Piercing the water surface with a blade: Singularities of the contact line
Mars M. Alimov and Konstantin G. Kornev

Citation: Physics of Fluids 28, 012102 (2016); doi: 10.1063/1.4938171
View online: http://dx.doi.org/10.1063/1.4938171
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/28/1?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
Pulsating flow driven alteration in moving contact-line dynamics on surfaces with patterned wettability gradients

Toward a description of contact line motion at higher capillary numbers

Effects of inertia on the hydrodynamics near moving contact lines
Phys. Fluids 11, 3209 (1999); 10.1063/1.870182

The effects of thin films on the hydrodynamics near moving contact lines
Phys. Fluids 10, 1793 (1998); 10.1063/1.869700

Effects of gravity on capillary motion of fluid
PHYSICS OF FLUIDS 28, 012102 (2016)

Piercing the water surface with a blade: Singularities of the contact line

Mars M. Alimov1 and Konstantin G. Kornev2
1Kazan Federal University, Kazan 420008, Russia
2Department of Materials Science & Engineering, Clemson University, Clemson, South Carolina 29634, USA

(Received 9 March 2015; accepted 30 November 2015; published online 7 January 2016)

An external meniscus on a narrow blade with a slit-like cross section is studied using the hodograph formulation of the Laplace nonlinear equation of capillarity. On narrow blades, the menisci are mostly shaped by the wetting and capillary forces; gravity plays a secondary role. To describe a meniscus in this asymptotic case, the model of Alimov and Kornev [“Meniscus on a shaped fibre: Singularities and hodograph formulation,” Proc. R. Soc. A 470, 20140113 (2014)] has been employed. It is shown that at the sharp edges of the blade, the contact line makes a jump. In the wetting case, the contact line sitting at each side of the blade is lifted above the points where the meniscus first meets the blade edges. In the non-wetting case, the contact line is lowered below these points. The contours of the constant height emanating from the blade edges generate unusual singularities with infinite curvatures at some points at the blade edges. The meniscus forms a unique surface made of two mirror-symmetric sheets fused together. Each sheet is supported by the contact line sitting at each side of the blade. © 2016 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4938171]

I. INTRODUCTION

The most popular method of characterization of the wetting properties of films and plates assumes their immersion into a liquid and studies the meniscus formed on both sides of the plate.1 It appears that the meniscus height and meniscus shape are very sensitive to the surface chemistry and roughness of the plate. In order to eliminate the end effects, wide plates and films are typically used in experiments. In this case, one can use the Laplace solution to the 1D equation of capillarity to describe the shape of the meniscus on a planar wall.2 However, for the slender plates or narrow blades this solution cannot be used anymore and the interpretation of the wetting experiment becomes problematic. Moreover, since the contact angle at the sharp edge is not defined,1,3–6 the meniscus behavior is difficult to predict. Hence this problem deserves special attention.

Concus and Finn7 discovered that menisci formed inside V-shaped grooves may form steep fingers running vertically along the grooves. This fingering effect is observed only under certain special conditions named the Concus–Finn conditions, constituting a relation between the contact angle and the V-angle. Experimental observations confirm these conditions.3,8 Capillary rise of the external meniscus on the V-shaped edges is not widely studied. A thorough mathematical analysis of different scenarios of meniscus behavior was proposed in Ref. 6 where the Concus–Finn condition was partially generalized to this case. Recently, this problem has been revisited by several groups.5,9,10 We theoretically analyzed a phase diagram of wetting of the chemically homogeneous and inhomogeneous corners and experimentally discovered that the external meniscus may possess a quite unusual singularity,10 the contact line can jump along the sharp edge as shown in Fig. 1. Water was loaded with a fluorescent dye, rhodamine, enabling one to distinguish the meniscus profile and contact line. The picture was taken when the tungsten blade was placed by its wider part in front of the camera. The blade has an elliptical cross-section with 1300 ± 10 µm widths and 30 ± 5 µm thickness. The meniscus approaching the edge of the blade first touches it at point $E_A$ where the
FIG. 1. (a) Fluorescent micrograph of the right half of meniscus climbing the tungsten blade. The metal surface is completely wettable by the aqueous solution of rhodamine. A jump of the contact line at the blade edge is clearly seen. (b) Bright field image of a hexadecane meniscus climbing a glass slide (Courtesy of Zhang and Vekselman).

contact line nucleates and then steeply rises up to point $E_B$. This behavior was also observed with a piece of a glass slide which was 1.1 mm wide and 0.15 mm thick. Hexadecane forms zero contact angle with the glass film. It appears that the contact line jumped 0.411 mm up from the points $C_A$ and $E_A$ where it first meets the edges of this slide.

Thus, the meniscus behavior shown in Fig. 1 suggests that the edge singularity is universal for thin blades with rounded and rectangular edges. When the thickness of a blade is much smaller than its width, it seems natural to neglect this thickness by modeling the blade as a slit. Results shown in Fig. 1 favor this idea. Graphene sheets putatively fall into this category of materials, however, the applicability of the continuum fluid mechanics yet needed to be checked experimentally. Clay platelets and crystalline whiskers used in nanocomposite manufacturing vary in thickness from a few nanometers to hundreds of nanometers to microns. Continuum mechanics does work for these cases, hence they all fall into a category of the slit-like blades when considering their immersion into the liquid by the edge down. At the edges, menisci of these materials are expected to form steep surfaces. Even if the edge is round, implying that the free surface should approach the edge at a finite contact angle, the capillary pressure at sharp edges is very high hence the meniscus surface inevitably forms steep slopes. Therefore, finiteness of the contact angle does not necessarily guarantee that all spatial derivatives of the meniscus height remain of the first order.

