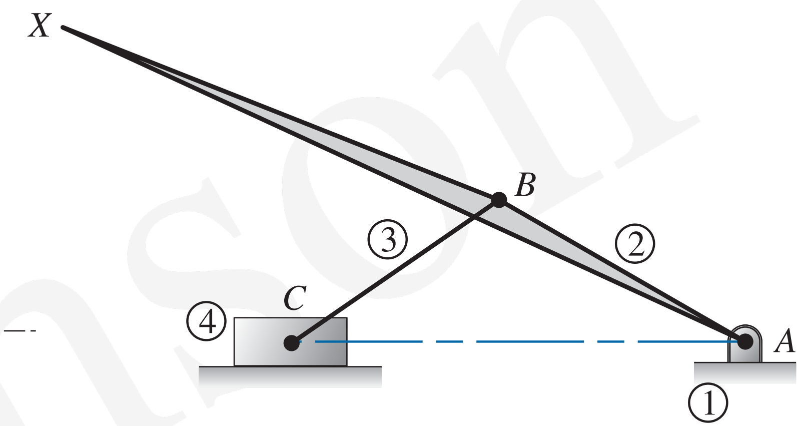
Thus, $C = 180° - A - B = 180° - 30° - 52.5° \Rightarrow \boxed{C = 97.5°}$.

$$\sin B = \frac{b\sin A}{a} = \frac{4\sin 30°}{2.52} = 0.7937 \quad \Rightarrow \quad B = 52.5° \text{ or } 127.5°$$

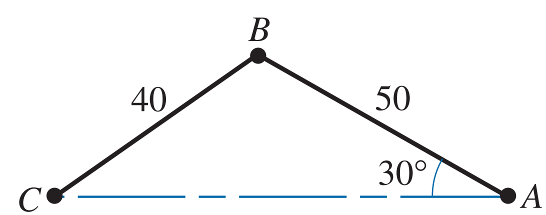
Example 4.3: Toggle Clamp used to securely hold parts.

Determine the displacement of the clamp surface as the handle rotates downward, 15 deg.

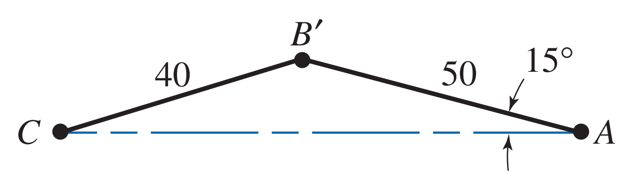




(a)

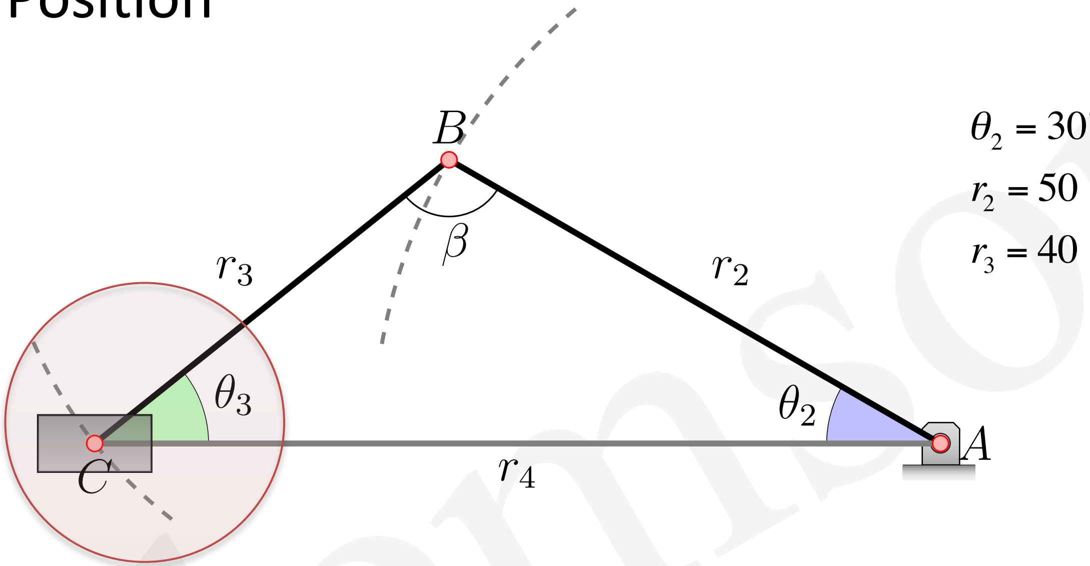


(b)



(c)

Original Position

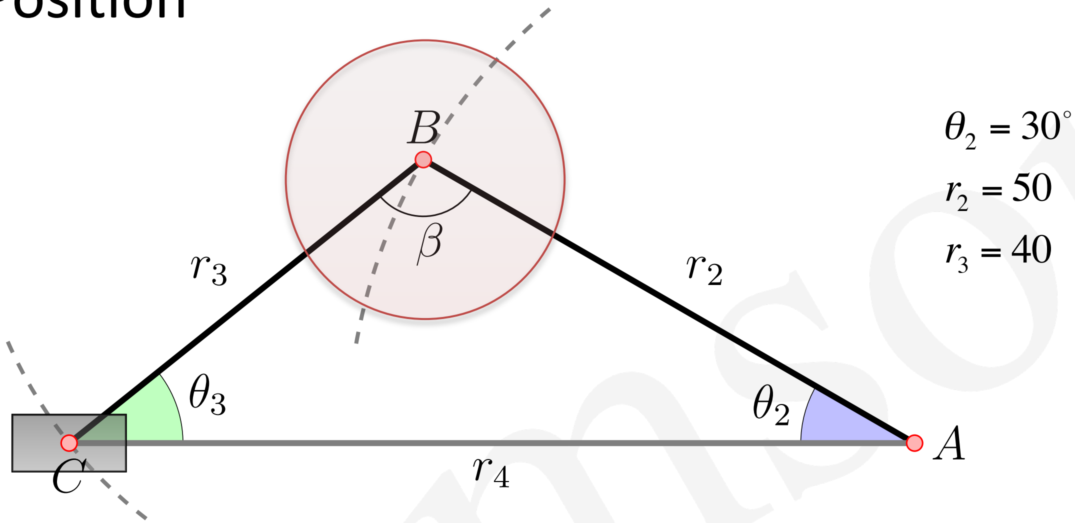


The internal angle at joint C, can be determined from the law of sines since one angle is known with opposite side lengths.

$$\frac{\sin\theta_2}{r_3} = \frac{\sin\theta_3}{r_2},$$

$$\theta_3 = \sin^{-1}\left[\frac{r_2}{r_3}\sin\theta_2\right] = \sin^{-1}\left[\frac{50}{40}\sin 30\right] = 38.68^\circ$$

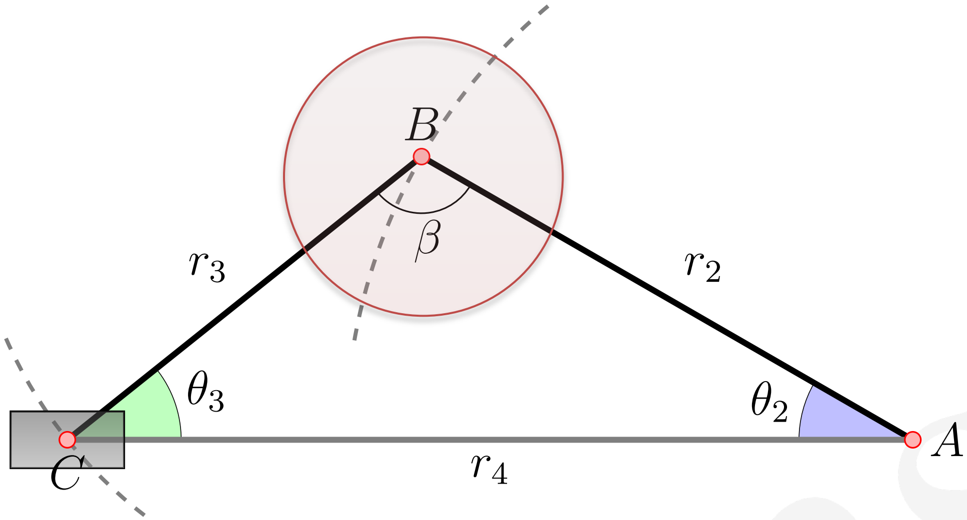
Original Position



The interior angle at joint B, β, can now be determined since the other two angles are now known and the sum of all interior angles in any triangle must total 180°.

$$\beta = 180^\circ - (\theta_2 + \theta_3) = 180^\circ - (30^\circ + 38.68^\circ) = 111.32^\circ$$

CAUTION when using the Law of Sines to find the beta angle, why?



$$\beta = 111.32^o$$

$$\theta_2 = 30^\circ$$
$$r_2 = 50$$
$$r_3 = 40$$

CAUTION when using the Law of Sines to find the beta angle

Failure mode: $\sin(\beta) = \sin(180^\circ - \beta) \rightarrow$ arcsin returns the acute (< 90) solution even when geometry requires obtuse (> 180).

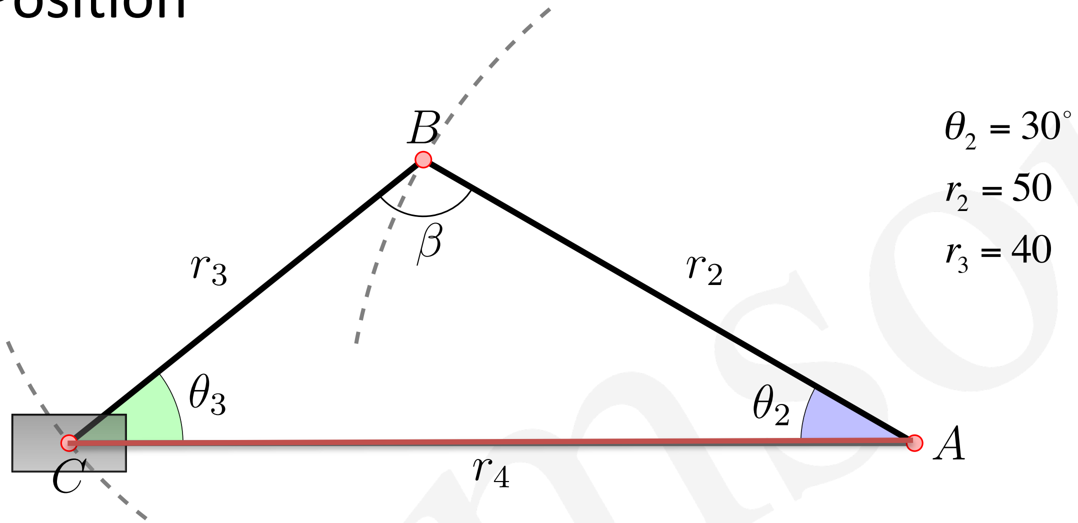
Remedies

•Prefer **Law of Cosines** to determine the angle first when possible (no ambiguity).

•If using Law of Sines, enforce configuration by:

- checking the sketch (obtuse vs acute),
- using angle sum in triangle, or
- comparing predicted point location against the mechanism configuration ("open/crossed").

Original Position

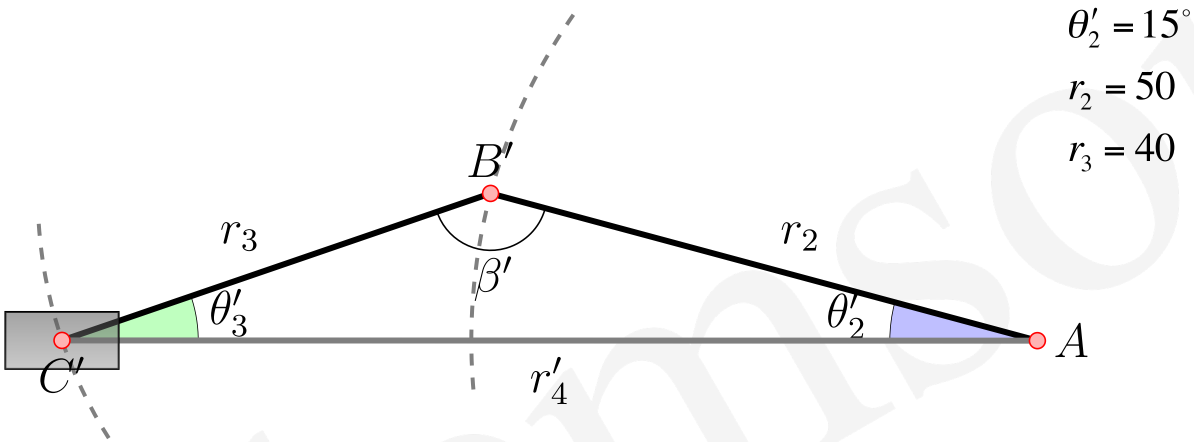


The length of side AC, r_4, represents the original position of the slider and can be determined from the law of cosines, since the angle β is known with adjacent side lengths,

$$r_4 = \sqrt{r_2^2 + r_3^2 - 2r_2r_3\cos\beta}$$

Substituting values gives $s = 74.52$ mm.

Displaced Position



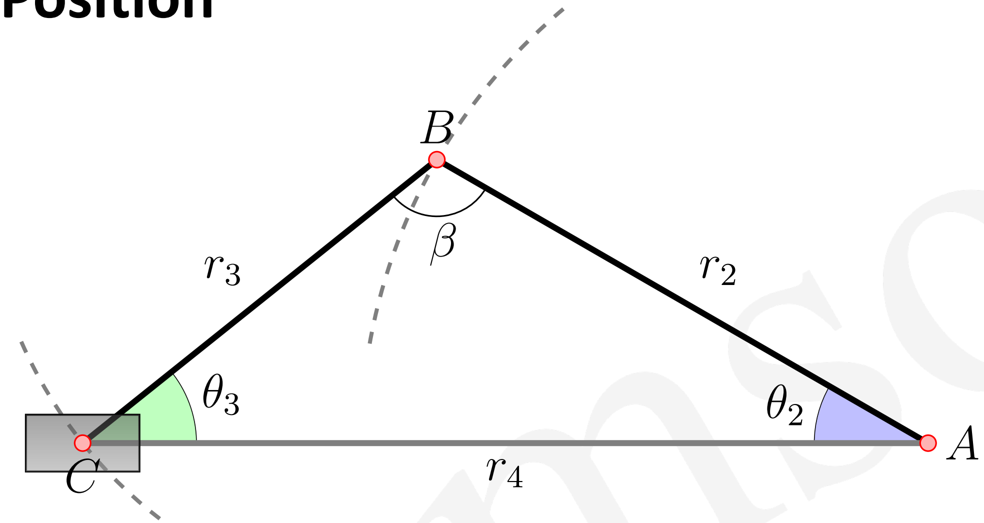
$$\theta_3' = \sin^{-1}\left[\frac{r_2}{r_3}\sin\theta_2'\right] = \sin^{-1}\left[\frac{50}{40}\sin 15\right] = 18.88^\circ$$

$$\beta' = 180^\circ - (\theta_2' + \theta_3') = 180^\circ - (15^\circ + 18.88^\circ) = 146.12^\circ$$

