The Equality of Riemann Integrals and the Integration of Jumps

James K. Peterson

Department of Biological Sciences and Department of Mathematical Sciences
Clemson University

February 22, 2018

Outline

Equality of Riemann Integrals

Integration with Jumps
Recall, we have shown

**Lemma**

\( f \) *Zero On* \((a, b)\) *Implies Zero Riemann Integral*

Let \( f \in B[a, b] \), with \( f(x) = 0 \) on \((a, b)\). Then \( f \) is integrable on \([a, b]\) and \( \int_a^b f(x) \, dx = 0 \).

**Theorem**

Let \( f, g \in RI[a, b] \) with \( f(x) = g(x) \) on \((a, b)\). Then
\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.
\]

**Theorem**

Let \( f, g \in RI[a, b] \), and assume that \( f = g \) except at finitely many points \( c_1, \ldots, c_k \). Then
\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.
\]

Next, we look at integrability of discontinuous functions.

**Theorem**

If \( f \) is bounded on \([a, b]\) and continuous except at one point \( c \) in \([a, b]\), then \( f \) is Riemann integrable.
Proof
For convenience, we will assume that \( c \) is an interior point, i.e. \( c \) is in \((a, b)\). We will show that \( f \) satisfies the Riemann Criterion and so it is Riemann integrable.

Let \( \epsilon > 0 \) be given. Since \( f \) is bounded on \([a, b]\), there is a real number \( M \) so that \( f(x) < M \) for all \( x \) in \([a, b]\). We know \( f \) is continuous on \([a, c - \epsilon/(6M)]\) and \( f \) is continuous on \([c + \epsilon/(6M), b]\). Thus, \( f \) is integrable on both of these intervals and \( f \) satisfies the Riemann Criterion on both intervals.

For this \( \epsilon \) there is a partition \( \pi_0 \) of \([a, c - \epsilon/(6M)]\) so that

\[
U(f, P) - L(f, P) < \epsilon/3, \quad \text{if } \pi_0 \leq P
\]

and there is a partition \( \pi_1 \) of \([c + \epsilon/(6M), b]\) so that

\[
U(f, Q) - L(f, Q) < \epsilon/3, \quad \text{if } \pi_1 \leq Q.
\]

Proof
Let \( \pi_2 \) be the partition we get by combining \( \pi_0 \) with the points \( \{c - \epsilon/(6M), c + \epsilon/(6M)\} \) and \( \pi_1 \). Then, we see

\[
U(f, \pi_2) - L(f, \pi_2) = U(f, \pi_0) - L(f, \pi_0)
\]

\[
+ \left( \sup_{x \in [c - \epsilon/(6M), c + \epsilon/(6M)]} f(x) \right) \epsilon/(3M)
\]

\[
+ U(f, \pi_1) - L(f, \pi_1)
\]

\[
< \epsilon/3 + M\epsilon/(3M) + \epsilon/3 = \epsilon
\]

Then if \( \pi_2 \leq \pi \) on \([a, b]\), we have

\[
U(f, \pi) - L(f, \pi) < \epsilon
\]
Proof
This shows $f$ satisfies the Riemann criterion and hence is integrable if the discontinuity $c$ is interior to $[a, b]$. The argument at $c = a$ and $c = b$ is similar but a bit simpler as it only needs to be done from one side.
Hence, we conclude $f$ is integrable on $[a, b]$ in all cases. 

It is then easy to extend this result to a function $f$ which is bounded and continuous on $[a, b]$ except at a finite number of points $\{x_1, x_2, \ldots, x_k\}$ for some positive integer $k$.

**Theorem**

If $f$ is bounded on $[a, b]$ and continuous except at finitely many points $\{x_1, x_2, \ldots, x_k\}$ in $[a, b]$, then $f$ is Riemann integrable.

Proof
We may assume without loss of generality that the points of discontinuity are ordered as $a < x_1 < x_2 < \ldots < x_k < b$. Then $f$ is continuous except at $x_1$ on $[a, x_1]$ and hence by the previous Theorem $f$ is integrable on $[a, x_1]$. Now apply this argument on each of the subintervals $[x_{k-1}, x_k]$ in turn. 

□
Now let’s look at the Riemann integral of functions which have points of discontinuity. We are going to find the $F$’s in the FTOC for various $f$’s with discontinuities.