This effect of sharp edges is very difficult to analyze numerically. For example, the numerical procedure of Ref. 15 falls short of describing menisci on ribbon-like fibers with elliptical cross sections for the moderate contact angles $\gamma \approx \pi/3$. The method works well only when the fiber aspect ratio (defined as the ratio of the ellipse semi-axes) was varied within the range $[0.5, 1]$. This range of the fiber aspect ratios does not cover the important case of an elliptical blade when the major axis of the ellipse is much greater than the minor axis.

In this paper, we consider the behavior of meniscus on a slit-like blade. This case serves as the leading order term of the general Van Dyke asymptotic scheme designed to describe any physical field around a slender body by transferring the boundary condition to a slit. The transfer of the Young-Laplace boundary condition to the slit is accomplished by Taylor expanding this
boundary condition about the slit faces. As Van Dyke demonstrated, this leading order approximation adequately describes the field outside a small vicinity of the rounded edge. However, for correction of the leading order approximation, one needs to introduce a boundary layer situated near the edge. The boundary layer asymptotics are beyond the scope of this paper.

Thus, following Van Dyke’s asymptotic scheme and using the leading order term, we discuss the effect of sharp edges on meniscus shape. The gravity effect is assumed negligible relative to the wetting and capillary effects. The Bond number sets the metrics for this assumption to be valid. We define the Bond number as \( \varepsilon = L_m^2/L_c \ll 1 \) where \( L_c = \sqrt{\sigma / \rho g} \) is the capillary length, \( \sigma \) is the surface tension of the liquid, \( \rho \) is its density, and \( g \) is acceleration due to gravity; \( L_m \) is the half-width of the blade. For example, the Bond number for the water meniscus on a blade of the 100 \( \mu \)m width is estimated as \( \varepsilon \approx 10^{-3} \ll 1 \). As a biological example, the diameters of insect feeding devices, such as proboscises of butterflies, moths, and biting flies, vary in the range from a few microns to hundreds of microns. For these natural fibers, the Bond number is always less than one.

For small Bond numbers, the problem of a capillary rise of a meniscus on a blade can be reformulated as a problem of minimal surfaces with a special boundary condition at infinity. This model will be used in this paper.

II. WETTING SCENARIOS

There are two physically plausible configurations of the meniscus on a slit as shown in Figs. 2(a) and 2(b). When the jump is small, the experimental distinction between the scenario in Fig. 2(a) and that in Fig. 2(b) is very difficult to perform: any optical method brings about an uncertainty of the jump detection.

This paper aims to show that the most general scenario in Fig. 2(b) captures all mathematical features of the leading order asymptotics compatible with the macroscopic picture of wetting described by the Laplace theory. A jump of the contact line on a slit is a natural consequence of the Laplace equation of capillarity with the Laplace–Young boundary condition of the general form with an arbitrary contact angle. For the sake of generality, we will discuss Fig. 2(b) in detail and conclude that two points \( C^A \) and \( C^B \) (and \( E^A \) and \( E^B \)) cannot merge in the general case. However, in some particular cases, especially when the contact angle is close to \( \pi/2 \), the jump can be very small. We will discuss this case separately.

One has to mention that for very thin blades, the thickness of the films on the edges is controlled by intermolecular forces. Hence, the model has to be augmented by a disjoining pressure term. In classical capillarity, though, one can consider liquid films connecting points \( C^A \) and \( C^B \) as well as \( E^A \) and \( E^B \) to have zero thickness.

III. PROBLEM FORMULATION

Meniscus profile and the blade are described by the dimensionless Cartesian system of coordinates \( x = X/L_m, \ y = Y/L_m, \ z = Z/L_m \) with the \( z \)-axis directed perpendicularly to the free horizontal surface, \( z = 0 \) as \( r \to \infty \), where \( r = \sqrt{x^2 + y^2} \). Menisci are described by the surfaces \( \Sigma \)

\[
\begin{align*}
z &= h(x,y), \quad (1) \\
\nabla \cdot \left[ (1 + |\nabla h|^2)^{-1/2} \nabla h \right] - \varepsilon h &= 0. \quad (2)
\end{align*}
\]

As shown in Ref. 21, when the Bond number is small \( \varepsilon \ll 1 \), one may drop the second term in the Laplace equation. However, this truncated equation cannot satisfy the boundary condition requiring that at infinity the meniscus surface has to merge with the horizontal free surface.
FIG. 2. Schematic of the potential meniscus configurations with (a) no jump at the blade edges, and (b) with a jump at the blade edges. In (b), the curve $C^B E^B$ corresponds to the contact line, and the curves $C^A A$ and $E^A A$ correspond to the free surface profile obtained by cutting the surface with the plane $x = 0$. (c) a schematic configuration of the contour $\Lambda$ of a constant height $H$; these contours can be obtained by imaginary sectioning of the meniscus with the plane $z = H$. The angle $\alpha$ is formed by the tangent to the contour of a constant height and the $y$-axis at point $M$. (d) Top view of the blade with the meniscus where points $C A$ and $E A$ cannot be distinguished and hence we mark them $C$ and $E$, respectively. In the wetting case, vector $J$ points toward infinity, and in the non-wetting case, vector $J$ points toward the greatest dimple depth at the blade center $(0, 0)$.

Therefore, we developed a matched asymptotic procedure to satisfy this requirement. This procedure suggests that in the blade vicinity, the meniscus profile is a part of a minimal surface, a solution to the truncated Laplace equation with $\varepsilon = 0$ and a special logarithmic behavior at infinity.