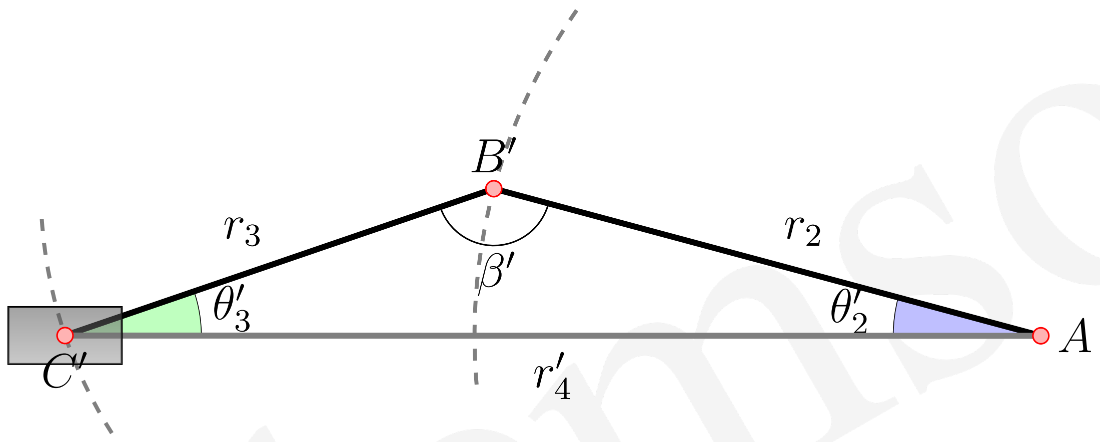
The length of the displaced position of the slider is

$$r_4' = \sqrt{r_2^2 + r_3^2 - 2r_2r_3\cos\beta'} = 86.14\,\text{mm}$$

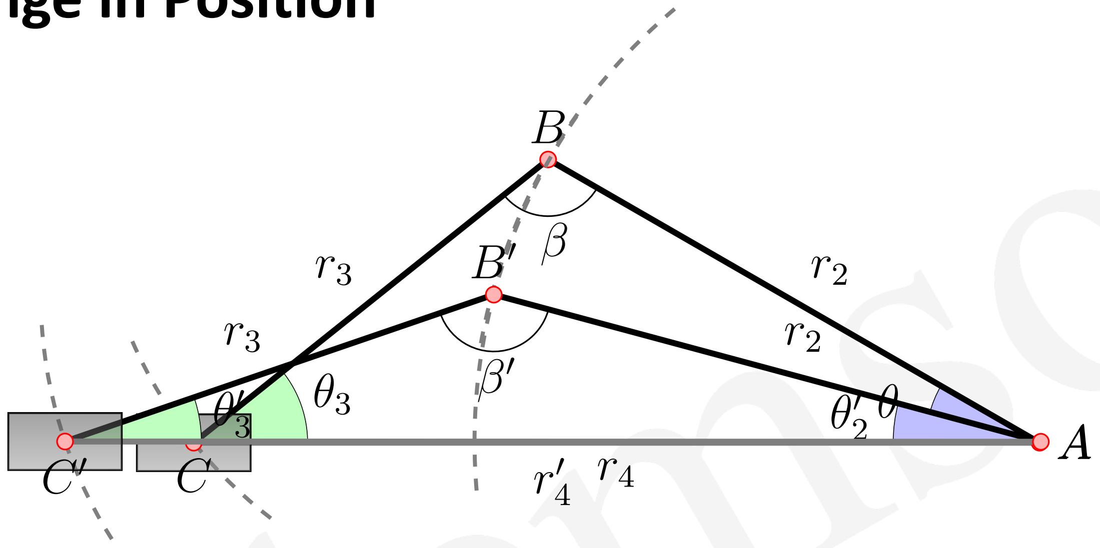
Original Position



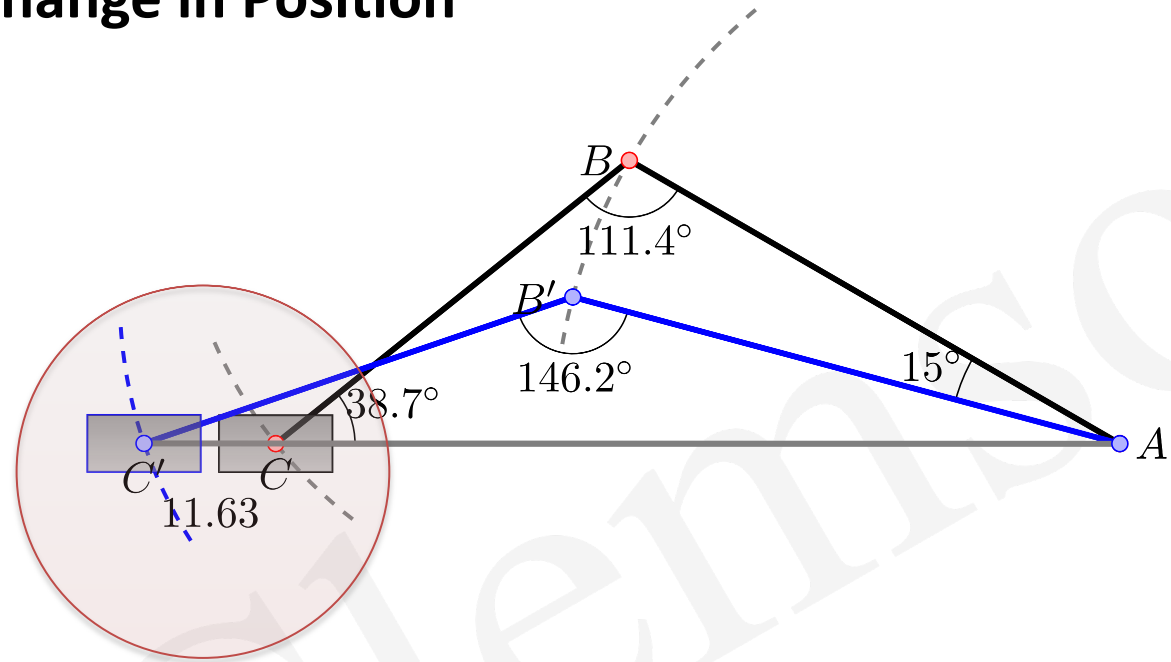
Displaced Position



Change in Position



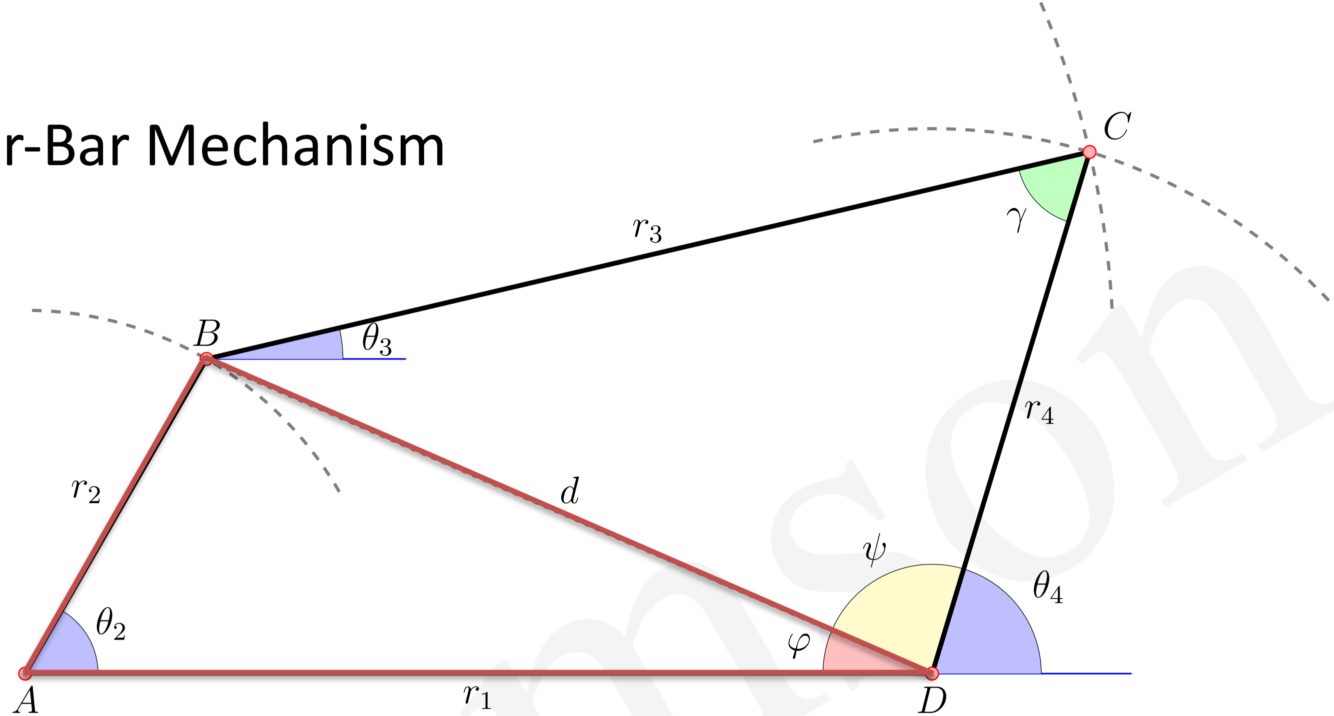
Change in Position



Sanity Checks:

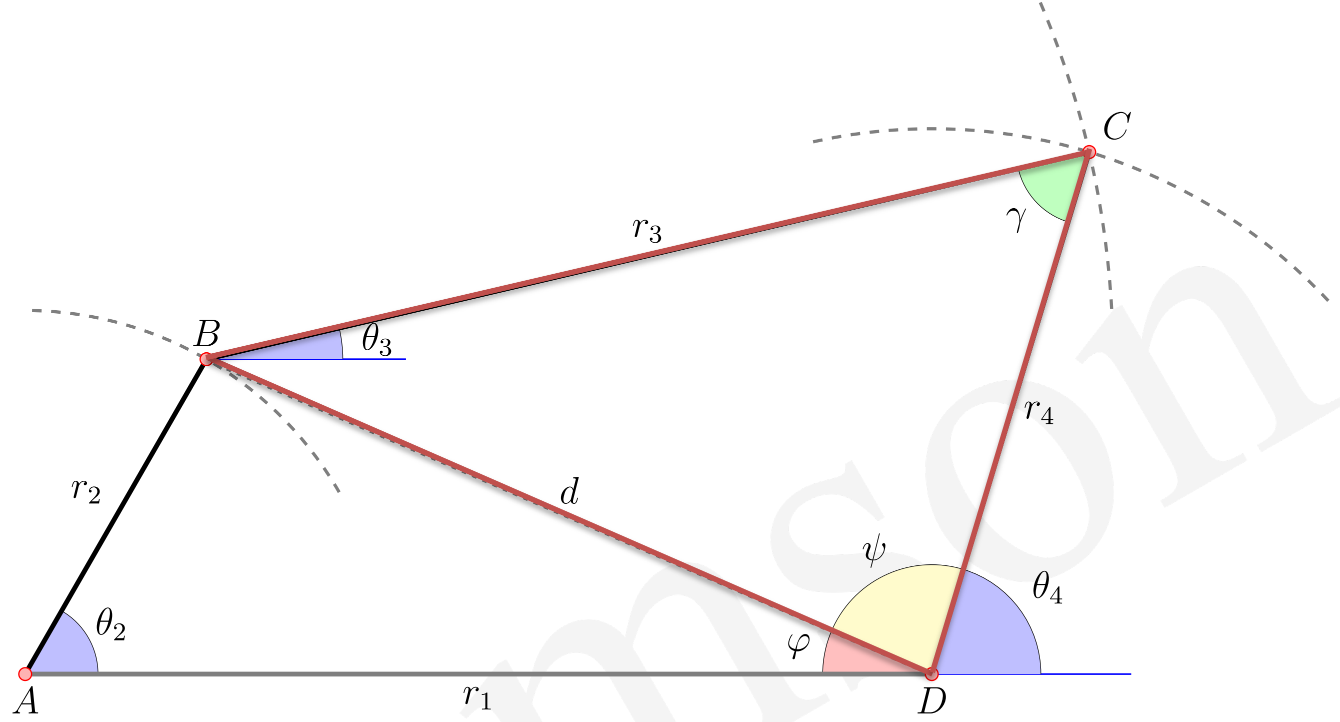
- Does the displacement direction match the picture?
- Are computed lengths consistent with the triangle inequality?
- Does the new point location satisfy link length constraints?

Four-Bar Mechanism



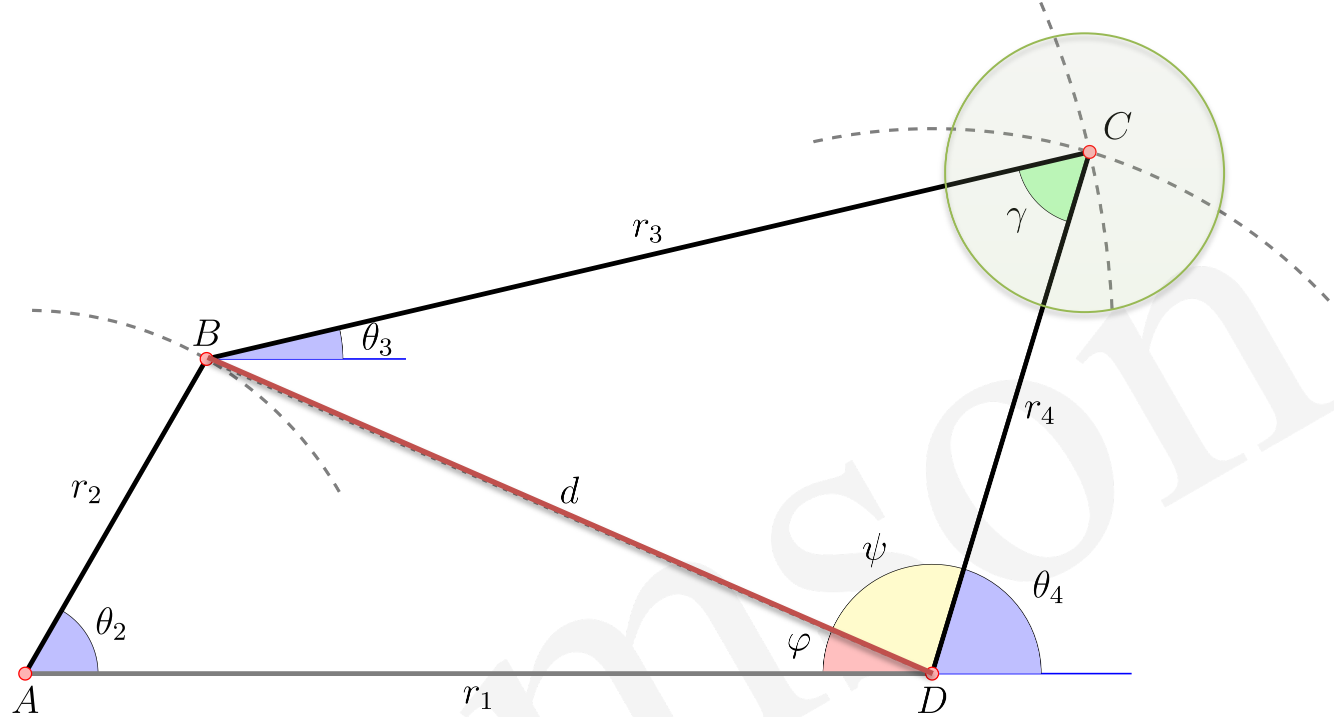
From the lower triangle $\angle BAD$, we can use the law of cosines to determine the diagonal distance, d, of BD, with known crank angle θ_2.

