# Project One: $C^\infty$ Bump functions

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# Outline

Let's recall what the bump functions are: Define

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0 \end{cases}$$

Let $(x_n)$ be any sequence with $x_n \leq 0$ which converges to 0. Then $f(x_n) \to 0$ also. So 0 is in $S(0)$. Let $(x_n)$ with $x_n > 0$ be any sequence with $x_n \to 0$. Then $f(x_n) = e^{-1/x_n}$. It is easy to see $-1/x_n$ with $x_n > 0$ always has an unbounded limit which we usually call $-\infty$. If you recall from calculus, $e^y \to 0$ as $y \to -\infty$. Thus, $f(x_n) \to 0$ here for these sequences.

If $(x_n)$ was a sequence with infinitely many terms both positive and negative which still converged to 0, then the subsequence of positive terms, $(x_n^1)$, converges to 0 and the subsequence of negative terms, $(x_n^2)$, also converges to 0. Further, $f(x_n^1) \to 0$ and $f(x_n^2) \to 0$ as well. With a little work we can see $f(x_n) \to 0$ too.

We see the only cluster point at 0 is 0. Thus, $\underline{\lim}_{x_n \to 0} f = \overline{\lim}_{x_n \to 0} f = 0$.

This function has many properties which you are going to prove!

- This function is continuous at 0
- This function is infinitely differentiable and $f^{(n)}(0) = 0$ for all $n$. Note this tells us all the derivatives of $f$ are actually continuous.

Note

- $f$ is so flat at 0 its Taylor Series expansion at 0 does not match $f$ at all!
- Even though 0 is a global minimum of $f$ occurring at 0 and other points, the second derivative test will fail there.

So now we can build some more things.

$$f_a(x) = \begin{cases} 0, & x \leq a \\ e^{-1/(x-a)}, & x > a \end{cases}$$

A similar analysis shows $f_a$ has a zero $n^{th}$ order derivative at $a$ for all $n$ and has a continuous $n^{th}$ order derivative at $a$ for all $n$.

$$g_b(x) = \begin{cases} 0, & x \geq b \\ e^{1/(x-b)}, & x < b \end{cases}$$

A similar analysis shows $g_b$ has a zero $n^{th}$ order derivative at $b$ for all $n$ and has a continuous $n^{th}$ order derivative at $b$ for all $n$.

We can then show for any $a < b$,

$$h_{a\,b}(x) = f_a(x)\, g_b(x) = \begin{cases} 0, & x \leq a \\ e^{\frac{1}{x-b}}\, e^{-\frac{1}{x-a}}, & a < x < b \\ 0, & x \geq b \end{cases}$$

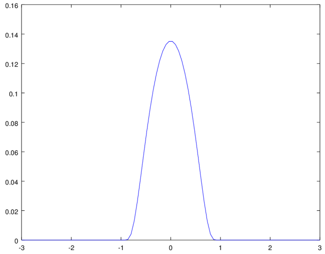
$$= \begin{cases} 0, & x \leq a \\ e^{\frac{b-a}{(x-a)(x-b)}}, & a < x < b \\ 0, & x \geq b \end{cases}$$

A similar analysis shows $h_{ab}$ has a zero $n^{th}$ order derivative at $a$ and $b$ for all $n$ which is continuous $a$ and $b$ for all $n$.

Now multiply $h_{ab}$ by $e^{\frac{2}{a+b}}$ to get $H_{ab}(x) = e^{\frac{2}{a+b}}\, f_{ab}(x)$. $H_{ab}(x)$ is an infinitely differentiable function whose value is 0 off of the interval $(a, b)$ and whose maximum value is $+1$. It is called a $C^\infty$ **bump** function. The closure of the set of $x$ values where $H_{ab}(x) > 0$ is then $[a, b]$ which is a compact set. This closure is called the **support** of $H_{ab}(x)$ and so $H_{ab}(x)$ is a $C^\infty$ function with compact support. Very important!

]fragile] To plot $h$ use the code below in the function `Plottingh(a,b,eps,P)`. The arguments for the function are `a` and `b` which are used to define $f_a$ and $g_b$, `eps` which is used to stay away for the points $a$ and $b$ in the plots and `P` which is used to set up the `linspace` command.

In the MatLab window type `Plottingh(-1,1,.01,72);`. This generates the plot




which needs labels and a title, of course, which you will have to add.

Just type in this function and save as the file `Plottingh.m` in your current directory.

```
function Plottingh (a,b,eps,P)
  fa = @(x,a) exp(-1/(x-a));
  gb = @(x,b) exp(1/(x-b));
  XA = linspace(a-2,a-eps,P/3);
  XM = linspace(a+eps,b-eps,P/3);
  XB = linspace(b+eps,b+2,P/3);
  X = [XA,XM,XB];
  [m,n] = size(X);
  for i=1:n
    if(X(i) < a)
      h(i) = 0;
    end
    if(X(i) > b)
      h(i) = 0;
    end
    if(X(i) >a & X(i) < b)
      h(i) = fa(X(i),a)*gb(X(i),b);
    end
  end
  plot(X,h);
end
```

## Due Date: Friday April 28, 2017

You are going to do the work to prove the stuff we talk about in the background slides. Do a great job on this. I want to see you put all the stuff we have been doing together. Write well on one side of the paper and treat this like a full report.

1. $f_a$
    - Prove $f_a$ is differentiable for all $x$ and in particular $f_a'(a) = 0$
    - Prove $f_a$ is twice differentiable for all $x$ and in particular $f_a''(a) = 0$
    - Now look at what you have done and see if you can figure out how to scale your argument up to show $f_a$ has an $n^{th}$ derivative with $f_a^{(n)}(a) = 0$. You might want to look at the third derivative too to get a better idea what to do.
2. Do the same thing for $g_b$.
3. Do the same thing for $h_{ab}$.
4. Plot a nice selection of $h_{ab}$'s to get the feel of how they look. You will have to be careful with the plots as you can't ask MatLab etc to plot near the place where $h_{ab}$ become 0.