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An External Meniscus on a Thin Fiber Whose Profile Has Separate Rectification Points

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Abstract—We clarify the limits of applicability of the previously developed asymptotic method for defining the configuration of an external meniscus of a liquid on a thin fiber. As was found out earlier, if the fiber material is fully wet, while the fiber profile has rectilinear parts, then the mentioned method gives unsatisfactory results when being applied near the fiber, because it leads to the infinite growth of the contact line. We prove that in the case when a smooth convex fiber profile contains only separate rectification points, the asymptotic approach predicts the meniscus shape, as a whole, satisfactory. However, it inadequately describes the behavior of the contact line near rectification points of the profile, namely, it gives a line with breaks instead of an expected smooth one.

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Introduction. A single fiber, being immersed in a liquid, causes its capillary rise that changes the shape of the liquid free surface, forming an external meniscus. In the paper [1], we propose an asymptotic approach for determining the configuration of such a meniscus for a fiber with an arbitrary profile, provided that its characteristic size is small compared to the capillary length. This approach was later used for determining the shape of concrete external menisci on fibers with various profiles. Namely, in [2] this problem is solved for a smooth convex ovoid or elliptic profile in the case of the full wetting of the material (the contact angle equals zero) and in [3] this is done for a nonsmooth profile representing a segment (a ribbon-like fiber) with arbitrary values of the contact angle.

Analyzing solutions obtained in [2] and [3], we conclude that two factors play an important role. The first factor is the smallness of the contact angle and the full wetting of the fiber at the limit. The second one is the fact that the fiber profile has parts with small curvature and rectilinear (at the limit) parts. The asymptotic approach proposed in [1] gives satisfactory results when only one of these factors or even no one takes place. However, if both factors are fulfilled, i.e., the profile contains rectilinear parts, while the contact angle takes on a small value, then the approach proposed in [1] leads to unsatisfactory results, namely, the height of the liquid rise on the fiber grows as the logarithm of the value inverse to the contact angle [3].

One can explain this by the fact that in the case of full wetting, near rectilinear parts of the profile there occurs a zone where the gradient of the function of the liquid rise height takes on large values. In this zone, gravitational term of the capillarity differential equation becomes significant, while the approximation of the minimal surface used in the asymptotics does not adequately describe the real shape of the meniscus. In this case, together with the use of the asymptotic approach proposed in [1], one should solve the complete capillarity equation in the mentioned zone. However, one can hardly deduce this equation in the analytic form because of its essential nonlinearity. Therefore, it makes sense to clearly indicate the limits of applicability of the approach proposed in [1].

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In this connection, we can imagine a case when a convex smooth profile has no rectilinear part, though there exist separate profile rectification points. The goal of this paper is to answer the question whether the occurrence of such points is critical for the applicability of the asymptotic approach proposed in [1] in the case of full wetting.

1. The problem. See Fig. 1, *a*) for the external meniscus Σ formed by immersing a fiber in a liquid. Here Γ_c is the contact line at the interface of three media, namely, air — liquid — fiber material. The normal section of the fiber has a profile Γ; evidently, it coincides with the projection of the contact line Γ_c to the plane x, y. We assume that the profile Γ is smooth, convex, and has at least one axis of symmetry (in Fig. 1, *b*), this is the axis x).