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2$$



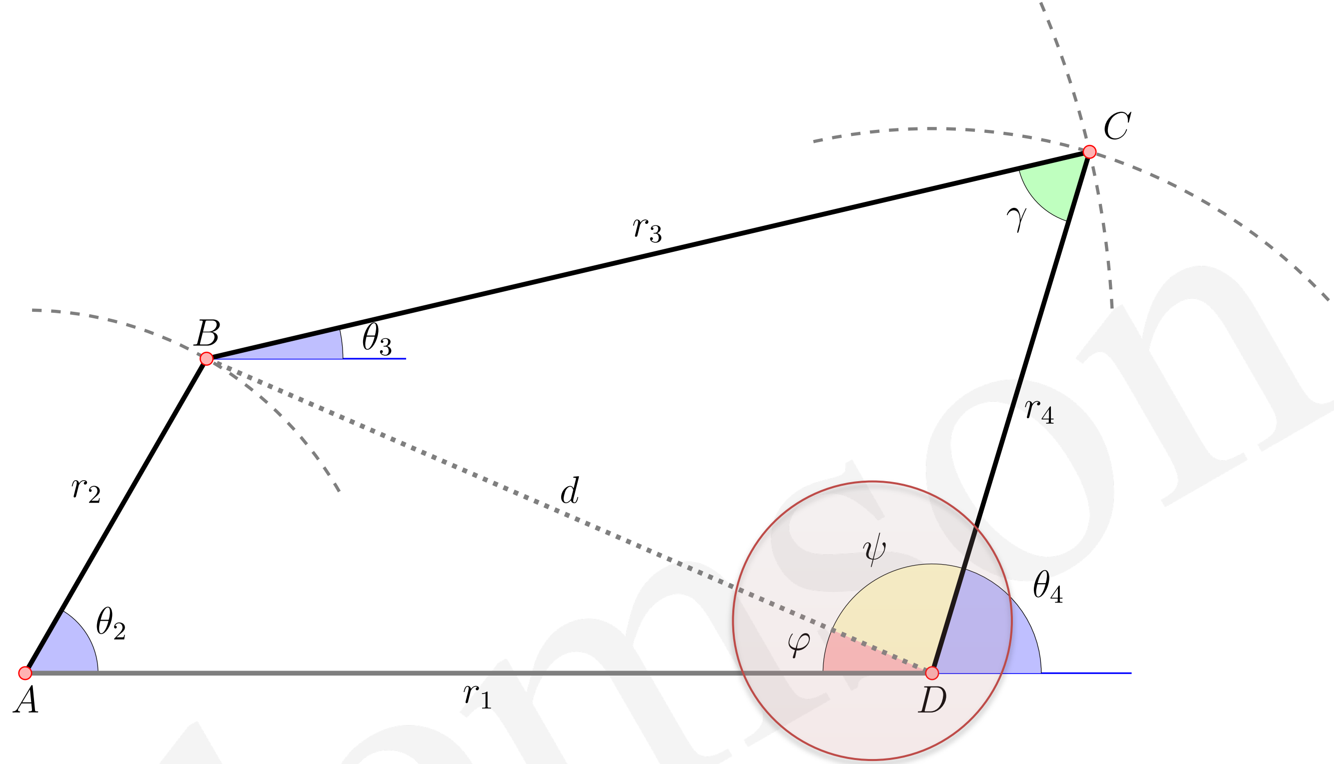
Again, for the upper triangle $\angle BCD$, the law of cosines relating γ and the diagonal BD gives,

$$d^2 = r_3^2 + r_4^2 - 2r_3r_4\cos\gamma$$



Equating the diagonal of BD, we can solve for the interior angle at C,

$$\gamma = \cos^{-1}\left(\frac{r_3^2 + r_4^2 - d^2}{2r_3r_4}\right)$$

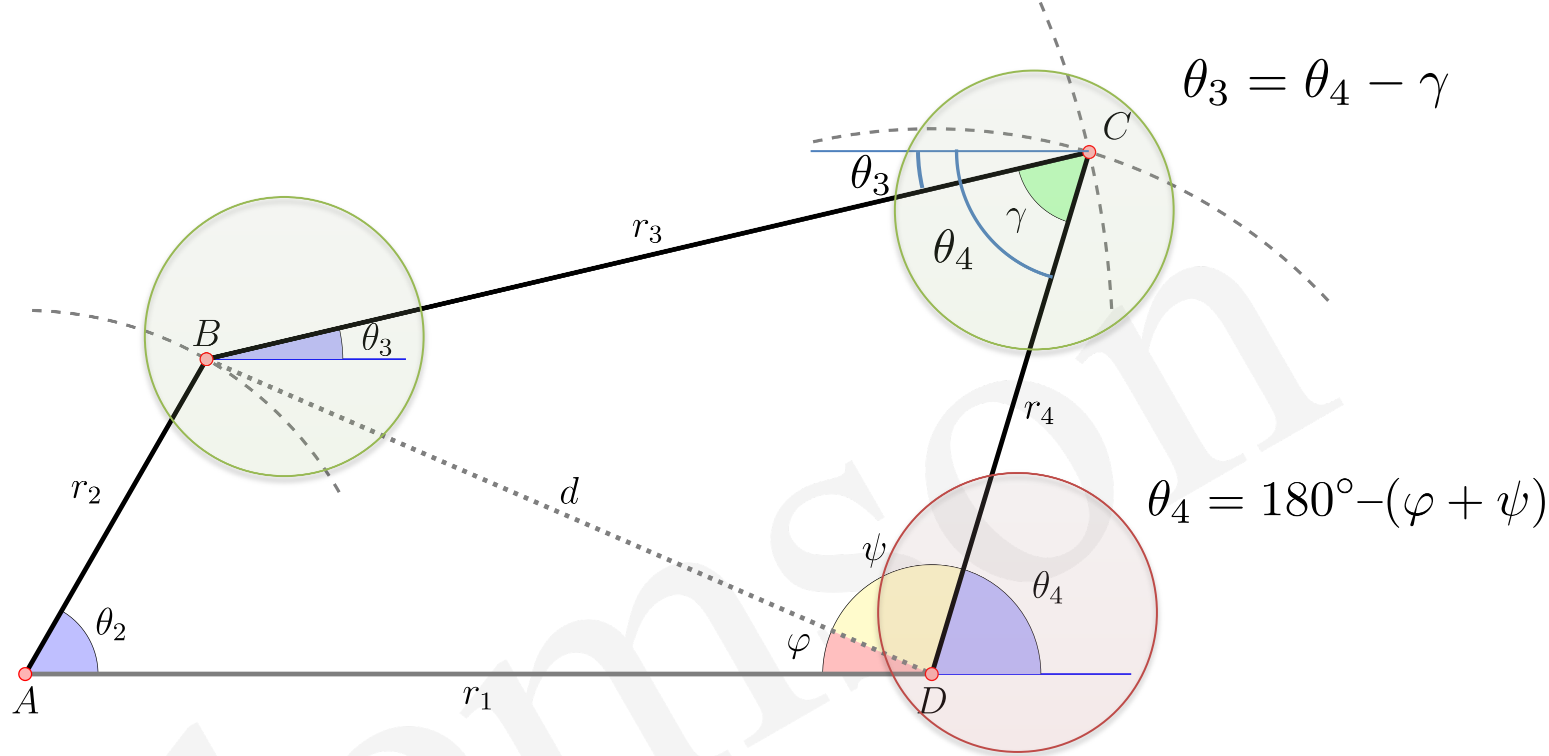


The other interior angles for the two triangles can be found using either the law of cosines or law of sines. The interior angle of $\angle BDA$ of the lower triangle can be determined from the law of sines:

$$\varphi = \sin^{-1}\left(\frac{r_2}{d}\sin\theta_2\right)$$

Similarly, the interior angle $\angle CDB$ for the upper triangle can be determined

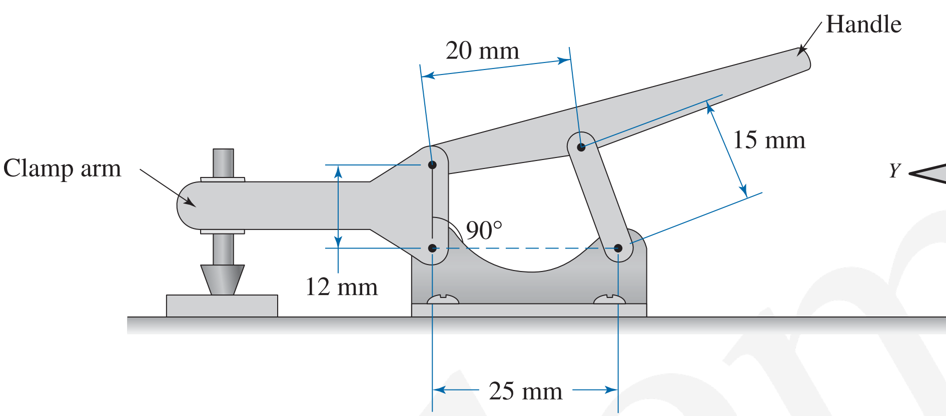
$$\psi = \sin^{-1}\left(\frac{r_3}{d}\sin\gamma\right)$$

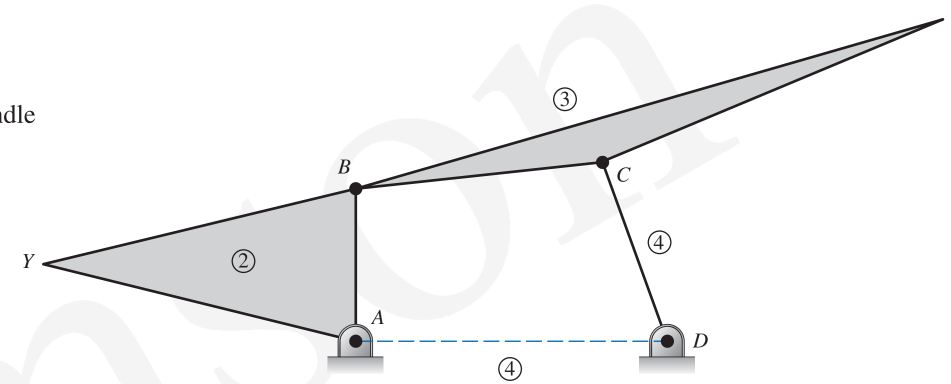


When taking the inverse sine, the angle should be checked for less than or greater than 90 deg. The total interior angle at joint D is $\varphi + \psi$, and the rocker angle of link 4 with respect to the x-axis is $\theta_4 = 180 - (\varphi + \psi)$. The angle of link 3 above the horizontal can be determined by $\theta_3 = \theta_4 - \gamma$ or by totaling the interior angles of the two triangles at joint B, and subtracting $180 - \theta_2$.

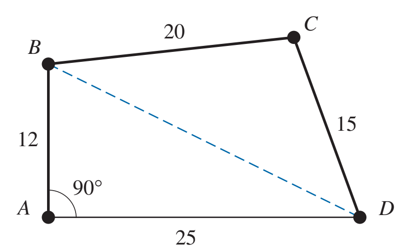
Example 4.5: Toggle Clamp.

Determine the angle that the handle must be displaced in order to lift the clamp arm 30°, clockwise.

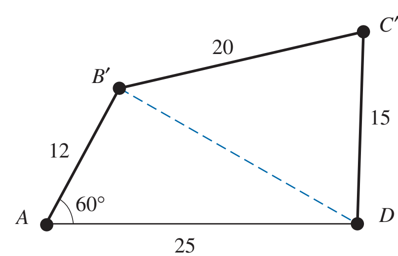




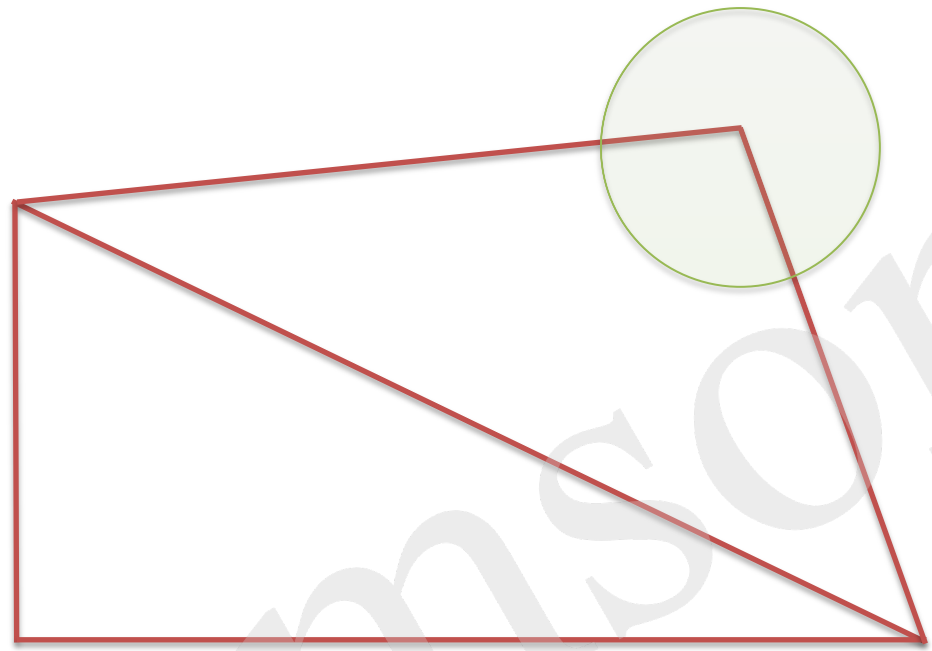
(a) Kinematic diagram



(b) Original configuration



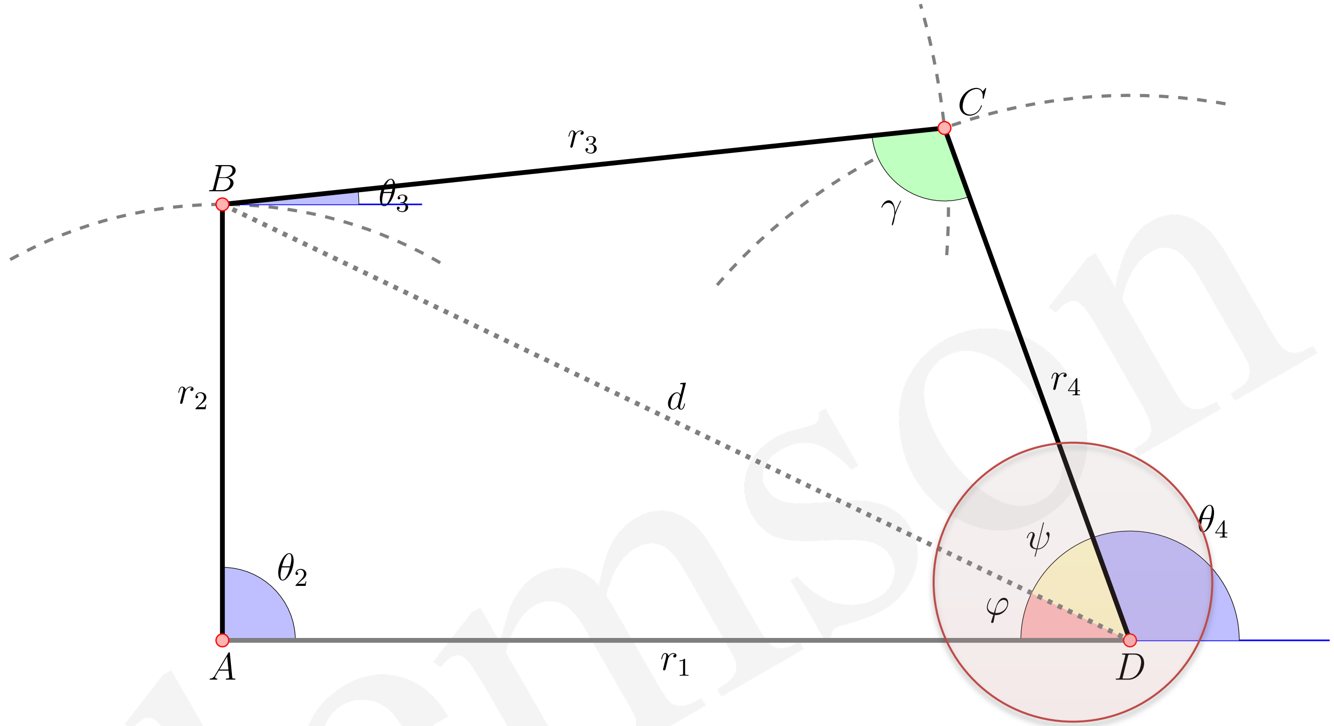
(c) Displaced configuration



$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2} = \sqrt{(12)^2 + (25)^2} = 27.73\,\text{mm}$$