For the particular case of a meniscus forming the contact angle $\gamma$ with each surface of the blade, this leading order approximation reads

$$h(x, y) = H(x, y) + \frac{l \cos \gamma}{2\pi} \left[ K_0(r\sqrt{\varepsilon}) + \ln r + \ln \left( e^{E\sqrt{\varepsilon}/2} \right) \right] + o(1),$$  \hspace{1cm} (3)

where $l = P/L_m$, and $P$ is the perimeter of the blade cross section, $r = \sqrt{x^2 + y^2}$, $K_0(u)$ is the modified Bessel function of the second kind and $E = 0.577$ is the Euler constant. The function $H(x, y)$ is selected as a solution to the following model:

$$\nabla \cdot \left[ (1 + |\nabla H|^2)^{-1/2} \nabla H \right] = 0,$$  \hspace{1cm} (4)

$$A : H = -l(2\pi)^{-1} \cos \gamma \ln \left( e^{E\sqrt{\varepsilon}r/2} + O(r^{-1}) \right)_{r \to \infty},$$  \hspace{1cm} (5)

$$\Gamma : (1 + |\nabla H|^2)^{-1/2} \nabla H \cdot n = -\cos \gamma,$$  \hspace{1cm} (6)

where $\nabla = (\partial/\partial x, \partial/\partial y)$ is the 2D gradient operator, and $n$ is the external normal to the blade surface $\Gamma$. For the slit, we have to use the following formulas:

$$l = 4, \quad n = (1, 0).$$  \hspace{1cm} (7)
As shown recently, the contact line at the cusped ends should behave singularly, i.e., $|\nabla H| \rightarrow \infty$ at the end points. As a result, the solution of model (3)–(7) should behave singularly at the slit ends. This behavior can be captured analytically for an important case of contact angles close to $\pi/2$. In fiber-based microfluidics, this case corresponds to the wetting transition from philic to phobic regimes (and vice versa, from phobic to philic regimes) when the fiber surface is made of the phobic/philic patches and the meniscus crosses the boundary of these distinct patches. A wetting dichotomy was also discovered on butterfly proboscises: it appears that the external surface of the proboscis closer to its tip is hydrophilic while the remaining external surface of the proboscis is hydrophobic. This allows the insect to keep its proboscis clean by hiding the hydrophilic tip in the coil when butterfly is not feeding. It is expected that the effect of wetting dichotomy plays a significant role in the insect world. Therefore, it is instructive to study this effect of phobic/philic dichotomy for the slit as a popular model of many feeding and attachment devices of insects as well as a popular shaped fiber.

**IV. BEHAVIOR OF THE MENISCUS WITH THE CONTACT ANGLE CLOSE TO $\pi/2$**

In this case, the contact angle $\gamma$ changes in the vicinity of $\pi/2$, hence

$$|\cos \gamma| = \delta \ll 1. \quad (8)$$

Parameter $\delta \rightarrow 0$ provides an upper estimate of the meniscus slope far away from the slit edges: the meniscus is supposed to meet the fiber at angle $\gamma$ and its slope should gradually increase from an infinite distance to the fiber surface. Therefore, we tentatively assume that the slope of the meniscus at any point $x, y$ outside the slit is of the order $|\nabla H| = O(\delta)$. This assumption allows one to construct a regular asymptotic expansion of the non-linear model (3)–(7) by Taylor expanding the Laplace equation of capillarity and the Young–Laplace boundary over the quadratic $(\nabla H)^2 \propto \delta^2$ terms. Thus, dropping the quadratic $(\nabla H)^2 \propto \delta^2$ terms, we have the following problem in the leading order:

$$\Delta H = 0, \quad (9)$$

$$\Gamma : \nabla H \cdot \mathbf{n} = -\cos \gamma. \quad (10)$$

The solution to problem (5), (7)–(10) is analogous to the problem of 2D electrostatics of determining a potential $H$ generated by a system of charges placed onto the slit with the density $\cos \gamma$. This problem is solved by introducing the logarithmic potential as

$$H(x, y) = -\frac{\cos \gamma}{2\pi} \int_{-1}^{1} \ln\left[x^2 + (y - \eta)^2\right] d\eta + C_0, \quad (11)$$

where constant $C_0$ has to be specified from the boundary condition at infinity (5). Taking integral (11), one obtains the meniscus height as

$$H(x, y) = -\frac{\cos \gamma}{2\pi} \left[2x \left[\arctan\left(\frac{1 - y}{x}\right) + \arctan\left(\frac{1 + y}{x}\right)\right] + (1 - y) \ln \left(r^2 - 2y + 1\right) + (1 + y) \ln \left(r^2 + 2y + 1\right)\right] + C_0. \quad (12)$$

Moving along the axis $y = 0$ toward infinity $x \rightarrow +\infty$, we have from Eq. (12)

$$y = 0, \quad x \rightarrow +\infty : H(x, y) = -\frac{\cos \gamma}{2\pi} \left[4x \arctan\left(\frac{1}{x}\right) + 4 \ln x + 2 \ln(1 - x^{-2})\right]_{x \rightarrow +\infty} + C_0 = -\frac{2\cos \gamma}{\pi} \ln x - \frac{2\cos \gamma}{\pi} + C_0 + O(\sqrt{x}/x). \quad (13)$$

On the other hand, taking the same limit in Eq. (5), $y = 0, \ x \rightarrow +\infty$, and substituting $l = 4$, we obtain

$$y = 0, \quad x \rightarrow +\infty : H(x, y) = -\frac{\cos \gamma}{\pi} \ln x - \frac{2\cos \gamma}{\pi} \ln(e^{\sqrt{\pi}/\ln x}) + O(x^{-1}). \quad (14)$$
Equations (13) and (14) will result in the same asymptotic behavior if one chose the constant to be

\[ C_0 = \frac{2 \cos \gamma}{\pi} \left[ 1 - \ln \left( e^{E\sqrt{\varepsilon}/2} \right) \right]. \tag{15} \]

It is instructive to study the slopes of the meniscus surface by differentiating Eq. (12):

\[ \frac{\partial H}{\partial x} = -\frac{\cos \gamma}{\pi} \left[ \arctan \left( \frac{1 - y}{x} \right) + \arctan \left( \frac{1 + y}{x} \right) \right], \tag{16} \]

\[ \frac{\partial H}{\partial y} = -\frac{\cos \gamma}{2\pi} \ln \left( \frac{r^2 + 2y + 1}{r^2 - 2y + 1} \right). \tag{17} \]

