Bolzano Weierstrass Theorems I

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Outline

1. The Bolzano Weierstrass Theorem

2. Bounded Infinite Sets

3. Homework
The Bolzano Weierstrass Theorem

Every bounded sequence with an infinite range has at least one convergent subsequence.

Proof

As discussed, we have already shown a sequence with a bounded finite range always has convergent subsequences. Now we prove the case where the range of the sequence of values \( \{a_1, a_2, \ldots, \} \) has infinitely many distinct values. We assume the sequences start at \( n = k \) and by assumption, there is a positive number \( B \) so that \( B \leq a_n \leq B \) for all \( n \geq k \). Define the interval \( J_0 = [\alpha_0, \beta_0] \) where \( \alpha_0 = -B \) and \( \beta_0 = B \). Thus at this starting step, \( J_0 = [-B, B] \). Note the length of \( J_0 \), denoted by \( \ell_0 \) is \( 2B \).

Let \( S \) be the range of the sequence which has infinitely many points and for convenience, we will let the phrase infinitely many points be abbreviated to IMPs.
Proof

Step 1:
Bisect \([\alpha_0, \beta_0]\) into two pieces \(u_0\) and \(u_1\). That is the interval \(J_0\) is the union of the two sets \(u_0\) and \(u_1\) and \(J_0 = u_0 \cup u_1\). Now at least one of the intervals \(u_0\) and \(u_1\) contains IMPS of \(S\) as otherwise each piece has only finitely many points and that contradicts our assumption that \(S\) has IMPS. Now both may contain IMPS so select one such interval containing IMPS and call it \(J_1\). Label the endpoints of \(J_1\) as \(\alpha_1\) and \(\beta_1\); hence, \(J_1 = [\alpha_1, \beta_1]\). Note \(\ell_1 = \beta_1 - \alpha_1 = \frac{1}{2} \ell_0 = B\) We see \(J_1 \subseteq J_0\) and

\[-B = \alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 = B\]

Since \(J_1\) contains IMPS, we can select a sequence value \(a_{n_1}\) from \(J_1\).

Step 2:
Now bisect \(J_1\) into subintervals \(u_0\) and \(u_1\) just as before so that \(J_1 = u_0 \cup u_1\). At least one of \(u_0\) and \(u_1\) contain IMPS of \(S\).
Choose one such interval and call it \( J_2 \). Label the endpoints of \( J_2 \) as \( \alpha_2 \) and \( \beta_2 \); hence, \( J_2 = [\alpha_2, \beta_2] \). Note \( \ell_2 = \beta_2 - \alpha_2 = \frac{1}{2} \ell_1 \) or \( \ell_2 = (\frac{1}{4})\ell_0 = (\frac{1}{2^2})\ell_0 = (\frac{1}{2})B \). We see \( J_2 \subseteq J_1 \subseteq J_0 \) and

\[
-B = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \beta_2 \leq \beta_1 \leq \beta_0 = B
\]

Since \( J_2 \) contains IMPS, we can select a sequence value \( a_{n_2} \) from \( J_2 \). It is easy to see this value is different from \( a_{n_1} \), our previous choice. You should be able to see that we can continue this argument using induction.

**Proposition:**
\[ \forall p \geq 1, \exists \text{ an interval } J_p = [\alpha_p, \beta_p] \text{ with the length of } J_p, \ell_p = B/(2^{p-1}) \text{ satisfying } J_p \subseteq J_{p-1}, J_p \text{ contains IMPS of } S \text{ and} \]

\[
\alpha_0 \leq \ldots \leq \alpha_{p-1} \leq \alpha_p \leq \beta_p \leq \beta_{p-1} \leq \ldots \leq \beta_0
\]

. Finally, there is a sequence value \( a_{n_p} \) in \( J_p \), different from \( a_{n_1}, \ldots, a_{n_{p-1}} \).
Proof

We have already established the proposition is true for the basis step $J_1$ and indeed also for the next step $J_2$.

**Inductive:** We assume the interval $J_q$ exists with all the desired properties. Since by assumption, $J_1$ contains IMPs, bisect $J_1$ into $u_0$ and $u_1$ like usual. At least one of these intervals contains IMPs of $S$. Call the interval $J_{q+1}$ and label $J_{q+1} = [\alpha_{q+1}, \beta_{q+1}]$. We see immediately that

$$\ell_{q+1} = (1/2)\ell_q = (1/2)(1/2^{q-1})B = (1/2^q)B$$

with $\ell_{q+1} = \beta_{q+1} - \alpha_{q+1}$ with

$$\alpha_q \leq \alpha_{q+1} \leq \beta_{q+1} \leq \beta_q.$$

This shows the nested inequality we want is satisfied.