For the upper triangle $\angle BCD$, the law of cosines relating the interior angle at C, γ and the diagonal BD gives,

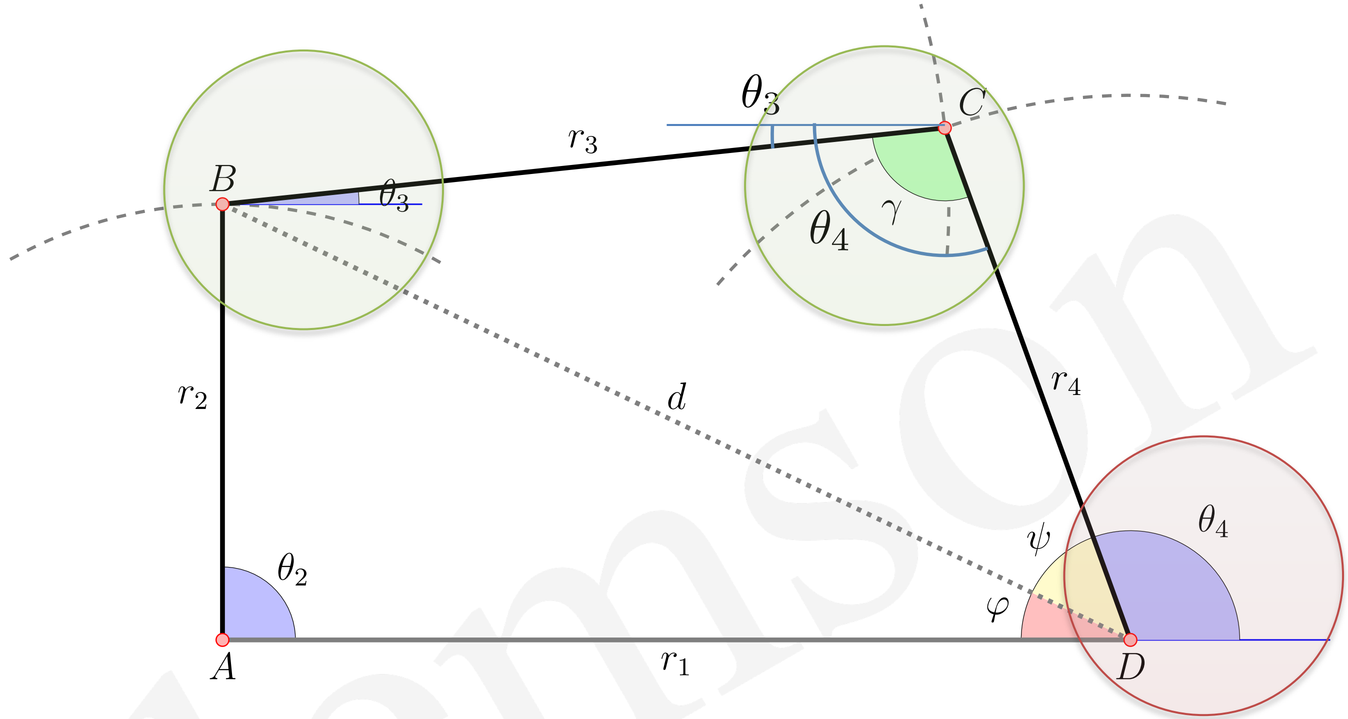
$$\gamma = \cos^{-1}\left(\frac{r_3^2 + r_4^2 - d^2}{2r_3 r_4}\right) = \cos^{-1}\left(\frac{(20)^2 + (15)^2 - (27.73)^2}{2(20)(15)}\right) = 103.9^\circ$$



$$\varphi = \cos^{-1}\left(\frac{r_1}{d}\right) = \cos^{-1}\left(\frac{25}{27.73}\right) = 25.6^\circ$$

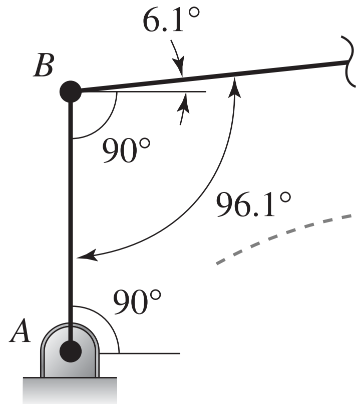
The interior angle $\angle CDB$ of the upper triangle can be determined by the law of sines:

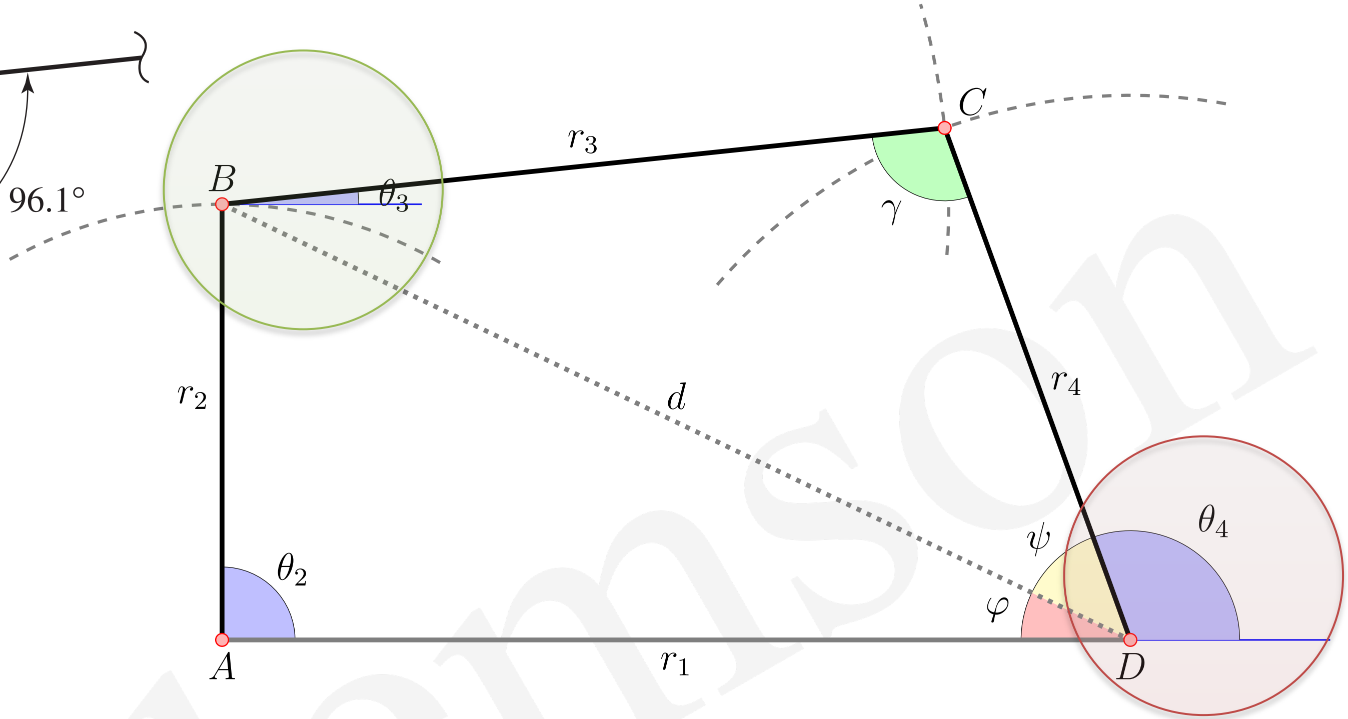
$$\psi = \sin^{-1}\left(\frac{r_3}{d}\sin\gamma\right) = \sin^{-1}\left(\frac{20}{27.73}\sin 103.9^\circ\right) = 44.4^\circ$$



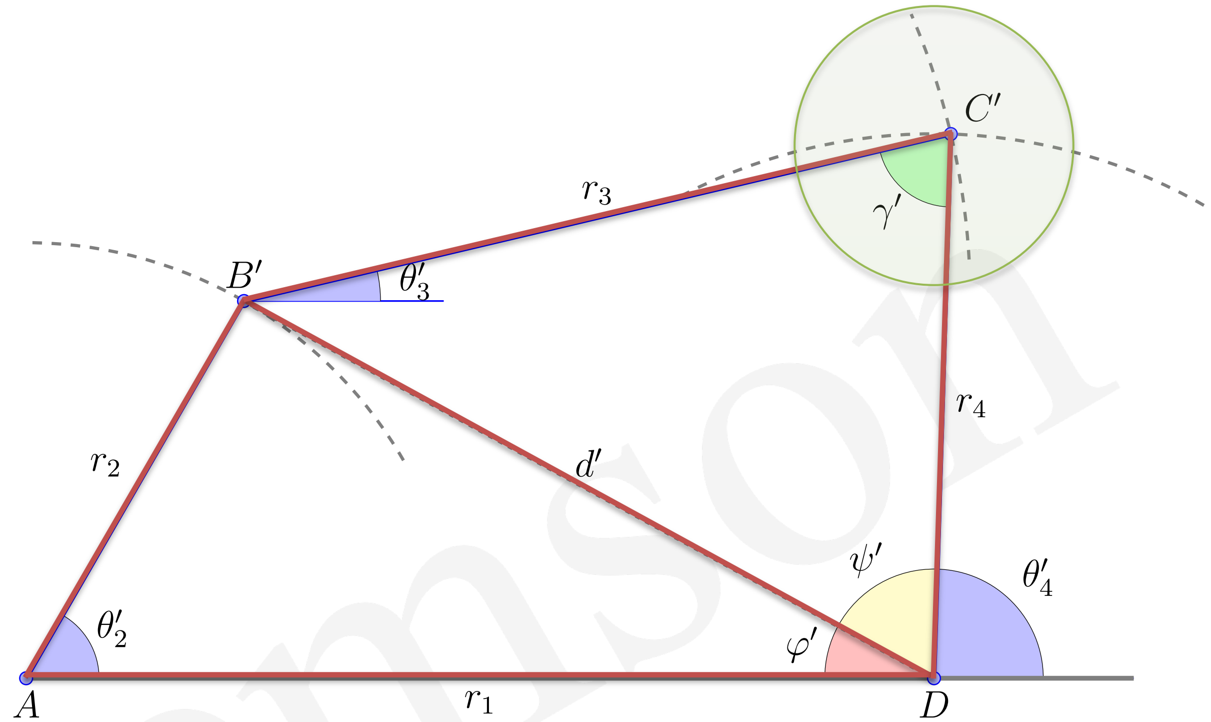
$$\theta_4 = 180^\circ - (\varphi + \psi) = 180 - (25.6 + 44.4) = 110^\circ$$

$$\theta_3 = \theta_4 - \gamma = 110^\circ - 103.9^\circ = 6.1^\circ$$





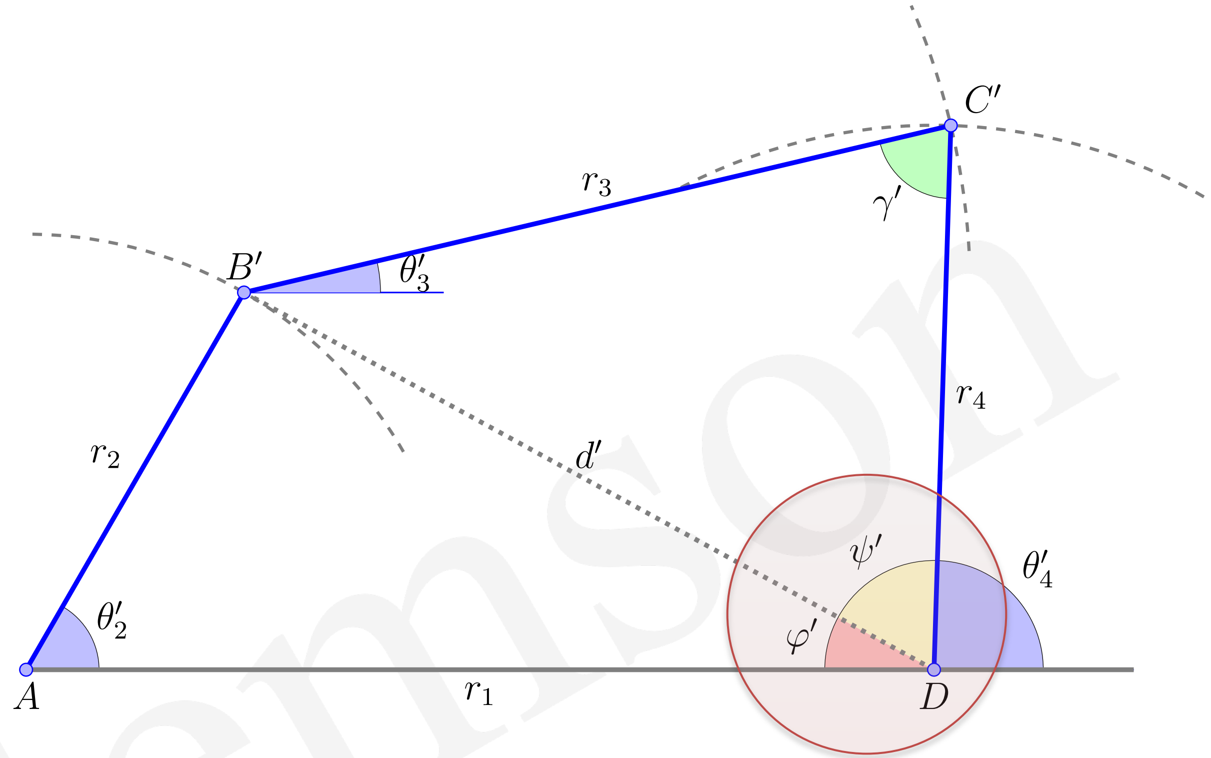
The total interior angle at joint D (between links 1 and 4) is then $25.6° + 44.4° = 70°$. The rocker angle of link 4 with respect to horizontal is $\theta_4 = 180 - 70 = 110°$. The angle of link 3 above the horizontal can be determined by $\theta_3 = \theta_4 - \gamma = 110 - 103.9 = 6.1°$. Alternatively, the interior angle at joint B can first be determined adding the interior angles of the two triangles at B. The angle of link 3 above the horizontal can then be found by subtracting $\theta_2 = 90$. The interior angles $\angle ABD$ and $\angle CBD$ can be found since the sum of all interior triangles in any triangle must total $180°$, i.e. $90 - 25.6 = 64.4°$, and $180 - (103.9 + 44.4) = 31.7°$, respectively. The total interior angle at joint B (between links 2 and 3) is then $64.4° + 31.7° = 96.1°$, and then the angle of link 3 above the horizontal is $\theta_3 = 96.1° - \theta_2 = 6.1°$.