As follows from these equations, the meniscus is mirror-symmetric with respect to the axes. In particular, along the line \( x = 0 \) we have \( \partial H/\partial x = 0 \), as \( |y| > 1 \) and along the line \( y = 0 \) we have \( \partial H/\partial y = 0 \). Equation (17) gives the slope of the contact line along the slit as

\[ x = 0 : \frac{\partial H}{\partial y} = -\frac{\cos \gamma}{\pi} \ln \left( \frac{y + 1}{y - 1} \right). \tag{18} \]

It is evident that the slope \( \partial H/\partial y \) goes to infinity at the end points \( y = \pm 1 \) and this behavior is universal and does not depend on the contact angle. It immediately follows that the leading order solution with respect to \( \delta = |\cos \gamma| \ll 1 \) falls short in the exponentially small vicinity of the slit ends (at the distance of the order of \( O(e^{-1/\delta}) \)). Thus, in order to analyze the fine structure of the meniscus in the boundary layer, one needs to go beyond regular asymptotics. In the Secs. V–VIII, we will discuss the meniscus behavior near the slit ends in detail.

Matching the behavior of the meniscus height at infinity with Eqs. (12) and (15) as dictated by Van Dyke formula (3), we obtain the meniscus shape as

\[ h(x, y) = -\frac{\cos \gamma}{2\pi} \left\{ 2x \left[ \arctan \left( \frac{1 - y}{x} \right) + \arctan \left( \frac{1 + y}{x} \right) \right] + \right. \]

\[ \left. + (1 - y) \ln \left( r^2 - 2y + 1 \right) + (1 + y) \ln \left( r^2 + 2y + 1 \right) \right\} + \frac{2 \cos \gamma}{\pi} \left[ K_0(r\sqrt{\varepsilon}) + \ln r + 1 \right]. \tag{19} \]

This formula allows one to study meniscus profiles at different sections, in particular, the cross sections \( x = 0 \) and \( y = 0 \). A part \( y \in [-1, 1] \) of the cross section \( x = 0 \) describes the shape of the contact line \( \Gamma_c \). Substituting first \( x = 0 \) and then \( y = 0 \) in Eq. (19), we obtain two cross sections of the meniscus as

\[ h(0, y) = \frac{\cos \gamma}{\pi} \left[ 2K_0(|y|\sqrt{\varepsilon}) + 2 \ln |y| - (1 - y) \ln |1 - y| - (1 + y) \ln |1 + y| + 2 \right], \tag{20} \]

\[ h(x, 0) = \frac{\cos \gamma}{\pi} \left[ 2K_0(|x|\sqrt{\varepsilon}) + 2 \ln |x| - \ln (x^2 + 1) - 2x \arctan \left( \frac{1}{x} \right) + 2 \right]. \tag{21} \]

All logarithmic singularities are canceled out and these formulas describe profiles with no singularities. Two examples of these profiles are shown in Fig. 3, where we used \( \varepsilon = 10^{-3} \). In this regular asymptotic expansion, as \( \delta \to 0 \) the meniscus follows the wetting scenario in Fig. 2(a). However, as mentioned earlier, this asymptotic solution has to be corrected at the slit ends.

As evident from Fig. 3, the meniscus height is not constant. The contact line nucleates at the edges, points \( C \) and \( E \), and these points are located below point \( B \) corresponding to the maximum height of the meniscus. Following this analysis, the curvature of the meniscus surface at the blade edges is very high, implying high capillary pressure. Thus, the liquid is squeezed out from the edges toward the blade’s edges. This illustrative and physically meaningful analytic solution allows one to demonstrate that the derivatives of the nonlinear model (3)–(7) should behave singularly at the endpoints of the slit. Formula (18) explicitly introduces the logarithmic character of this singularity confirming the general statements of Refs. 6 and 10. With this result in hand, we turn to the analysis of non-linear model (3)–(7) taking into account this singular behavior of the height derivatives \( |\nabla h| \to \infty \) at the slit ends.
Profiles of the meniscus and contact line $h(x, y) = H(x, y)/\cos \gamma$ obtained at the leading order of a regular asymptotic expansion with respect to $|\cos \gamma| = \delta \ll 1$: (a) at the cross section $x = 0$ and (b) at the cross section $y = 0$; the Bond number is $\varepsilon = 10^{-3}$. In this approximation, the slope is assumed to be small, $|\nabla h| = O(\delta)$. The contact line CBE at $y \in [-1, 1]$, continuously meets the meniscus at the slit edges. The dashed lines mark the submersed slit.

V. CHAPLYGIN–SOKOLOVSKY TRANSFORMATION

Looking at the meniscus from the top, one cannot recognize points $C^A$, $C^B$ and $E^A$, $E^B$. Hence we mark them $C$ and $E$, respectively. Due to the mirror symmetry of the blade with respect to the $x$ and $y$ axes, the surface of the meniscus should follow the same symmetry. As follows from the analysis of Sec. IV, the meniscus is bulged up along $y = 0$ where we should have $\partial H/\partial y = 0$, for all $x$. Meniscus profile at the axis $x = 0$ for all $|y| > 1$ should also have zero slope, $\partial H/\partial x = 0$ implying that the two semi-infinite sheets of the meniscus surfaces are continuously fused along $x = 0$. These symmetry arguments suggest to consider only one quadrant $\Omega = ABCA$ in the $x, y$ plane, Fig. 2(d).

Non-linear model (3)–(7) can be solved analytically using the hodograph formulation. A mathematical analogy of the non-linear equation of minimal surfaces with the Chaplygin gas equations or equations describing flows of non-Newtonian fluids through porous media allows one to use a rich arsenal of methods of complex analysis to obtain the explicit solution of this problem.