Finally, since $J_{q+1}$ contains IMPs, we can choose $a_{n_{q+1}}$ distinct from the other $a_{n_i}$'s. So the inductive step is satisfied and by the POMI, the proposition is true for all $n$.  \[\Box\]
Proof

- From our proposition, we have proven the existence of three sequences, \((\alpha_p)_{p \geq 0}\), \((\beta_p)_{p \geq 0}\) and \((\ell_p)_{p \geq 0}\) which have various properties.
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- The sequence \(\ell_p\) satisfies \(\ell_p = (1/2)\ell_{p-1}\) for all \(p \geq 1\). Since \(\ell_0 = 2B\), this means \(\ell_1 = B\), \(\ell_2 = (1/2)B\), \(\ell_3 = (1/2^2)B\) leading to \(\ell_p = (1/2^{p-1})B\) for \(p \geq 1\).
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\[-B = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p \leq \ldots \leq \beta_p \leq \ldots \leq \beta_2 \leq \ldots \leq \beta_0 = B\]
Proof

- From our proposition, we have proven the existence of three sequences, \((\alpha_p)_{p \geq 0}, (\beta_p)_{p \geq 0}\) and \((\ell_p)_{p \geq 0}\) which have various properties.

- The sequence \(\ell_p\) satisfies \(\ell_p = (1/2)\ell_{p-1}\) for all \(p \geq 1\). Since \(\ell_0 = 2B\), this means \(\ell_1 = B, \ell_2 = (1/2)B, \ell_3 = (1/2^2)B\) leading to \(\ell_p = (1/2^{p-1})B\) for \(p \geq 1\).

- 
  \[
  -B = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_p \\
  \leq \ldots \leq \\
  \beta_p \leq \ldots \leq \beta_2 \leq \ldots \leq \beta_0 = B
  \]

- Note \((\alpha_p)_{p \geq 0}\) is bounded above by \(B\) and \((\beta_p)_{p \geq 0}\) is bounded below by \(-B\). Hence, by the completeness axiom, \(\inf (\beta_p)_{p \geq 0}\) exists and equals the finite number \(\beta\); also \(\sup (\alpha_p)_{p \geq 0}\) exists and is the finite number \(\alpha\).
Proof

If you think about the construction process here (there is a picture of it in the notes too that might help), at level $p$ we have an interval $J_p = [\alpha_p, \beta_p]$. At the next step, we pick one of the halves of $J_p$ and so $J_{p+1}$ shares one endpoint with $J_p$. However at the next steps, $J_{p+2}$, $J_{p+3}$ and so on, the new subintervals we get do not share any endpoints of $J_p$. Hence form $J_{p+2}$ on, we know $J_{p+2}$ are on are strictly contained in $J_p$. 

Thus, $\alpha$ and $\beta$ are in $[\alpha_p, \beta_p] = J_p$ for all $p$. Next we show $\alpha = \beta$. 


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So if we fix $p$, it should be clear the number $\beta_p$ is an upper bound for all the $\alpha_p$ values (look at our inequality chain again and think about this). Thus $\beta_p$ is an upper bound for $(\alpha_p)_{p \geq 0}$ and so by definition of a supremum, $\alpha \leq \beta_p$ for all $p$. Of course, we also know since $\alpha$ is a supremum, that $\alpha_p \leq \alpha$. Thus, $\alpha_p \leq \alpha \leq \beta_p$ for all $p$. 
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- So if we fix $p$, it should be clear the number $\beta_p$ is an upper bound for all the $\alpha_p$ values (look at our inequality chain again and think about this). Thus $\beta_p$ is an upper bound for $(\alpha_p)_{p \geq 0}$ and so by definition of a supremum, $\alpha \leq \beta_p$ for all $p$. Of course, we also know since $\alpha$ is a supremum, that $\alpha_p \leq \alpha$. Thus, $\alpha_p \leq \alpha \leq \beta_p$ for all $p$.

- A similar argument shows if we fix $p$, the number $\alpha_p$ is an lower bound for all the $\beta_p$ values and so by definition of an infimum, $\alpha_p \leq \beta \leq \beta_p$ for all the $\alpha_p$ values.