$$d' = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta'_2} = \sqrt{(12)^2 + (25)^2 - 2(12)(25)\cos 60^\circ} = 21.66\,\mathrm{mm}$$

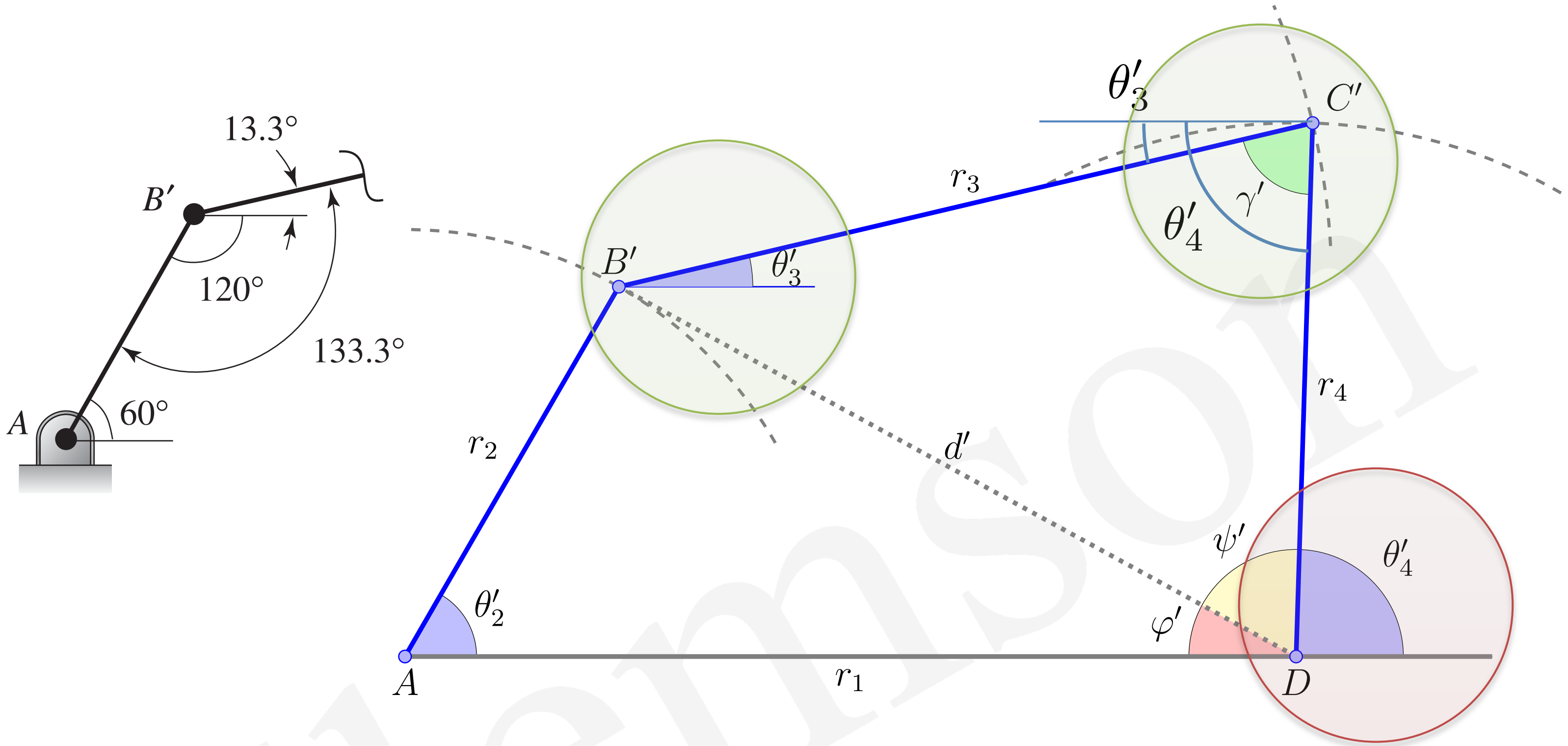
The law of cosines relating the interior angle at C', is then

$$\gamma' = \cos^{-1}\left(\frac{r_3^2 + r_4^2 - d'^2}{2r_3r_4}\right) = \cos^{-1}\left(\frac{(20)^2 + (15)^2 - (21.66)^2}{2(20)(15)}\right) = 74.9^\circ$$

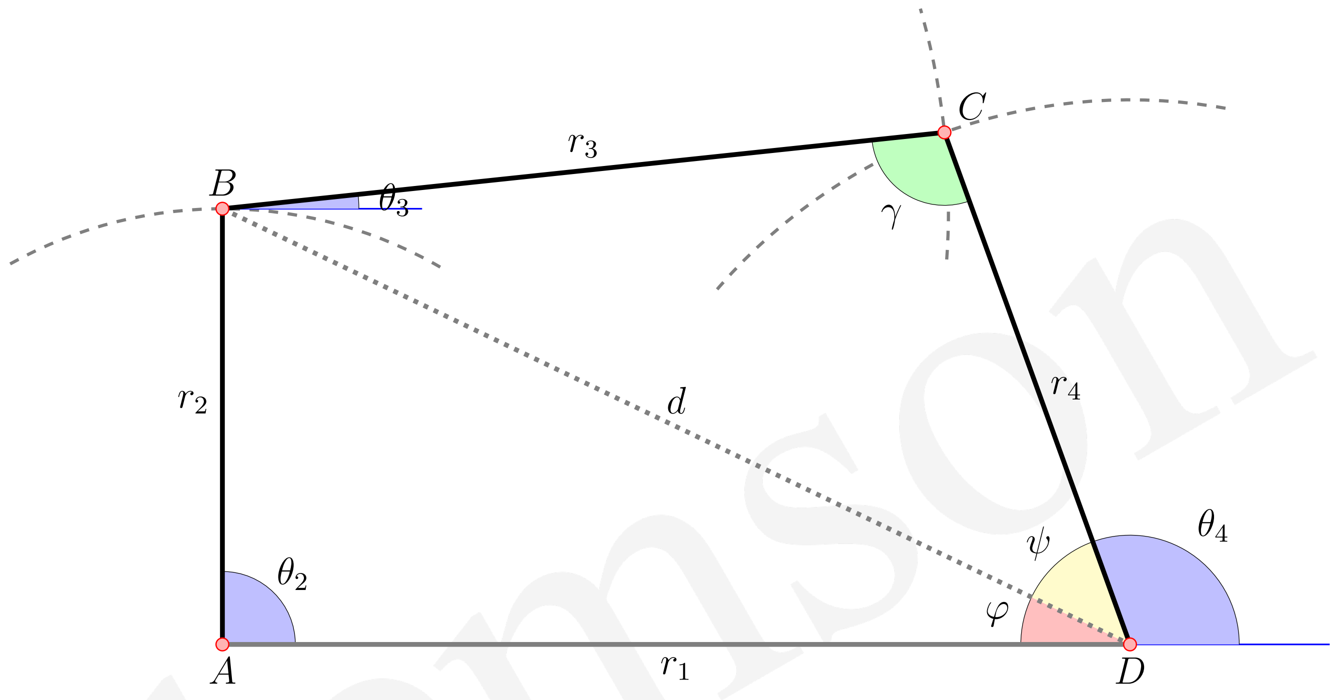


$$\varphi' = \sin^{-1}\left(\frac{r_2}{d'}\sin\theta_2\right) = \sin^{-1}\left(\frac{12}{21.66}\sin 60^\circ\right) = 28.7^\circ$$

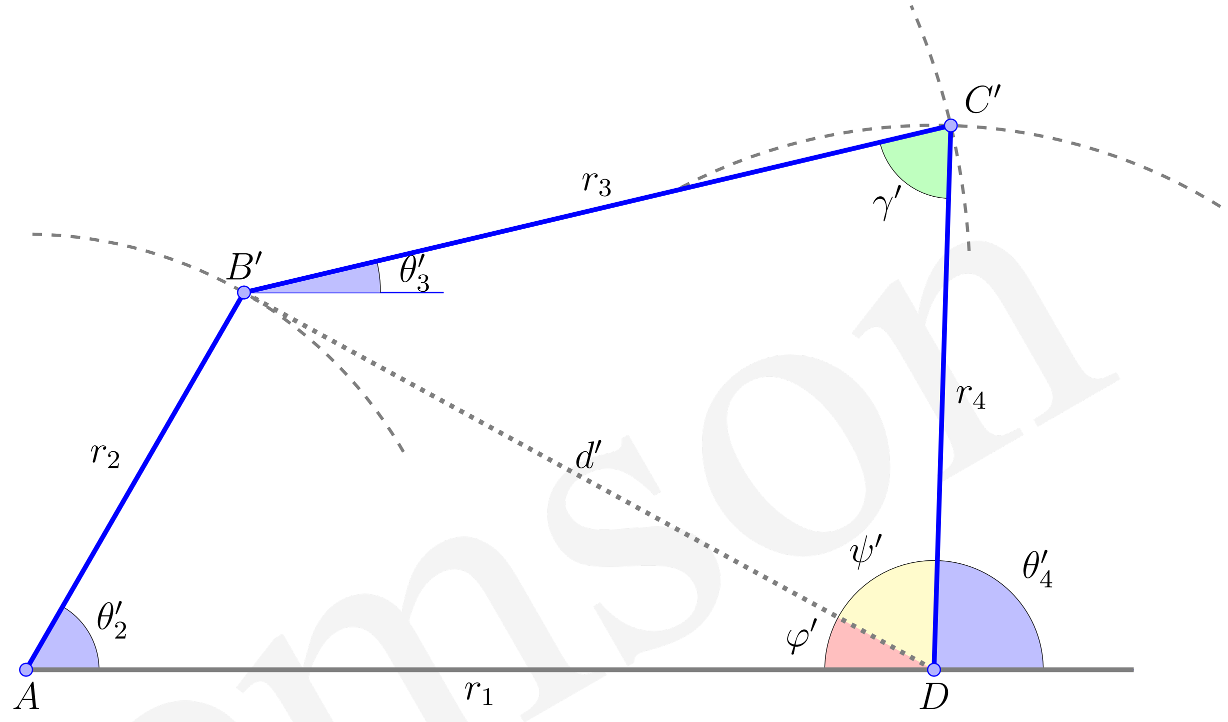
$$\psi' = \sin^{-1}\left(\frac{r_3}{d'}\sin\gamma'\right) = \sin^{-1}\left(\frac{20}{21.66}\sin 74.9^\circ\right) = 63.1^\circ$$



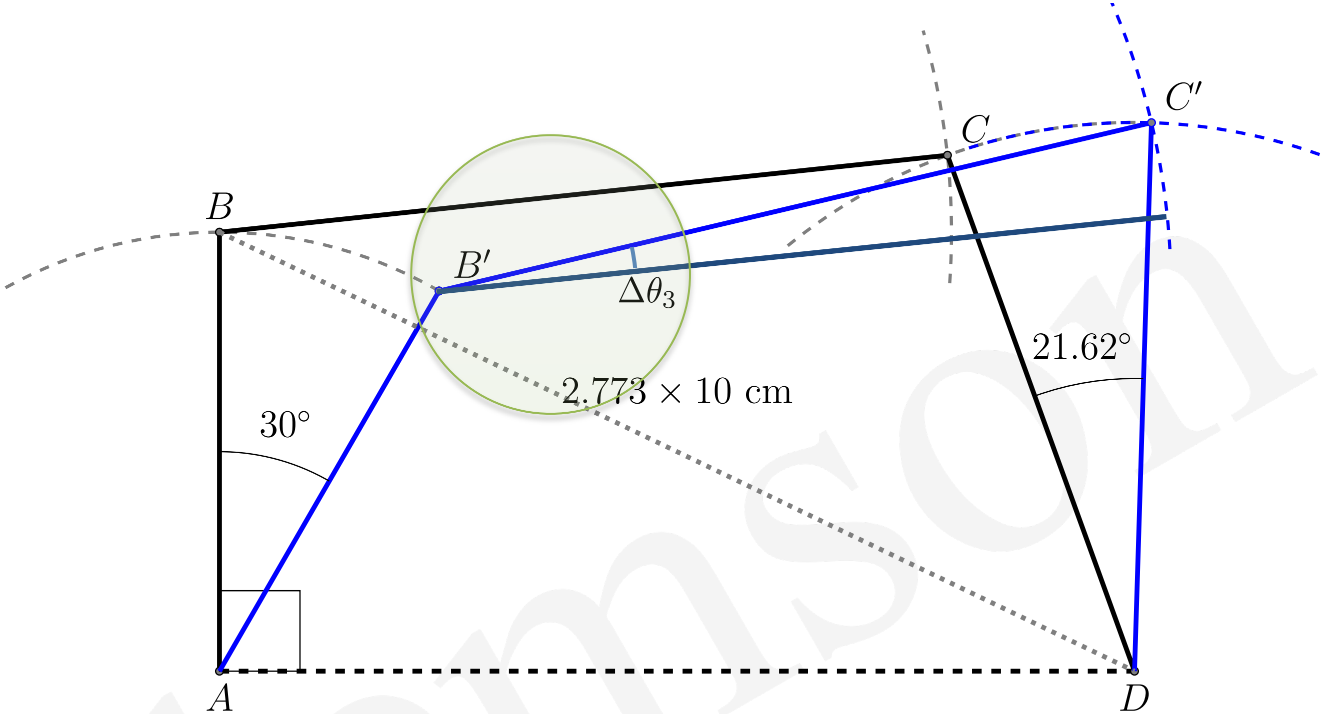
The total interior angle at joint D (between links 1 and 4) is then $28.7° + 63.1° = 91.8°$. The rocker angle of link 4 with respect to horizontal is $\theta_4' = 180 - 91.8 = 88.2°$. For the displaced configuration, the angle of link 3 above the horizontal is then $\theta_3' = \theta_4' - \gamma' = 88.2° - 74.9° = 13.3°$. Finally, the angular displacement of the handle link 3 is determined by the difference of the displaced and original angles $\Delta\theta_3 = \theta_3' - \theta_3 = 13.3° - 6.1° = 7.2°$, counter-clockwise.