Following Chaplygin and Khristianovich, Eq. (4) is rewritten as a conservation equation for a fictitious flux $J$

$$\nabla \cdot J = 0, \quad \text{where} \quad J = -J|\nabla H|^{-1}\nabla H, \quad J = \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}. \quad (22)$$

While the slope $|\nabla H|$ can be infinitely large, this fictitious flux is always bounded from above $J \leq 1$ inside the domain $\Omega$ including its boundary.

Third equation (22) for the flux magnitude $J$ can be inverted as $|\nabla H| = \Phi(J), \ \Phi(J) = J(1 - J^2)^{-1/2}, \ \Phi(0) \geq 0, \ \Phi'(0) \geq 0$. It is convenient to express the Cartesian components of the flux vector through the flux magnitude and the angle $\theta$ that the flux $J$ makes with the $x$-axis, $J_x = J \cos \theta$ and $J_y = J \sin \theta$. In the 2D case, Eq. (22) is automatically satisfied by introducing the stream function $\psi(x, y)$ as

$$J_x = J \cos \theta = \frac{\partial \psi}{\partial y}, \quad J_y = J \sin \theta = -\frac{\partial \psi}{\partial x}. \quad (23)$$
Therefore, together with flux definition (22), one can rewrite the basic equation as a system of four partial differential equations of the first order,

\[
\begin{align*}
\frac{\partial H}{\partial x} &= -\Phi(J) \cos \theta, \\
\frac{\partial \psi}{\partial x} &= -J \sin \theta, \\
\frac{\partial H}{\partial y} &= -\Phi(J) \sin \theta, \\
\frac{\partial \psi}{\partial y} &= J \cos \theta.
\end{align*}
\] (24)

One can take one step further to eliminate the \( x \) and \( y \) variables from this system of equations assuming the following functional dependencies \( \psi = \psi(J, \theta) \), \( H = H(J, \theta) \), and applying the Chaplygin transformation to the system as

\[
\frac{\Phi(J) \partial \psi}{\Phi(J) \partial J} = \frac{-\partial H}{\partial \theta}, \quad \frac{\Phi(J) \partial \psi}{\partial \theta} = \frac{\partial H}{\partial J}.
\] (25)

As a result, one obtains the following set of basic equations, \( J(1 - J^2)^{1/2} \partial \psi / \partial J = -\partial H / \partial \theta \) and \( \partial \psi / \partial \theta = J(1 - J^2)^{1/2} \partial H / \partial J \). This nonlinear system of partial differential equations permits Sokolovsky’s transformation by introducing a new variable \( t \) through the equation: \( dt / dJ = -J^{-3}(1 - J^2)^{-1/2} \). Integrating this equation and expressing the function \( \Phi(J) \) through a new variable \( t \), one obtains, \( J = (\cosh t)^{-1}, \quad t = \text{arccosh} \left( J^{-1} \right), \quad \Phi(J) = (\sinh t)^{-1} \).

Thus, Eqs. (25) are rewritten in the Cauchy–Riemann form as \( \partial \psi / \partial t = \partial H / \partial \theta \) and \( \partial \psi / \partial \theta = -\partial H / \partial J \). As soon as one finds the solution to the Cauchy–Riemann equations with the corresponding boundary conditions, functions \( x(H, \psi) \) and \( y(H, \psi) \) are obtained by solving the Chaplygin ordinary differential equations:

\[
\begin{align*}
dx &= -\cos \theta \sinh t \, dH - \sin \theta \cosh t \, d\psi, \\
dy &= -\sin \theta \sinh t \, dH + \cos \theta \cosh t \, d\psi.
\end{align*}
\] (27)

VI. COMPLEX VARIABLES

A. Complex variables and hodograph plane

The Cauchy–Riemann equations naturally generate the complex-valued functions \( W \) and \( \chi \) as

\[
W = -H + i\psi, \quad \chi = t + i\theta.
\] (28)

According to the Cauchy–Riemann equations, function \( W(\chi) \) is an analytic function defined within the plane \( \chi \). Following the tradition of fluid mechanics, we will call the function \( W \) the complex potential of a fictitious gas flow generated by the constitutive Equation (22). The \( \chi \)-function describes the so-called flow hodograph.

An image of the physical domain \( \Omega \) on the hodograph plane \( \chi \) is denoted as \( \Omega_{\chi} \). The meniscus surface \( \Sigma \) is supported by the contact line situated at the blade boundary \( \Gamma = BC \), Figs. 2(b) and 2(d). As follows from the constitutive Eq. (22), in the wetting case when the meniscus rises at the blade, the flux vector is directed toward infinity; in the non-wetting case, when the meniscus forms a dimple at the blade, the flux is directed toward the blade. The boundary conditions are as follows:

\[
\begin{cases}
AB : \theta = 0, & AC : \theta = -\pi/2, \quad \text{(wetting case)} \\
AB : \theta = \pi, & AC : \theta = \pi/2, \quad \text{(non-wetting case)}
\end{cases}
\] (29)

The boundary condition at the blade surface \( \Gamma \) takes on the form \( J \cdot \mathbf{n} = \cos \gamma \), which is rewritten in terms of \( J \) and \( \theta \) as

\[
\Gamma : J \cos \theta = \cos \gamma,
\] (30)

or, using Eq. (26), it is written in terms of \( \theta \) and \( t \) as

\[
\Gamma : \cos \theta = \cos \gamma \cosh t.
\] (31)
Equation (5) provides the following behavior of the meniscus height at infinity \( r = \sqrt{x^2 + y^2} \to \infty \): \( \nabla H = -2 \cos \gamma \frac{r}{(\pi r^2)} + O(r^{-2}) \). Consequently, the flux magnitude \( J \) decays at infinity as

\[
A : J = \left. \frac{2 \cos \gamma}{\pi r} + O(r^{-2}) \right|_{r \to \infty} \to 0. \tag{32}
\]

Thus, all boundary conditions are rewritten for the pair \((J, \theta)\).