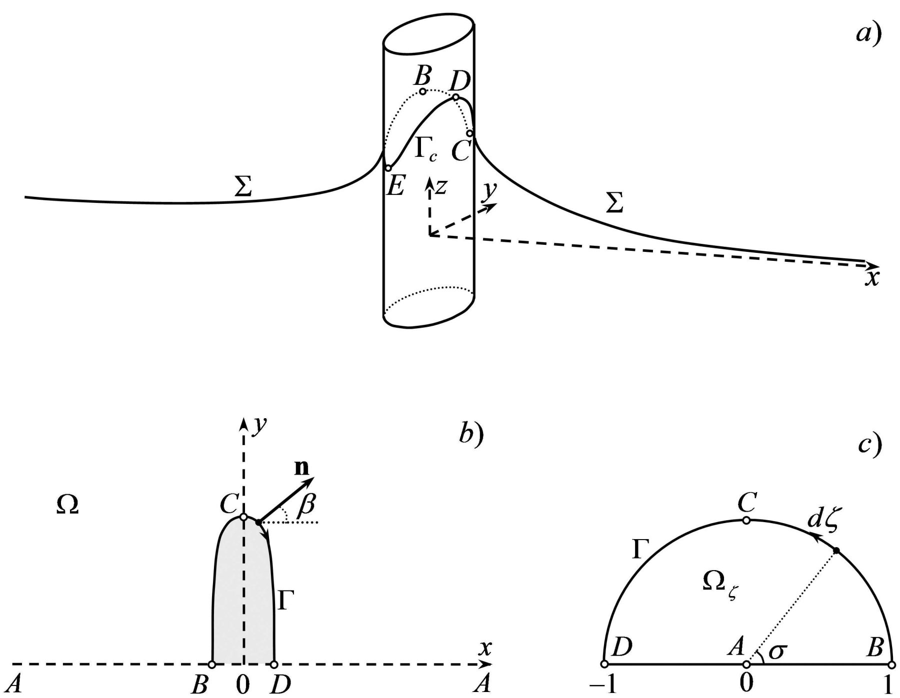

Fig. 1. *a*) the meniscus Σ and the contact line Γ_c in the space, *b*) the upper half of the physical plane x, y, *c*) the auxiliary plane ζ.

Choose the half of the maximal thickness of the fiber profile as the characteristic length L_m and introduce the dimensionless Cartesian coordinate system x, y, z so as to make the axis z be the central axis of the fiber, axes x, y be orthogonal to z, and the plane x, y be tangential to the meniscus surface at infinity $r \to \infty$ with $r^2 = x^2 + y^2$. The capillary rise is described by the dimensionless Bond number

$$\varepsilon = \frac{\rho g}{\sigma_s} L_m^2,$$

where σ_s is the surface tension, ρ is the density of the liquid, and g is the gravitational acceleration. Let the surface of the meniscus Σ obey the function

$$z = h(x, y),$$

where h is the height of the liquid column (measured from the level of the liquid at infinity) at the point (x, y). Taking into account the symmetry of the profile with respect to the axis x, we can introduce an element of the symmetry of the physical plane x, y, namely, the area $\Omega = ABDA$ (see Fig. 1, *b*)). We can write the mathematical model of the capillary rise of the liquid on the fiber in the case of full wetting of the fiber material as the following contact problem for the function

$h(x, y)$ on the area Ω [1]:

$$\Omega: \quad \nabla \cdot \left[\left(1 + |\nabla h|^2\right)^{-1/2} \nabla h \right] - \varepsilon h = 0, \tag{1}$$

$$A: \quad h|_{r \to \infty} \to 0, \tag{2}$$

$$AB \cup DA: \quad \partial h / \partial y = 0, \tag{3}$$

$$\Gamma = BD: \quad \left(1 + |\nabla h|^2\right)^{-1/2} \nabla h \cdot \mathbf{n} = -1; \tag{4}$$

here $\mathbf{n}$ is the external normal to the boundary of the fiber profile Γ.

2. The asymptotic approach to the analysis of problem (1)–(4). If the fiber is thin, then the Bond number is small ($\varepsilon \ll 1$), which allows us to apply asymptotic methods, in particular, the matching of asymptotic expansions. The inner asymptotic expansion $h^{(i)}(x, y)$ of the function $h(x, y)$ with respect to ε is valid in the fiber neighborhood $r \sim 1$, while the external one $h^{(o)}(x, y)$ is valid near infinity $r \sim \varepsilon^{-1}$; moreover, the principal term of the outer expansion for any fiber profile takes the form [1]

$$h^{(o)}(x, y) = \frac{l}{2\pi} K_0 \left(r\sqrt{\varepsilon}\right),$$

where l is the dimensionless perimeter of the fiber profile and K_0 is the Bessel function of the second kind.