C
B
r_3
θ_3
γ
r_2
d
r_4
ψ
θ_4
θ_2
φ
A
r_1
D



C'
r_3
γ'
B'
θ_3'
r_4
r_2
d'
ψ'
θ_4'
θ_2'
φ'
A
r_1
D

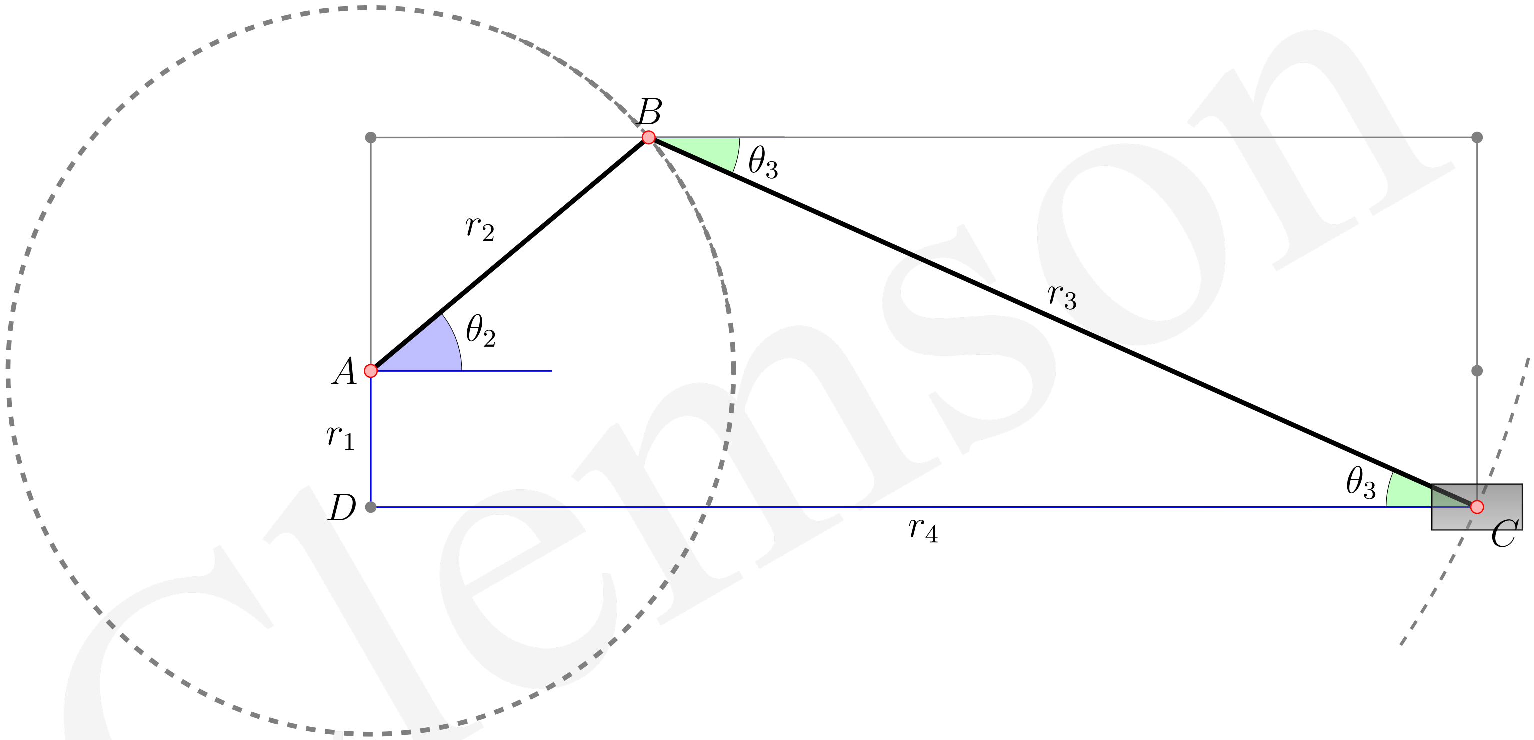


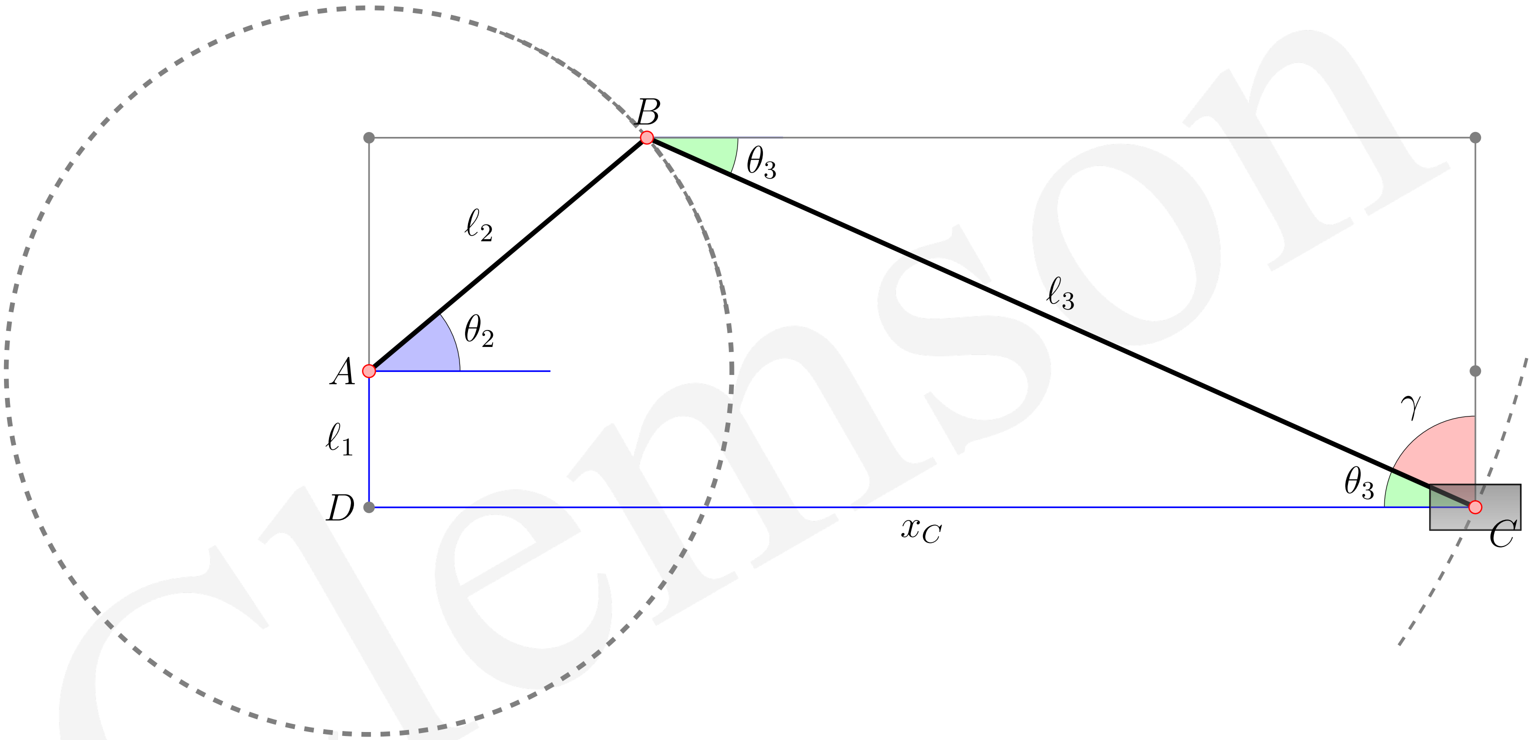
$$\Delta\theta_3 = \theta_3' - \theta_3 = 13.3 - 6.1 = 7.2^\circ \text{ CW}$$

Sanity Checks:

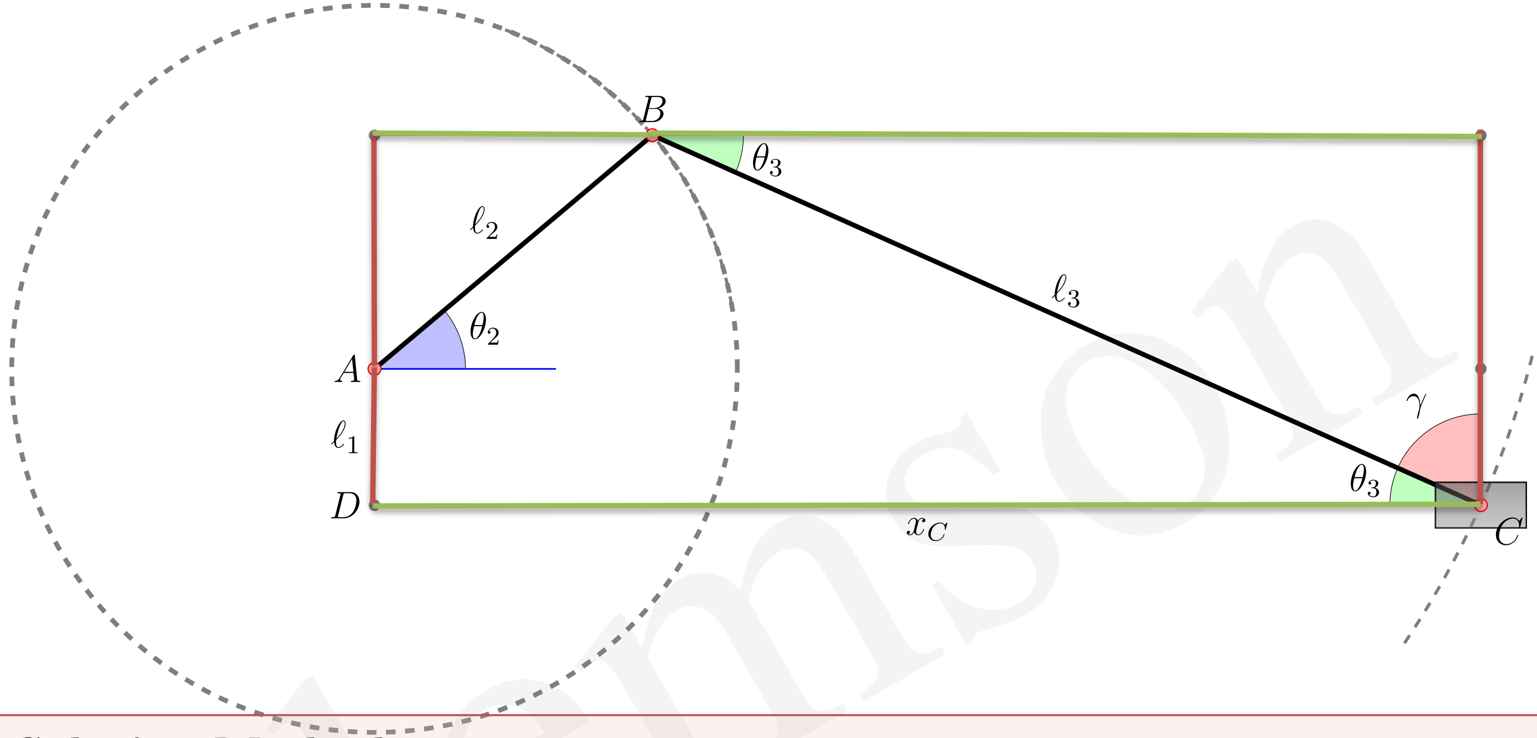
- Do the angle directions match the picture?
- Are computed angles consistent with the triangle inequality?
- Does the new point location satisfy link length constraints?

Offset Crank-Slider





B
θ_3
ℓ_2
ℓ_3
θ_2
A
γ
ℓ_1
θ_3
D
x_C
C



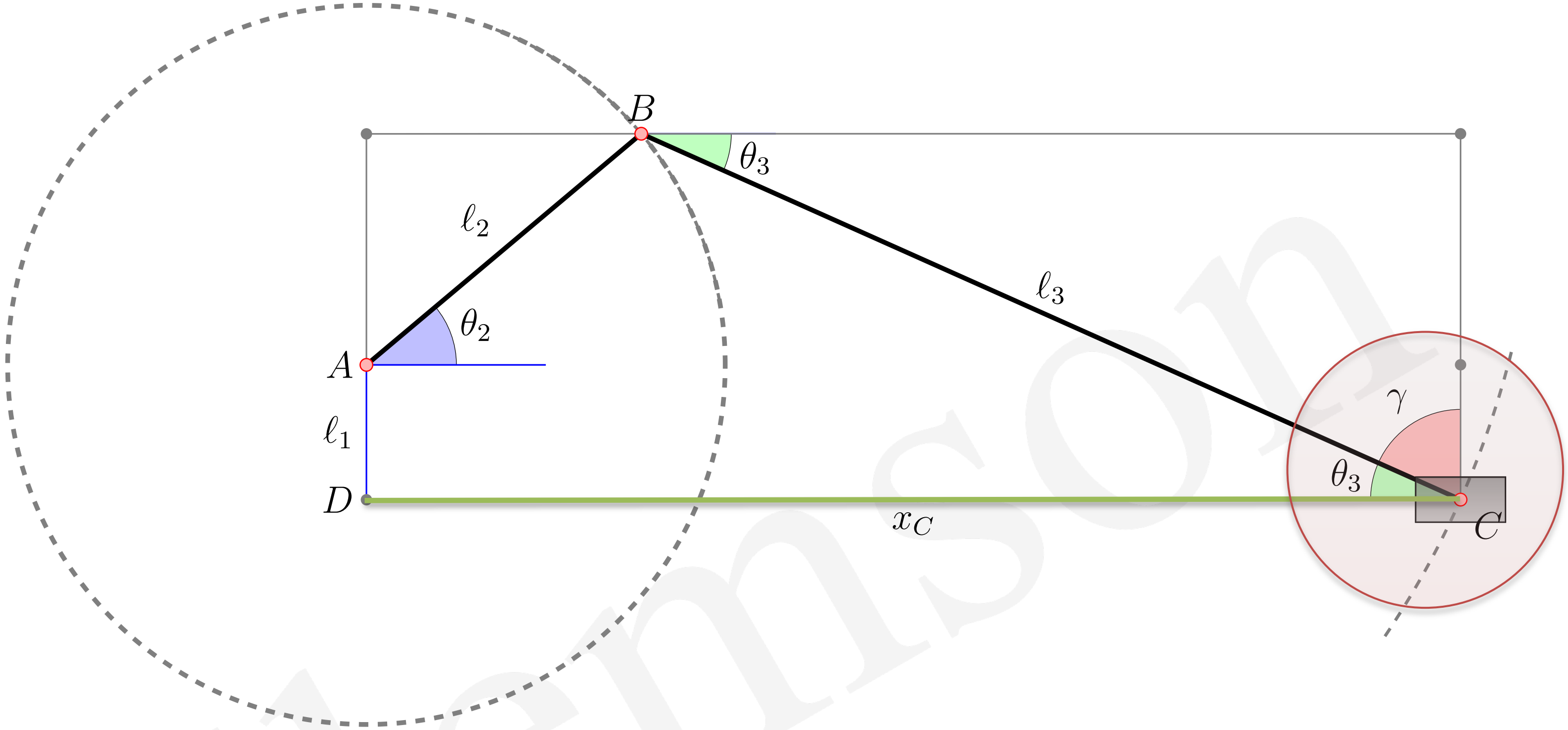
Solution Method 1

Using the extended right-triangles in Figure 1 to equate component lengths. In the perpendicular offset direction,