As follows from Eq. (29), the boundaries \( AB \) and \( AC \) in the hodograph plane \( \chi \) are transformed into the horizontal lines. Thereafter, we consider only the wetting case, the first line of Eq. (29). In order to locate point \( A \) on the \( \chi \)-plane, one needs to analyze the behavior of the meniscus slope at infinity. With the aid of Eq. (32), one infers that the flattening of the meniscus surface implies that \( J \) goes to zero. From the second Eq. (26), it follows that the limit \( J \to 0 \) implies \( t \to \infty \). Hence, point \( A \) is located at infinity. Thus, the plane \( \chi \) extends to infinity.

While the two lines \( AB \) and \( AC \) have already been specified, locations of the points \( C^A, C^B \), and \( B \) are not known yet. Using our hypothesis, we assume the following singularity \( |\nabla H| \to \infty \) at points \( C^A, C^B \). Consider first the behavior of the flux at the point \( C^A \). When the meniscus height sharply increases, \( |\nabla H| \to \infty : J \to 1 \), the \( t \)-variable goes to zero, \( t \to 0 \). Therefore, the point \( C^A \) on the \( \chi \)-plane must be located at \( \chi = -i\pi/2 \).

Since the point \( C^B \) is situated at the blade edge, the same condition \( |\nabla H| \to \infty, J \to 1 \) and \( t \to 0 \) holds for this point as well. Thus, at the line \( C^A C^B \) we have \( t = 0 \). However, the angle \( \theta \) changes along the edge and this variation is not known yet.

In order to find the boundary value of angle \( \theta \) at \( C^B \), we turn to the boundary condition on the blade surface, Eq. (31), taking the limit as \( t \to 0 \). Thus, the location of the point \( C^B \) on line \( t = 0 \) is defined by the solution of the equation \( \cos \theta = \cos \gamma \). It becomes clear that the angle \( \theta \) depends on the wetting properties of the blade surface,

\[
C^B: \begin{cases} 
\theta = -\gamma, & \text{wetting case, } \gamma < \pi/2, \\
\theta = \gamma, & \text{non-wetting case, } \gamma > \pi/2.
\end{cases} \tag{33}
\]

At the same time, the point \( C^A \) belongs to the symmetry line, hence Eq. (29) holds true. We therefore obtain

\[
\begin{cases} 
C^A A : \theta = -\pi/2, & \\
C^B : \theta = -\gamma, & \text{wetting case, } \gamma < \pi/2, \\
C^A A : \theta = \pi/2, & \\
C^B : \theta = \gamma, & \text{non-wetting case, } \gamma > \pi/2.
\end{cases} \tag{34}
\]

Position of point \( B \) in the hodograph plane \( \chi \) is defined by Eq. (31) as

\[
B : \theta_B = 0, \quad t_B = \arccosh(|\cos \gamma|^{-1}). \tag{35}
\]

To determine the angle at which the image of the contact line approaches points \( B \) and \( C^B \), we employ the definition of the contact line \( \cos \theta(t) = \cos \gamma \cosh t \) and differentiate this equation with respect to \( t \),

\[
\Gamma : \frac{d\theta}{dt} = -\cos \gamma \frac{\sinh t}{\sin \theta}. \tag{36}
\]

Substituting the particular values of parameters, one can infer the behavior of the contact line near points \( B, C^B \),

\[
B : \left. \frac{d\theta}{dt} \right|_{t \to 0} = -\cos \gamma \frac{\sinh t_B}{\sin \theta} \to \infty, \quad C^B : \left. \frac{d\theta}{dt} \right|_{t \to 0} = \cos \gamma \frac{\sinh t}{\sin \gamma} = 0. \tag{37}
\]

Thus, the image of the contact line in the hodograph plane forms the right angles at points \( B, C^B \). This analysis allows one to draw the hodograph plane, Fig. 4(a).

**B. Plane of the complex potential**

According to the Cauchy–Riemann equations, the map \( \chi \to W \) is conformal. Therefore, since the piece \( C^A C^B \) of the boundary \( \Omega_+ \) in the hodograph plane is finite (i.e., the points \( C^A \) and \( C^B \).
FIG. 4. (a) Hodograph plane for the wetting case, and (b) plane of complex potential for the wetting case. For the non-wetting case, the domain $\Omega_x$ is obtained by shifting the whole domain $\Omega_x$ in (a) vertically by $\pi$ so that the horizontal line $C^A A$ is moved to the line $\theta = \pi/2$ and the line $BA$ is moved to the line $\theta = -\pi$. For the non-wetting case, the domain $\Omega_W$ is obtained by shifting the whole domain $\Omega_W$ in (b) vertically by $(-2\cos \gamma)$ so that the horizontal line $C^B C^A A$ is moved to the line $\psi = \cos \gamma$ and the line $BA$ is moved to the line $\psi = 2\cos \gamma$.

are separated, then the piece $C^A C^B$ of the boundary $\Omega_W$ in the $W$-plane must also be finite and the corresponding points must also be separated, i.e., we should allow the function $H$ to change between $H_{C^A}$ and $H_{C^B}$.

The domain $\Omega_W = AC^A C^B BA$ on the $W$-plane is unknown and its construction deserves special attention. To evaluate the behavior of the stream function at the boundary $\Omega_W$, we follow the definition of $\psi(x, y)$ through the flux components $J_x(x, y)$ and $J_y(x, y)$, Eq. (23). First, one can see that the $x$-component of the flux is zero along the line $AC$. Integrating Eq. (23) along this line, we obtain

$$ AC : \psi(y, 0) = \int_{-\infty}^y J_x dy + \psi_{AC}. $$

(38)

where $\psi_{AC}$ is the integration constant. As seen from the hodograph plane, the $x$-component of the flux along the blade edge $C^A C^B$ is bounded from above, $J_x \leq J \leq 1$. Therefore, the integral goes to zero everywhere along the line $AC$ including the point $C$. Therefore, the stream function $\psi$ is constant everywhere along the line $AC^A C^B$ on the $W$-plane, $\psi = \psi_{AC}$.