The principal term $h^{(u)}(x, y)$ of the asymptotic expansion $h(x, y)$ with respect to ε, which uniformly approximates the solution $h(x, y)$ to problem (1)–(4) in the whole plane x, y, obeys the formula [1]

$$h^{(u)}(x, y) = h^{(i)}(x, y) + \frac{l}{2\pi} \left[K_0 \left(r\sqrt{\varepsilon}\right) + \ln\left(\frac{r\sqrt{\varepsilon}}{2}\right) + E \right], \tag{5}$$

where $E = 0.57\ldots$ is the Euler constant.

The principal term of the inner asymptotic expansion $h^{(i)}(x, y)$ satisfies the equation of the minimal surface that represents the complete capillarity equation (1) without the gravitational term. In [1] we establish the fact that the problem for $h^{(i)}(x, y)$ is analogous to filtration problems for abnormally viscous liquids. In [2] we propose an effective solution method for this problem for convex smooth profiles Γ, whose parametric form is described by the positive definite function $F(\beta) = -s'(\beta)$ that represents the dependence of the curvature radius of the contour Γ on the angle β of inclination of the normal to the contour, where s is the contour arc abscissa measured from the point B.

We can find the function $h^{(i)}(x, y)$ indirectly by introducing an auxiliary parametric plane of complex variable $\zeta = \xi + i\eta$ and by defining three functions $h^{(i)}(\xi, \eta)$, $x(\xi, \eta)$, and $y(\xi, \eta)$. The plane ζ is the unit semicircle Ω_ζ with the correspondence of points indicated in Fig. 1, *c*). The boundary of the contour Γ corresponds to the unit semicircumference $\zeta = \exp(i\sigma)$, $\sigma \in [0, \pi]$, while the argument σ of the boundary point is directly connected with β ([2]), namely,

$$\Gamma: \ \sigma = \pi - \beta. \tag{6}$$

Let us first introduce an auxiliary function $\omega(\zeta)$ of complex variable ζ such that it is analytic inside the unit circle $|\zeta| < 1$ ([2]), can be defined in terms of the Schwartz integral, namely,

$$\omega(\zeta) = -\frac{1}{2\pi} \int_0^{2\pi} \left\{ F(\beta)|_{\beta = \pi - \sigma} \right\} \frac{\exp(i\sigma) + \zeta}{\exp(i\sigma) - \zeta} \, d\sigma, \tag{7}$$

and

$$\zeta = \exp(i\sigma): \quad \operatorname{Re}\omega(\zeta) = -\left\{ F(\beta)|_{\beta=\pi-\sigma} \right\}.$$

The function $h^{(i)}(\xi, \eta)$ represents the real part of the analytic function of complex variable defined with the help of the function $\omega(\zeta)$ ([2]) as follows:

$$W(\zeta) \equiv -h^{(i)}(\xi, \eta) + i\psi(\xi, \eta) = \int_0^\zeta \frac{\omega(\zeta)}{\zeta}\, d\zeta + \frac{l}{2\pi}\left[E + \ln\left(\frac{l\sqrt{\varepsilon}}{8\pi}\right)\right]. \tag{8}$$

Functions $x(\xi, \eta)$ and $y(\xi, \eta)$, in turn, can be obtained by integrating differential correlations [2]

$$dx = -\cos\theta \sinh t\, dh^{(i)} - \sin\theta \cosh t\, d\psi, \qquad dy = -\sin\theta \sinh t\, dh^{(i)} + \cos\theta \cosh t\, d\psi, \tag{9}$$

where symbols t and θ stand for functions $t(\xi, \eta) = -\ln|\zeta|$ and $\theta(\xi, \eta) = \pi - \arg\zeta$.

In [2] we also describe a certain class of functions $F(\beta)$ which parametrically define a smooth contour Γ of the ovoid type and allow us to get an exact closed form of the function $W(\zeta)$. However, this class does not contain the case of a convex contour with rectification points that we are interested in. Therefore, it makes sense to use a known contour with the necessary properties. For example, the contour of the McLeod bubble [4] and [5] is convex and contains separate rectification points.