$$\ell_1 + \ell_2 \sin\theta_2 = \ell_3 \sin\theta_3 \tag{1}$$

In the parallel direction to slider,

$$x_C = \ell_2 \cos\theta_2 + \ell_3 \cos\theta_3 \tag{2}$$

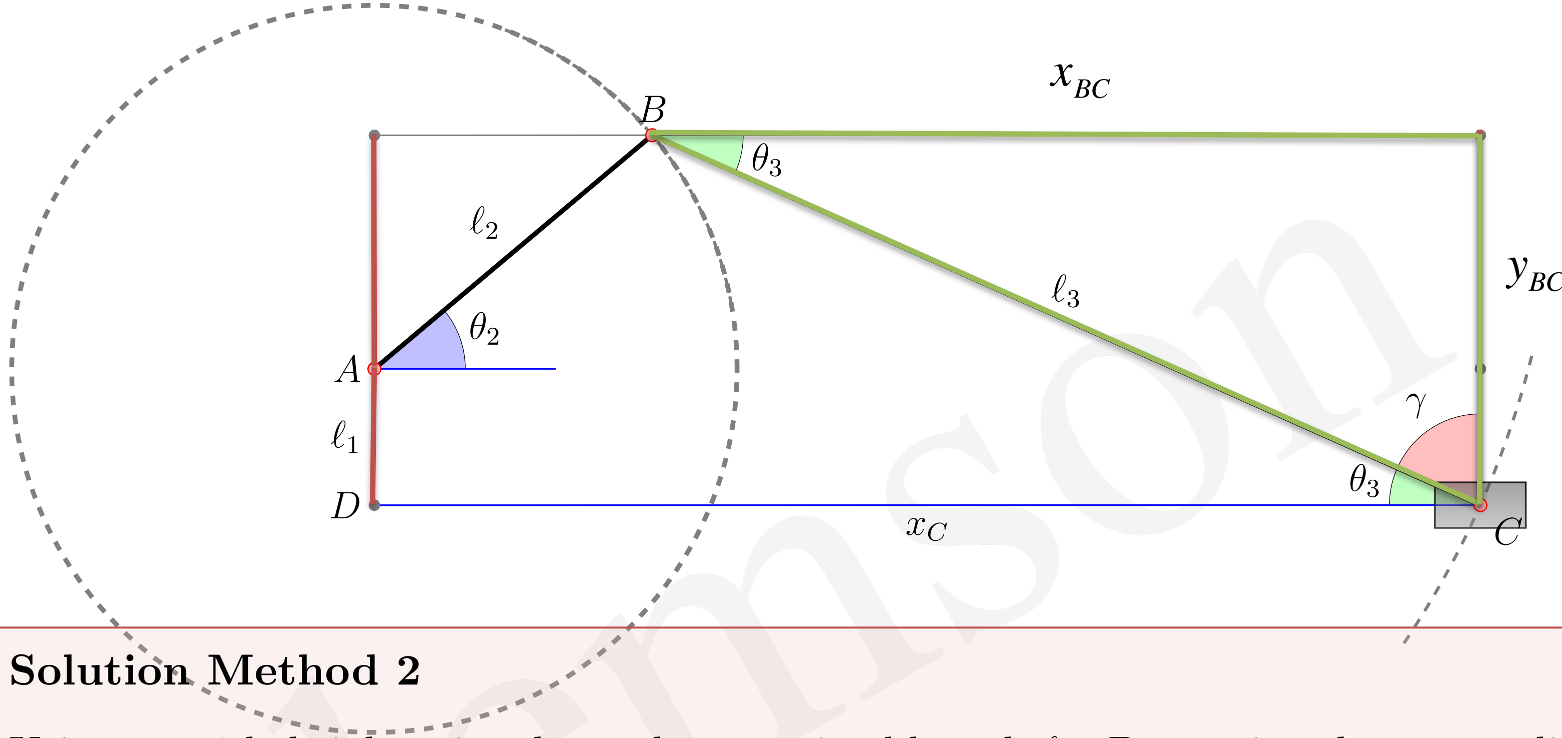


These orthogonal component equations give two equations for the two unknowns θ_3 and x_C. Using (1), solve for θ_3,

$$\theta_3 = \sin^{-1}\left(\frac{\ell_1 + \ell_2 \sin\theta_2}{\ell_3}\right)$$

Then use (2), solve for x_C.

$$x_C = \ell_2 \cos\theta_2 + \ell_3 \cos\theta_3$$



Solution Method 2

Using extended right-triangles and constrained length ℓ_3. Determine the perpendicular offset between B and C,

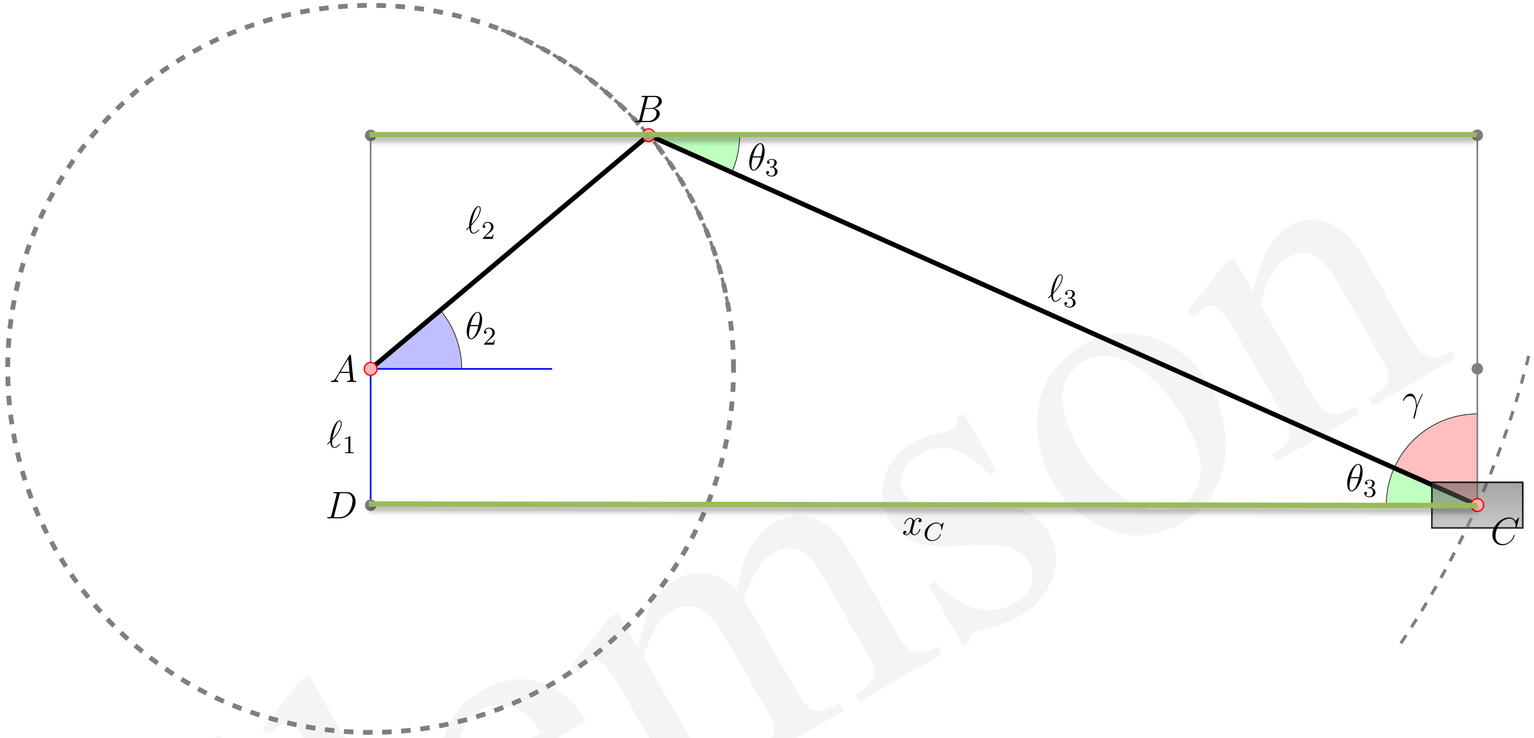
$$\ell_1 + \ell_2 \sin\theta_2 = y_{BC} \tag{3}$$

Construct the diagonal ℓ_3 with constrained length

$$(\ell_3)^2 = (y_{BC})^2 + (x_{BC})^2 = (\ell_1 + \ell_2 \sin\theta_2)^2 + (x_{BC})^2$$

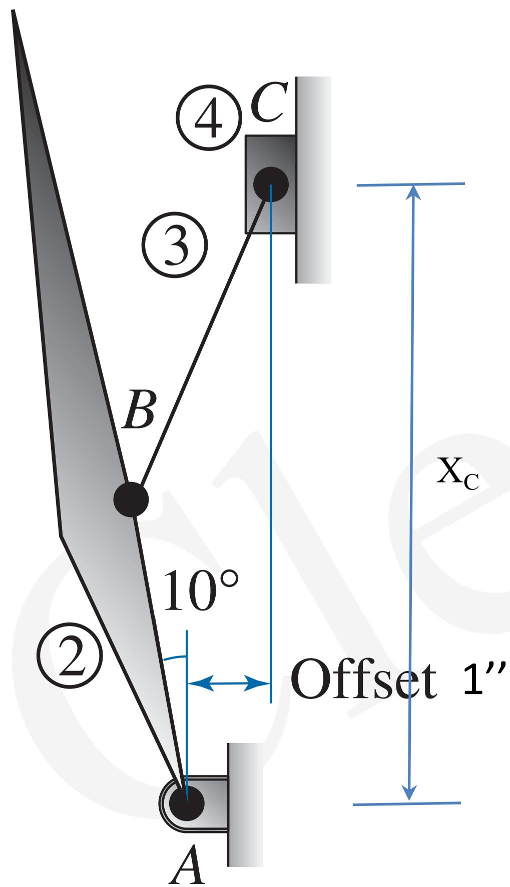
where x_{BC} is the parallel distance between B and C. Solve,

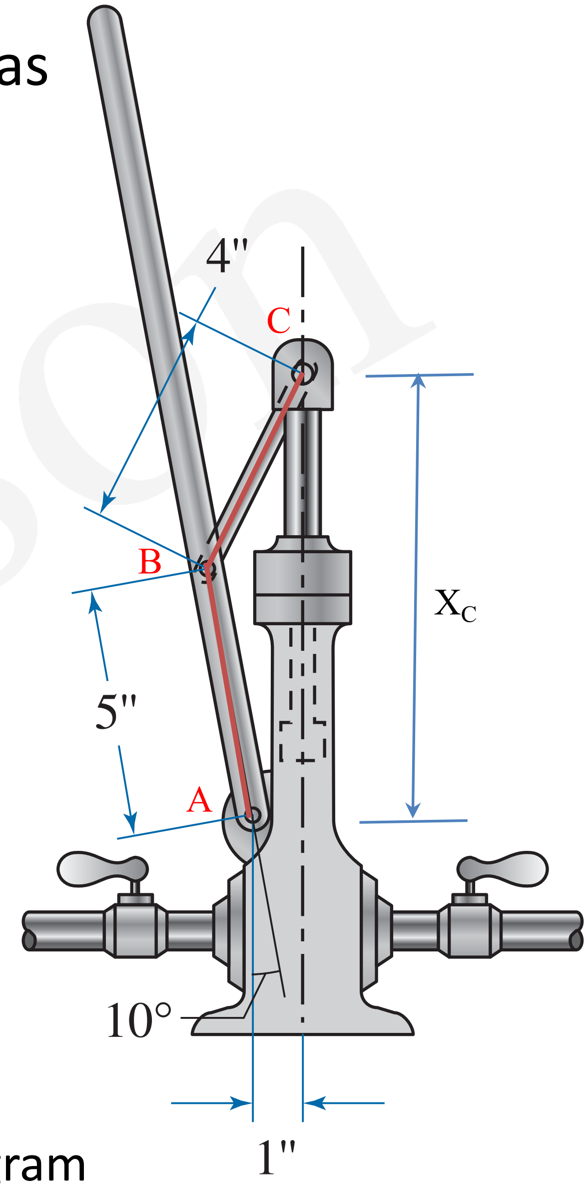
$$x_{BC} = \sqrt{(\ell_3)^2 - (\ell_1 + \ell_2 \sin\theta_2)^2}$$



$$x_C = \ell_2 \cos\theta_2 + x_{BC} = \ell_2 \cos\theta_2 + \sqrt{\ell_3^2 - (\ell_1 + \ell_2 \sin\theta_2)^2}$$

Determine the displacement of the piston as the handle rotates 15º, counterclockwise.



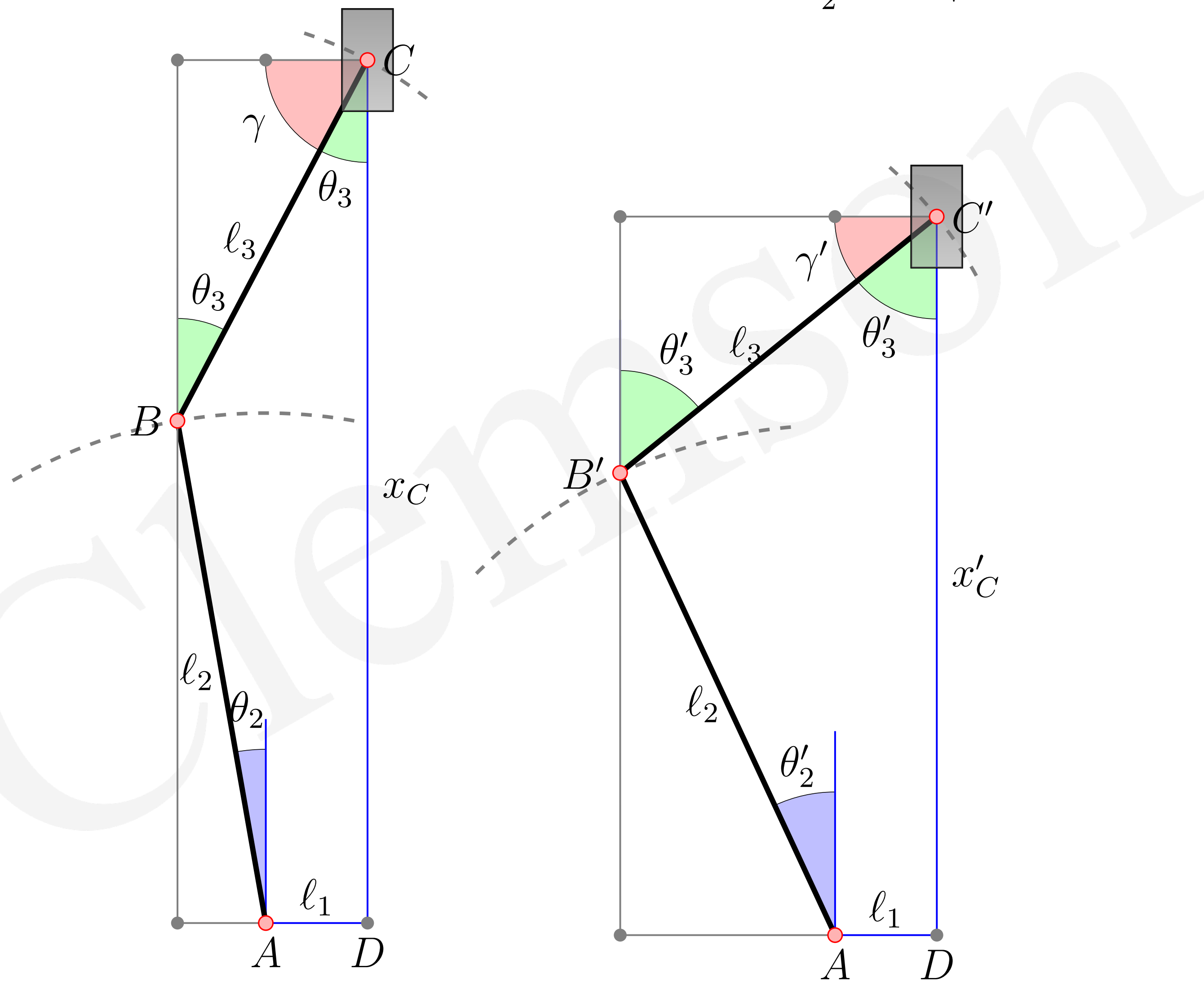


Step 1. Draw a Kinematic Diagram

(Left) Original Position. $\theta_2 = 10°$

(Right) Displaced Position.