On the other hand, along the line $AB$ the $y$-component of flux is zero, $\partial \psi / \partial x = 0$. Therefore, following the same argument, we obtain that this line is also a streamline where the stream function is constant, $\psi = \psi_{AB}$.

These two constants can be determined by invoking the boundary condition at infinity, Eq. (32), where $|\nabla H| \ll 1$. Rewriting this condition in the asymptotic form as $J \approx -\nabla H$, one can see that this form of the constitutive equation implies that the flow of a fictitious liquid at infinity is a potential flow with potential in Eq. (4) caused by a point source of discharge, $4 \cos \gamma$. Thus, for the symmetry quadrant of the total fictitious flow, we have

$$ AB : \psi = \psi_{AB} = 4 \cos \gamma, \quad AC^A C^B : \psi = \psi_{AC} = 3 \cos \gamma, \quad A : H \rightarrow -\infty. $$

(39)

Since the map $\chi \rightarrow W$ is conformal, the points $C^A$ and $C^B$ are separated and positioned along the stream line $AC^A C^B$. Thus, the contact line jumps along the blade edge, $H_{C^A} \neq H_{C^B}$. This conclusion eliminates the possibility of having a meniscus with a continuous contact line shown schematically in Fig. 2(a).

A complete construction of the $W$-plane cannot be done analytically, because the contact line $\Gamma$ connecting points $C^B$ and $B$ in the $W$-plane is not explicitly specified. The parametric equation of the contact line $\Gamma$ in the $W$-plane is given as $\psi = \psi_1(H)$ where function $\psi_1(H)$ is defined by the following differential equation $\Gamma : d\psi_1 / dH = (d\psi_1 / dy)(dH / dy)^{-1} = (d\psi / dy)(dH / dy)^{-1}$. Using Eqs. (24), (26), and (31), the latter is represented as

$$ \Gamma : \frac{d\psi_1}{dH} = -\frac{J \cos \theta}{\Phi(J) \sin \theta} = -\frac{\tanh t}{\tan \theta} = -\frac{\sinh t \cos \gamma}{\sqrt{1 - \cosh^2 \cos^2 \gamma}}. $$

(40)

Tracing the changes of the right hand side of this equation along $\Gamma$ in the $\Omega_x$ plane, one concludes that the function $\psi_1(H)$ is a monotonous function. In particular, one can immediately infer that
contact line \( \Gamma \) approaches line \( AB \) forming angle \( \pi/2 \), while it approaches line \( AC^B \) forming angle \( \pi \),
\[
B : \frac{d\psi_B}{dH} = -\frac{\tanh t_B}{\tan \theta} \bigg|_{\theta \to 0} \to -\infty; \quad C^B : \frac{d\psi_C}{dH} = \frac{\tanh t}{\tan \gamma} \bigg|_{\gamma \to 0} = 0. \tag{41}
\]
Schematic of the domain in the \( W \)-plane is shown in Fig. 4(b).

**VII. ANALYSIS OF MENISCUS SINGULARITIES**

One can go further and analyze the behavior of the meniscus surface in the vicinity of the blade edges. Consider any line \( \Lambda \) of constant height \( H(x, y) = H_M \) passing a point \( M \in C^A C^B \) at the blade edge, Fig. 2(c). Using Eqs. (27), one derives the differential equation of this line as
\[
\Lambda : \frac{dx}{dy} = -\tan \theta. \tag{42}
\]

Thus, the tangent lines to the contours \( \Lambda \) form angle \( \alpha = -\theta \) with the \( y \)-axis. When one moves along the contact line and approaches the blade edge, the angle \( \theta \) at which the meniscus surface approaches the blade monotonously changes from \( -\gamma \) at the uppermost point \( C^B \) to \(-\pi/2\) at the lowermost point \( C^A \) on the blade edge. Looking at the tangent line to the contour \( \Lambda \), corresponding to \( H(x, y) = H_{C^B} \), one concludes that this line emanates from the uppermost point \( C^B \) of the contact line situated at the blade edge and forms the angle \( \alpha = \gamma \). At the lowermost point \( C^A \) of the contact line situated at the blade edge, the contour \( \Lambda \) corresponding to \( H(x, y) = H_{C^A} \) forms the angle \( \alpha = \pi/2 \) with the \( y \)-axis.

One can go further and analyze the curvature of the contour \( \Lambda \) of constant height at an arbitrary point \( M \) sitting at the blade edge along the line \( C^A C^B \) in Fig. 2(c). Introducing arc length \( s \) measured from point \( M \) along the contour \( \Lambda \), the radius of curvature of this plane curve is written as \( R = |ds/d\theta| \). Using the definition of the arc length differential \( ds = \sqrt{(dx)^2 + (dy)^2} \) and Chaplygin formulas (27), we obtain
\[
\Lambda : ds = \sqrt{\sin^2 \theta \cosh^2 t + \cos^2 \theta \cosh^2 t} \, d\psi = \cosh t \, d\psi. \tag{43}
\]
As follows from the hodograph plane, Fig. 4(a), at point \( M \) we have \( t = 0 \), therefore the radius of curvature at point \( M \) is calculated as
\[
M \in \Lambda : R_M = \left| \frac{d\psi}{d\theta} \right|_M = \cosh t \left| \frac{d\psi}{d\theta} \right|_M = \left| \text{Re} \left( \frac{dW}{d\chi} \right) \right|_M. \tag{44}
\]
Since the map \( \chi \to W \) is conformal at any point \( M \) within \( C^A C^B \), the derivative \( dW/d\chi \big|_M \) cannot be zero or infinitely large. Therefore, the radius of curvature \( R_M \) is finite and cannot be zero. When we approach the points \( C^A \) and \( C^B \), where the map \( \chi \to W \) is no longer conformal, the function \( W(\chi) \) behaves as \( \chi \to \chi_{C^A B} : W(\chi) \sim (\chi - \chi_{C^A B})^2 \).\textsuperscript{44} Hence, the required derivative is estimated as
\[
\chi \to \chi_{C^A B} : \frac{dW}{d\chi} \sim (\chi - \chi_{C^A B}). \tag{45}
\]
Consequently, as point \( M \) moves toward points \( C^A \) and \( C^B \), the radius of curvature of contours \( \Lambda \) of the constant height goes to zero \( R_M = 0 \).