- This tells us $\alpha$ and $\beta$ are in $[\alpha_p, \beta_p] = J_p$ for all $p$. Next we show $\alpha = \beta$. 
Proof

Let $\epsilon > 0$ be arbitrary. Since $\alpha$ and $\beta$ are in $J_p$ whose length is $\ell_p = (1/2^{p-1})B$, we have $|\alpha - \beta| \leq (1/2^{p-1})B$. Pick $P$ so that $1/(2^{P-1}) < \epsilon$. Then $|\alpha - \beta| < \epsilon$. But $\epsilon > 0$ is arbitrary. Hence, by a previous proposition, $\alpha - \beta = 0$ implying $\alpha = \beta$. 
**Proof**

- Let $\epsilon > 0$ be arbitrary. Since $\alpha$ and $\beta$ are in $J_p$ whose length is $\ell_p = (1/2^{p-1})B$, we have $|\alpha - \beta| \leq (1/2^{p-1})B$. Pick $P$ so that $1/(2^{P-1}) < \epsilon$. Then $|\alpha - \beta| < \epsilon$. But $\epsilon > 0$ is arbitrary. Hence, by a previous proposition, $\alpha - \beta = 0$ implying $\alpha = \beta$.

- We now must show $a_{n_k} \to \alpha = \beta$. This shows we have found a subsequence which converges to $\alpha = \beta$. We know $\alpha_p \leq a_{n_p} \leq \beta_p$ and $\alpha_p \leq \alpha \leq \beta_p$ for all $p$. Pick $\epsilon > 0$ arbitrarily. Given any $p$, we have

$$
|a_{n_p} - \alpha| = |a_{n_p} - \alpha_p + \alpha_p - \alpha|, \quad \text{add and subtract trick}
$$
$$
\leq |a_{n_p} - \alpha_p| + |\alpha_p - \alpha|, \quad \text{triangle inequality}
$$
$$
\leq |\beta_p - \alpha_p| + |\alpha_p - \beta_p|, \quad \text{definition of length}
$$
$$
= 2|\beta_p - \alpha_p| = 2(1/2^{p-1})B.
$$

Choose $P$ so that $(1/2^{P-1})B < \epsilon/2$. Then, $p > P$ implies $|a_{n_p} - \alpha| < 2\epsilon/2 = \epsilon$. Thus, $a_{n_k} \to \alpha$. 

A more general type of result can also be shown which deals with sets which are bounded and contain infinitely many elements.

**Definition**

Let $S$ be a nonempty set. We say the real number $a$ is an **accumulation** points of $S$ if given any $r > 0$, the set

$$B_r(a) = \{x : |x - a| < r\}$$

contains at least one point of $S$ different from $a$. The set $B_r(a)$ is called the **ball** or **circle** centered at $a$ with radius $r$.

**Example**

$S = (0, 1)$. Then $0$ is an accumulation point of $S$ as the circle $B_r(0)$ always contains points greater than $0$ which are in $S$, Note $B_r(0)$ also contains points less than $0$. Note $1$ is an accumulation point of $S$ also Note $0$ and $1$ are not in $S$ so accumulation points don’t have to be in the set. Also note all points in $S$ are accumulation points too. Note the set of all accumulation points of $S$ is the interval $[0, 1]$. 
Example

$S = ((1/n)_{n \geq 1}$. Note 0 is an accumulation point of $S$ because every circle $B_r(0)$ contains points of $S$ different from 0. Also, if you pick a particular $1/n$ in $S$, the distance from $1/n$ to its neighbors is either $|1/n - 1/(n + 1)|$ or $1/n - 1/(n - 1)$. If you let $r$ be half the minimum of these two distances, the circle $B_r(1/n)$ does not contain any other points of $S$. So no point of $S$ is an accumulation point. So the set of accumulation points of $S$ is just one point, $\{0\}$. 
8.1 Let $S = (2, 5)$. Show 2 and 5 are accumulation points of $S$.

8.2 Let $S = (\cos(n\pi/4)_{n \geq 1}$. Show $S$ has no accumulation points.

8.3 This one is a problem you have never seen. So it requires you look at it right! Let $(a_n)$ be a bounded sequence and let $(b_n)$ be a sequence that converges to 0. Then $a_nb_n \rightarrow 0$. This is an $\epsilon - N$ proof. Note this is **not** true if $(b_n)$ converges to a nonzero number.

8.4 If you know $(a_nb_n)$ converges does that imply both $(a_n)$ and $(b_n)$ converge?