$$\theta_2' = 10 + 15 = 25°.$$



Step 2. Analyze Geometry of Original Position

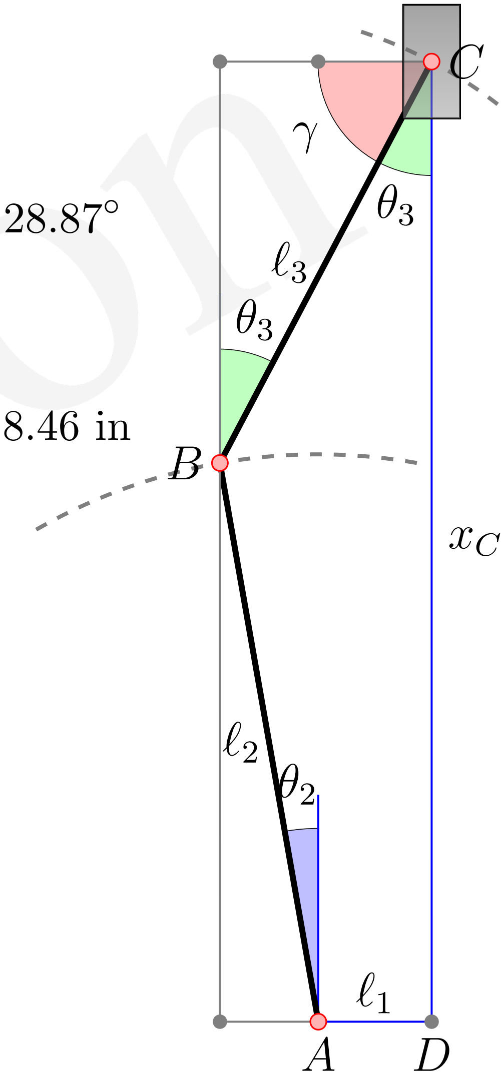
$$\ell_1 + \ell_2 \sin\theta_2 = \ell_3 \sin\theta_3$$

Solve for θ_3,

$$\theta_3 = \sin^{-1}\left(\frac{\ell_1 + \ell_2 \sin\theta_2}{\ell_3}\right) = \sin^{-1}\left(\frac{1 + 5\sin(10^\circ)}{4}\right) = 28.87^\circ$$

Then equating lengths in the direction parallel to the slider

$$x_C = \ell_2 \cos\theta_2 + \ell_3 \cos\theta_3 = 5\cos(10^\circ) + 4\cos(28.87^\circ) = 8.46 \text{ in}$$

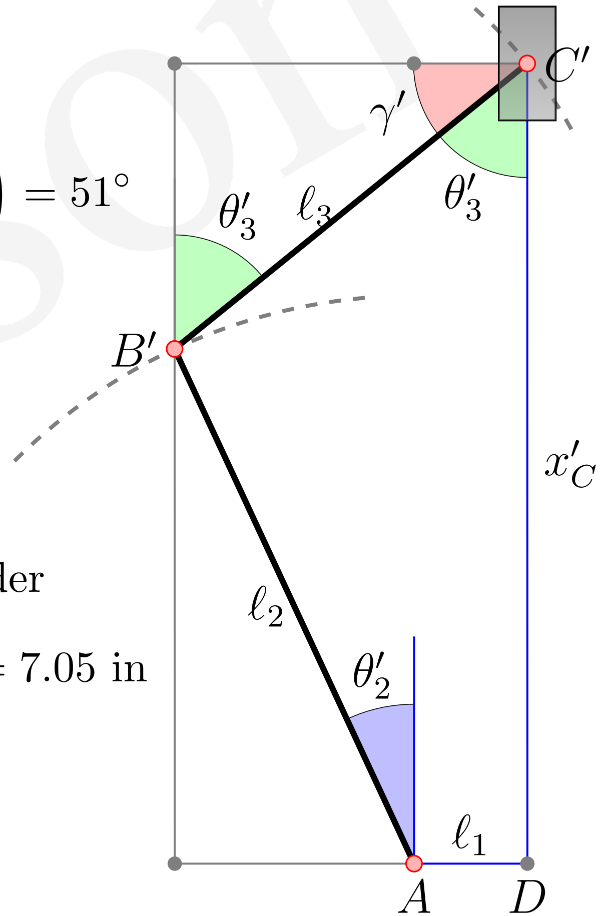


Step 3. Analyze Geometry in Displaced New Position

$$\ell_1 + \ell_2 \sin\theta_2' = \ell_3 \sin\theta_3'$$

Solve for θ_3',

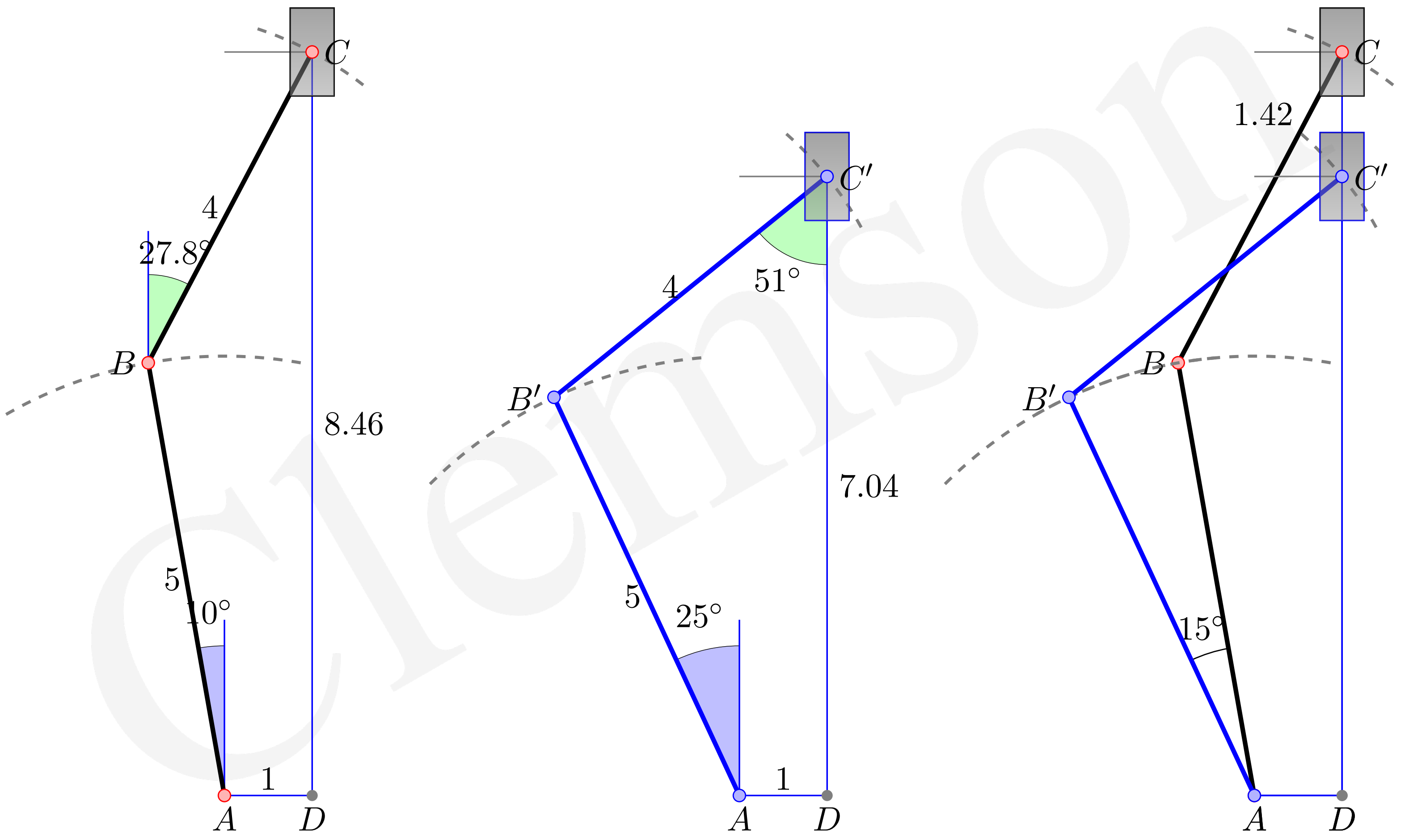
$$\theta_3' = \sin^{-1}\left(\frac{\ell_1 + \ell_2 \sin\theta_2'}{\ell_3}\right) = \sin^{-1}\left(\frac{1 + 5\sin(25^\circ)}{4}\right) = 51^\circ$$

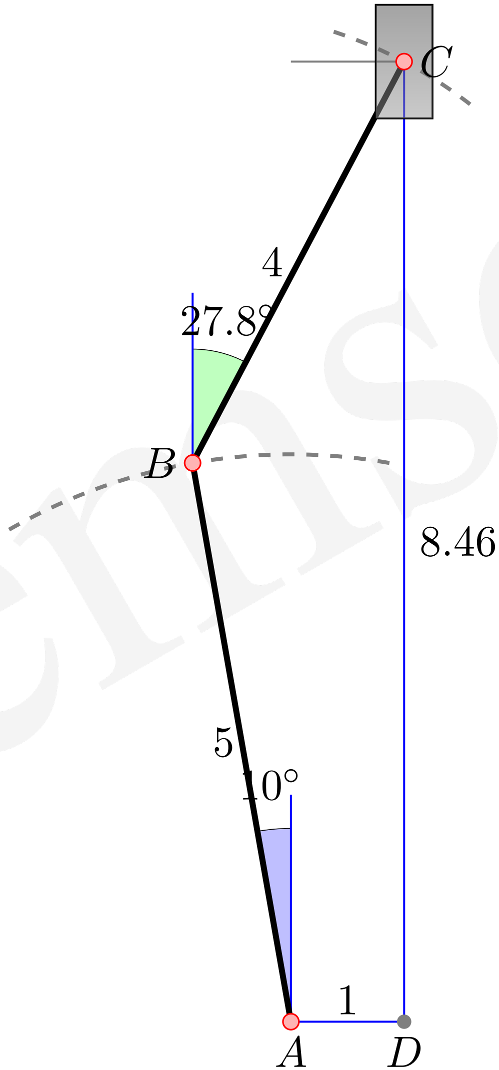


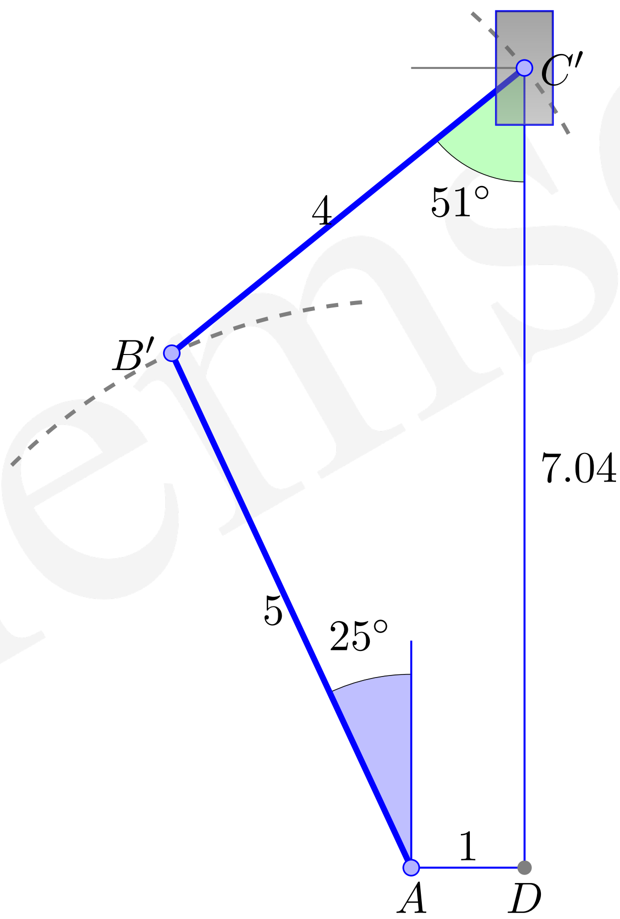
Then equating lengths in the direction parallel to the slider

$$x_C' = \ell_2 \cos\theta_2' + \ell_3 \cos\theta_3' = 5\cos(25^\circ) + 4\cos(51^\circ) = 7.05 \text{ in}$$

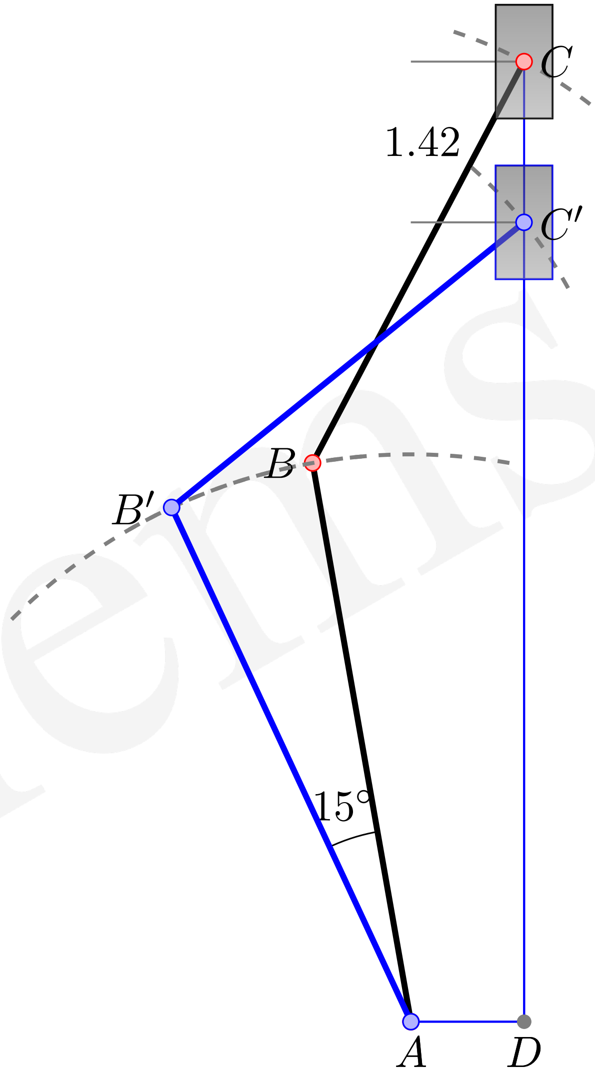
$$\Delta x_C = x'_C - x_C = 7.05 - 8.46 = -1.42 \text{ in} = 1.42 \text{ in} \downarrow$$

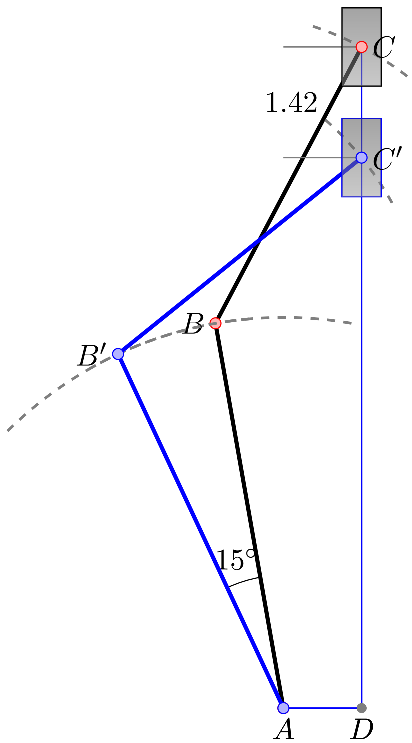



C
4
27.8°
B
8.46
5
10°
1
A
D


C′
4
51°
B′
7.04
5
25°
1
A
D

$$\Delta x_C = x'_C - x_C = 7.05 - 8.46 = -1.42 \text{ in} = 1.42 \text{ in} \downarrow$$





$$\Delta x_C = x'_C - x_C = 7.05 - 8.46 = -1.42 \text{ in} = 1.42 \text{ in} \downarrow$$

Sanity Checks:

- Does the displacement direction match the picture?
- Are computed lengths consistent with the triangle inequality?
- Does the new point location satisfy link length constraints?