We can also analyze the behavior of the meniscus profile when it approaches the blade edge along line \( AC \). Since this line is the streamline where the stream function is constant, we can use Eq. (27) to derive an equation of the meniscus profile as
\[
AC : \frac{dy}{dH} = -\sin \theta \sinh t \, dH. \tag{46}
\]
As one approaches the blade edge along the line \( AC \), \( t \to 0 \), the value \( dH/dy \) goes to infinity. Thus, the meniscus profile bends up (wetting case) or down (non-wetting case) approaching the blade edge vertically.
FIG. 5. Meniscus at the blade edge. The dry part of the blade is shown in white. The observer sees the blade at some angle. The contours of the equal height $z = H$ are shown as the white solid curves. The contact line jumps at the edge from point $C^A$ to point $C^B$. The bright spot below point $C^A$ illustrates a patch of sunlight bouncing off the meniscus surface.

When we move along the blade, Eq. (40) holds true and using the second Eq. (27), we obtain $d\psi_\Gamma/dH = -\cot \theta \tanh t$. Therefore, the Chaplygin equation is reduced to

$$
\Gamma : dy = \left( -\sin \theta \sinh t + \cos \theta \cosh t \frac{d\psi_\Gamma}{dH} \right) dH = -\frac{\sinh t}{\sin \theta} dH. 
$$

Calculating these derivatives at the blade edge, $t = 0$, $\theta = -\gamma$, we see that the contact line approaches the edge vertically along the blade side, $dH/dy \to -\infty$.

This analysis suggests that the behavior of the meniscus at the blade edge is universal: when one approaches the blade edge along line $AC$, the behavior of meniscus does not depend on the wetting properties of the blade. The contact line must jump up or down from the point where the meniscus touches the blade edge. Since the height $C^A C^B$ in hodograph plane, Fig. 4(a), depends on the contact angle $\gamma$, the jump height of the contact line at the blade edge depends on the contact angle as well. Also, the wetting properties of the blade show up on the blade sides, where the meniscus must approach the blade sides at the constant contact angle.

Using this quantitative analysis of the meniscus behavior at the blade edges, one can develop a numerical method enabling a resolution of the meniscus features at the edges. The basic ideas of this method have been explained in Ref. 21 and a detailed description of the method will be explained in a follow-up paper. In Figure 5, we show the level lines of the meniscus at the $C^A C^B$ edge. The viewer sees the edge at some angle so that one can observe the behavior of the lines of constant height $\Lambda$ (sketched in Fig. 2(c)) approaching the edge from the visible side of the blade and bending back to its non-visible side. The dry part of the blade is shown in white. All contours $\Lambda$ which do not touch the blade, i.e., the contours corresponding to the heights, $z = H \leq H_{CA}$, are corner free. The contour $z = H_{CA}$ is also corner-free because the full angle $2\alpha$ at the point $C^A$ is equal to $2\alpha = \pi$ as discussed earlier. The contours $H_{CA} < z < H_{CB}$ will always form corners at the blade edge, because the angle $2\alpha$, decreases there from $\pi$ at point $C^A$ to $2\gamma$ at point $C^B$.

VIII. CONCLUSION

Macroscopic model of the Laplace capillarity augmented by the Laplace–Young boundary condition at the contact line is examined in the case of a capillary rise of a meniscus on a thin blade. The Bond number is assumed to be small. A new type of meniscus singularity discovered in Ref. 10 was analyzed in detail using the problem formulation of Ref. 21. It is shown that the meniscus
and the contact line approach the blade edges vertically and this behavior is universal and does not depend on the contact angle. We proved that the macroscopic model selects only one wetting scenario shown in Fig. 2(b); the wetting scenario of Fig. 2(a) is not compatible with the Laplace model of capillarity. Within this scenario, the contact line jumps at the blade edges. However, when the contact angle is close to $\pi/2$, this jump is very small and is difficult to distinguish. The contours of the constant height have quite unusual singularities at the blade edges. At the uppermost points $C^B$, $E^B$, the curvature of the contour of constant height goes to infinity and the contour forms a corner with the internal angle $2\gamma$. At the lowermost points $C^A$ and $E^A$, the contour of constant height is corner free, but its curvature goes to infinity. Thus, the Laplacian capillarity and wetting of a thin blade generates a rich library of new singularities.

ACKNOWLEDGMENTS

We thank C. Zhang and V. Vekselman for providing images of the contact line shown in Fig. 1. M.M.A. was supported by the Russian Foundation for Basic Research, Projects Nos. 15-01-06029a and 13-01-00368. K.G.K. was supported by the National Science Foundation through the Grant No. PoLS 1305338.

2 P. S. Laplace, Mécanique Celeste (Encyclopédia Britannica, Paris, 1806).
4 B. Finn, Equilibrium Capillarity Surfaces (Springer-Verlag, New York, 1986).
22 B. V. Derjaguin, N. V. Churaev, and V. M. Muller, Surface Forces (Consultants Bureau, Plenum Press, New York, 1987).