**Example**

Consider the function $f$ defined on $[-2, 5]$ by

$$f(t) = \begin{cases} 
2t & -2 \leq t < 0 \\
1 & t = 0 \\
(1/5)t^2 & 0 < t \leq 5
\end{cases}$$

Let’s calculate $F(t) = \int_{-2}^{t} f(s) \, ds$. This will have to be done in several parts because of the way $f$ is defined.

**Solution**

(1): *On the interval $[-2, 0]$, note that $f$ is continuous except at one point, $t = 0$. Hence, $f$ is Riemann integrable. Also, the function $2t$ is continuous on this interval and hence is also Riemann integrable. Then since $f$ on $[-2, 0]$ and $2t$ match at all but one point on $[-2, 0]$, their Riemann integrals must match. Hence, if $t$ is in $[-2, 0]$, we compute $F$ as follows:*

$$F(t) = \int_{-2}^{t} f(s) \, ds = \int_{-2}^{t} 2s \, ds = s^2 \bigg|_{-2}^{t} = t^2 - (-2)^2 = t^2 - 4$$

(2): *On the interval $[0, 5]$, note that $f$ is continuous except at one point, $t = 0$. Hence, $f$ is Riemann integrable.*
Solution
(2): Continued:
Also, the function \((1/5)t^2\) is continuous on this interval and so is also
Riemann integrable. Then since \(f\) on \([0, 5]\) and \((1/5)t^2\) match at all but
one point on \([0, 5]\), their Riemann integrals must match. Hence, if \(t\) is in
\([0, 5]\), we compute \(F\) as follows:

\[
F(t) = \int_{-2}^{t} f(s) \, ds = \int_{-2}^{0} f(s) \, ds + \int_{0}^{t} f(s) \, ds
\]
\[
= \int_{-2}^{0} 2s \, ds + \int_{0}^{t} (1/5)s^2 \, ds
\]
\[
= \left[ s^2 \right]_{-2}^{0} + \left[ (1/15)s^3 \right]_{0}^{t}
\]
\[
= -4 + \frac{t^3}{15}
\]

Solution
Thus, we have found that

\[
F(t) = \begin{cases} 
    t^2 - 4 & -2 \leq t \leq 0 \\
    t^3/15 - 4 & 0 \leq t \leq 5 
\end{cases}
\]

Since \(f\) is Riemann Integrable on \([-2, 5]\), we know from the FTOC \(F\)
must be continuous. Let’s check. \(F\) is clearly continuous on either side of
0 and we note that \(\lim_{t \to 0^-} F(t)\) which is \(F(0^-)\) is \(-4\) which is exactly
the value of \(F(0^+)\). Hence, \(F\) is indeed continuous at 0.

What about the differentiability of \(F\)? The FTOC guarantees that \(F\) has
a derivative at each point where \(f\) is continuous and at those points
\(F'(t) = f(t)\). Hence, we know this is true at all \(t\) except perhaps at 0.
Solution
Note at those \( t \), we find

\[
F'(t) = \begin{cases} 
2t & -2 \leq t < 0 \\
(1/5)t^2 & 0 < t \leq 5 
\end{cases}
\]

which is exactly what we expect. Also, note \( F'(0^-) = 0 \) and \( F'(0^+) = 0 \) as well. Hence, since the right and left hand derivatives match, we see \( F'(0) \) does exist and has the value 0. But this is not the same as \( f(0) = 1 \). Note, \( F \) is not the antiderivative of \( f \) on \([-2, 5]\) because of this mismatch.

Example
Now consider the function \( f \) defined on \([-2, 5]\) by

\[
f(t) = \begin{cases} 
2t & -2 \leq t < 0 \\
1 & t = 0 \\
2 + (1/5)t^2 & 0 \leq t \leq 5 
\end{cases}
\]

Let’s calculate \( F(t) = \int_{-2}^{t} f(s) \, ds \). Again, this will have to be done in several parts because of the way \( f \) is defined.

Solution
(1): On the interval \([-2, 0]\), note that \( f \) is continuous except at one point, \( t = 0 \). Hence, \( f \) is Riemann integrable.
Solution

(1): (continued) Also, the function $2t$ is continuous on this interval and therefore is also Riemann integrable. Then since $f$ on $[-2, 0]$ and $2t$ match at all but one point on $[-2, 0]$, their Riemann integrals must match. Hence, if $t$ is in $[-2, 0]$, we compute $F$ as follows:

$$F(t) = \int_{-2}^{t} f(s) \, ds = \int_{-2}^{t} 2s \, ds = s^2 \bigg|_{-2}^{t} = t^2 - (-2)^2 = t^2 - 4$$

(2): On the interval $[0, 5]$, note that $f$ is continuous except at one point, $t = 0$. Hence, $f$ is Riemann integrable.