3. The case, when the fiber profile is the contour of the McLeod bubble. The mentioned contour is doubly symmetric with respect to coordinate axes (Fig. 2, *a*)). Taking into account the accepted normalization of the physical plane, we can represent its upper half in the parametric form [4], [5]

$$\Gamma: \quad x_\Gamma(\alpha) = \frac{9}{44}\cos\alpha - \frac{1}{44}\cos(3\alpha), \quad y_\Gamma(\alpha) = \frac{45}{44}\sin\alpha + \frac{1}{44}\sin(3\alpha), \tag{10}$$

where α is the representation parameter varying from $\alpha = 0$ at the point D to $\alpha = \pi$ at the point B, while the curvature of the contour Γ at points D and B equals zero. However, for applying formulas of item 2, we need to know the dependence of the curvature radius of the contour Γ on the angle β rather than representation (10). To this end, we proceed in representation (10) to the complex variable $Z_\Gamma = x_\Gamma + iy_\Gamma$, namely,

$$\Gamma: \quad Z_\Gamma(\alpha) = \frac{1}{44}\left[27\exp(i\alpha) - 18\exp(-i\alpha) - \exp(-3i\alpha)\right]. \tag{11}$$

Differentiating this expression with respect to α, we get the correlation

$$\Gamma: \quad Z'_\Gamma(\alpha) = \frac{3i}{44}\exp(i\alpha)\left[3 + \exp(-2i\alpha)\right]^2,$$

whence

$$\Gamma: \quad s'(\alpha) = -\left|Z'_\Gamma(\alpha)\right|, \qquad \beta = \arg\left\{Z'_\Gamma(\alpha)\right\} - \frac{\pi}{2}. \tag{12}$$

Substituting expression (11) in formula (12), we get correlations

$$\Gamma: \quad s'(\alpha) = -\frac{3}{22}\left[5 + 3\cos(2\alpha)\right], \quad \beta(\alpha) = \alpha - 2\arctan\left[\frac{\sin(2\alpha)}{3 + \cos(2\alpha)}\right]. \tag{13}$$

Integrating $s'(\alpha)$ with respect to α from 0 to π, we calculate the contour perimeter Γ: $l = 15\pi/11$. Furthermore, differentiating $\beta(\alpha)$, we get the formula

$$\Gamma: \quad \beta'(\alpha) = \frac{6\sin^2\alpha}{5 + 3\cos(2\alpha)}. \tag{14}$$

Dividing the first expression in (13) by (14), we get the desired form of the function $F(\beta)$

(Fig. 2, *b*))

$$F(\beta) = \frac{1}{44}\left[\frac{5+3\cos(2\alpha)}{\sin\alpha}\right]^2\Bigg|_{\alpha=\,\alpha(\beta)}, \tag{15}$$

where the function $\alpha(\beta)$ is inverse to $\beta(\alpha)$ (see the second formula in (13)).

Singularities of the function $F(\beta)$ at points D $(\alpha=\beta=0)$ and B $(\alpha=\beta=\pi)$ make the calculation of the Schwartz integral (7) by standard techniques inefficient. By finding the explicit form of these singularities of the function $F(\beta)$ and defining the corresponding principal part $\omega_0(\zeta)$ of the function $\omega(\zeta)$, we can essentially increase the effectiveness of these procedures. Then it remains only to restore (with the help of the Schwartz integral) the regular part $\omega_{\rm reg}(\zeta)$ of the function $\omega(\zeta)$ in the form

$$\omega(\zeta) = \omega_0(\zeta) + \omega_{\rm reg}(\zeta). \tag{16}$$

Let us estimate the behavior of the function $F(\beta)$ at the point D. To this end, we get the following estimate for the dependence $\beta(\alpha)$ (see the second formula in (13)) with small α:

$$\alpha \ll 1: \quad \beta(\alpha) \approx \frac{1}{4}\,\alpha^3 + \frac{23}{240}\,\alpha^5 + O(\alpha^7).$$

Inverting the function $\beta(\alpha)$, we estimate the dependence $\alpha(\beta)$ for small β, namely,

$$\beta \ll 1: \quad \alpha(\beta) \approx 4^{1/3}\beta^{1/3} - \frac{23}{45}\,\beta + O\left(\beta^{5/3}\right).$$

Substituting this bound in formula (15), we estimate the behavior of the function $F(\beta)$ near the point D $(\beta=0)$ as follows:

$$\beta \ll 1: \quad F(\beta) \approx \frac{4^{4/3}}{11}\,\beta^{-2/3} - \frac{656}{495} + O\left(\beta^{2/3}\right).$$

Evidently, the function $F(\beta)$ demonstrates a similar behavior near the point B $(\beta=\pi)$.

In accordance with the type of singularities of the function $F(\beta)$ and the correlation $\beta=\pi-\sigma$ that follows from formula (6), the principal part of the function $\omega(\zeta)$ can take the form

$$\omega_0(\zeta) = 2^{2/3}\frac{8}{11}\left[(1+\zeta)^{-2/3} - (1-\zeta)^{-2/3}\right]\zeta + \frac{296}{495}\,\zeta^2; \tag{17}$$

here one should choose single-valued branches of multivalued functions of raising to the fractional power $(-2/3)$ which have zero argument on the boundary DAB of the area Ω_ζ.

4. Definition of functions $\omega_{\rm reg}(\zeta)$ and $W(\zeta)$. Using formulas (7) and (16), we can restore the remaining regular part $\omega_{\rm reg}(\zeta)$ of the function $\omega(\zeta)$ with the help of the Schwartz integral

$$\omega_{\rm reg}(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi}\left\{\mathrm{Re}\ \omega_0(\zeta')\big|_{\zeta'=\exp(i\sigma)} + F(\beta)\big|_{\beta=\pi-\sigma}\right\}\frac{\exp(i\sigma)+\zeta}{\exp(i\sigma)-\zeta}\,d\sigma,$$

where the expression in curly braces vanishes with $\sigma=0,\ \pi,\ 2\pi$. One can calculate this expression (analogously to [2]) with the help of the fast Fourier transform and thus define real coefficients c_k, $k=0,\ldots,N$, of the expansion

$$\omega_{\rm reg}(\zeta) = -\frac{15}{22} + \sum_{k=1}^{N} c_k\zeta^k. \tag{18}$$