Solution

Also, the function $2 + (1/5)t^2$ is continuous on this interval and so is also Riemann integrable. Then since $f$ on $[0, 5]$ and $2 + (1/5)t^2$ match at all but one point on $[0, 5]$, their Riemann integrals must match. Hence, if $t$ is in $[0, 5]$, we compute $F$ as follows:

$$F(t) = \int_{-2}^{t} f(s) \, ds = \int_{-2}^{0} f(s) \, ds + \int_{0}^{t} f(s) \, ds$$
$$= \int_{-2}^{0} 2s \, ds + \int_{0}^{t} (2 + (1/5)s^2) \, ds$$
$$= s^2 \bigg|_{-2}^{0} + (2s + (1/15)s^3) \bigg|_{0}^{t}$$
$$= -4 + 2t + t^3/15$$
Solution

Thus, we have found that

\[ F(t) = \begin{cases} 
  t^2 - 4 & -2 \leq t \leq 0 \\
  -4 + 2t + t^3/15 & 0 \leq t \leq 5 
\end{cases} \]

We can see \( F \) is continuous at \( t = 0 \) as expected from the FTOC.

What about the differentiability of \( F \)? The Fundamental Theorem of Calculus guarantees that \( F \) has a derivative at each point where \( f \) is continuous and at those points \( F'(t) = f(t) \). Hence, we know this is true at all \( t \) except 0. Note at those \( t \), we find

\[ F'(t) = \begin{cases} 
  2t & -2 \leq t < 0 \\
  2 + (1/5)t^2 & 0 < t \leq 5 
\end{cases} \]

which is exactly what we expect.

However, when we look at the one sided derivatives, we find \( F'(0^-) = 0 \) and \( F'(0^+) = 2 \). Hence, since the right and left hand derivatives do not match, we see \( F'(0) \) does not exist. Finally, note \( F \) is not the antiderivative of \( f \) on \([-2,5]\) because of this mismatch.
17.1 Compute \( F(t) = \int_{-3}^{t} f(s) \, ds \) for
\[
f(t) = \begin{cases} 
3t & -3 \leq t < 0 \\
6 & t = 0 \\
(1/6)t^2 & 0 < t \leq 6
\end{cases}
\]

0.1 Graph \( f \) and \( F \) carefully labeling all interesting points.
0.2 Verify that \( F \) is continuous and differentiable at all points but \( F'(0) \) does not match \( f(0) \) and so \( F \) is not the antiderivative of \( f \) on \([-3, 6]\).

17.2 Compute \( F(t) = \int_{2}^{t} f(s) \, ds \) for
\[
f(t) = \begin{cases} 
-2t & 2 \leq t < 5 \\
12 & t = 5 \\
3t - 25 & 5 < t \leq 10
\end{cases}
\]

0.1 Graph \( f \) and \( F \) carefully labeling all interesting points.
0.2 Verify that \( F \) is continuous and differentiable at all points but \( F'(5) \) does not match \( f(5) \) and so \( F \) is not the antiderivative of \( f \) on \([2, 10]\).
Homework 17

17.3 Compute \( F(t) = \int_{-3}^{t} f(s) \, ds \) for

\[
f(t) = \begin{cases} 
3t & -3 \leq t < 0 \\
6 & t = 0 \\
(1/6)t^2 + 2 & 0 < t \leq 6
\end{cases}
\]

0.1 Graph \( f \) and \( F \) carefully labeling all interesting points.
0.2 Verify that \( F \) is continuous and differentiable at all points except 0 and so \( F \) is not the antiderivative of \( f \) on \([-3, 6]\)

Homework 17

17.4 Compute \( F(t) = \int_{2}^{t} f(s) \, ds \) for

\[
f(t) = \begin{cases} 
-2t & 2 \leq t < 5 \\
12 & t = 5 \\
3t & 5 < t \leq 10
\end{cases}
\]

0.1 Graph \( f \) and \( F \) carefully labeling all interesting points.
0.2 Verify that \( F \) is continuous and differentiable at all points except 5 and so \( F \) is not the antiderivative of \( f \) on \([2, 10]\)