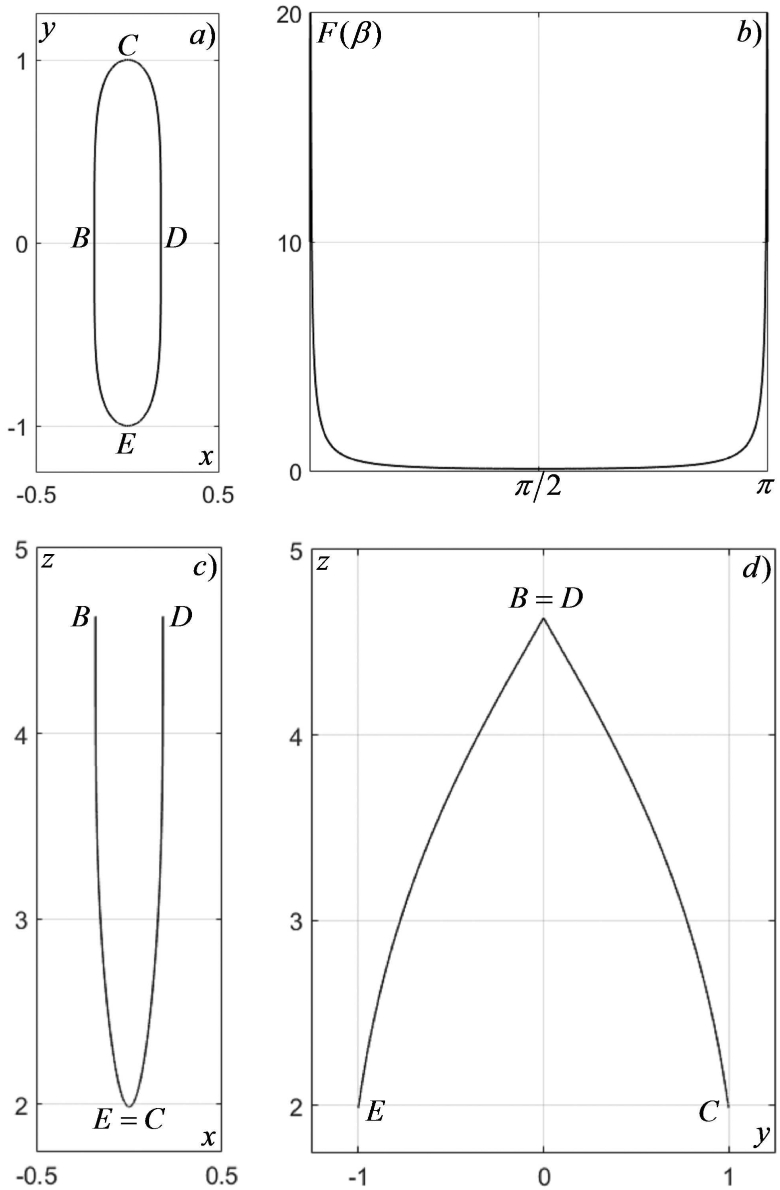

Fig. 2. *a*) The corresponding to representation (10) fiber profile Γ and *b*) function $F(\beta)$. *c*) Projections of the contact line Γ_c to planes x, z and *d*) y, z for $\varepsilon = 0.01$.

Then by using formulas (8) and (16)–(18) we can also find the function

$$
\begin{aligned}
W(\zeta) &= W_0(\zeta) + \sum_{k=1}^{N} \frac{c_k \zeta^k}{k} + \frac{15}{22}\left[E + \ln\left(\frac{15\sqrt{\varepsilon}}{88\zeta}\right)\right],\\
W_0(\zeta) &= 2^{2/3}\frac{24}{11}\left[(1-\zeta)^{1/3} + (1+\zeta)^{1/3} - 2\right] + \frac{148}{495}\,\zeta^2.
\end{aligned}
\tag{19}
$$

Furthermore, using formulas (5), (9), and (19), we can find the shape of the whole meniscus $\Sigma^{(u)}$ analogously to [2]. However, for answering the question stated in the Introduction, it suffices only to define the shape of the contact line Γ_c. To this end, we can use the initial parameterization (10) of the contour Γ with the help of the parameter α and find the distribution $h_\Gamma^{(i)}(\alpha)$ with the help of that

$$h_\Gamma^{(i)}(\sigma) = -\mathrm{Re}\, W(\zeta)\big|_{\zeta = \exp(i\sigma)},$$

where we have to proceed from σ to β and then to α, using formulas (6) and (13), namely,

$$\sigma = \pi - \alpha + 2\arctan\left[\frac{\sin(2\alpha)}{3+\cos(2\alpha)}\right].$$

5. Analysis of results. In accordance with the obtained formulas we have defined the shape of the contact line Γ_c (with $\varepsilon = 0.01$ and $N = 2^{12}$). Its projections to planes x, z and y, z are shown in Fig. 2, *c*) and *d*). The rise height of the liquid is finite, but at profile rectification points B and D the line Γ_c undergoes a break, whose angle depends on singularities of the function $\omega_0(\zeta)$ and equals $\pi/3$. At the same time, no break of the contact line has occurred in experiments with fibers with smooth convex profiles of bounded curvature. Therefore, as a result of direct verification we conclude that in the case, when the fiber material is fully wet, while the fiber itself has a smooth convex profile with separate rectification points, the asymptotic approach [1] satisfactory describes the meniscus configuration as a whole. However, in reality menisci do not necessarily have certain differential properties of the asymptotic approximation, namely, instead of the expected smooth contact line there occurs a line with breaks at profile rectification points.

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